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APPROXIMATION BY ZYGMUND SUMS IN THE CLASSES $C_\infty^{\overline{\psi}}$

In this working we consider the approximation of functions from the classes of $\overline{\psi}$-integrals by the Zygmund sums. In particularly, we obtain asymptotic formulas for the value $\mathcal{E}_n(C_\infty^{\overline{\psi}}, Z_n^s)_C = \sup\limits_{f\in C_\infty^{\overline{\psi}}} \|f(x)- -Z_n^s(f;x)\|_C$, under various conditions on functions $\psi_1(\cdot)$ and $\psi_2(\cdot)$.

1. Introduction. Let L denote the space of integrable 2π-periodic functions, and let

$$S[f] = \frac{a_o}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx) \equiv \sum_{k=0}^{\infty} A_k(f;x)$$

be the Fourier series of a function $f \in L$, i.e., for any $k = 0, 1, 2, \ldots$

$$a_k = a_k(f) = \frac{1}{\pi}\int\limits_{-\pi}^{\pi} f(t)\cos ktdt, \quad b_k = b_k(f) = \frac{1}{\pi}\int\limits_{-\pi}^{\pi} f(t)\sin ktdt.$$

The polynomials that have the form

$$Z_n^s(f;x) = \frac{a_0}{2} + \sum_{k=1}^{n-1}(1-(\frac{k}{n})^s)A_k(f;x), \quad s>0$$

are called the Zygmund sums, and Fejér sums in case of $s = 1$. In [1, Chpt.IV], $C_\infty^{\overline{\psi}}$ is class of 2π-periodic continuous functions which represented in the form of convolution

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$$f(x) = \frac{a_0}{2} + \frac{1}{\pi}\int_{-\pi}^{\pi} \varphi(x-t)\Psi(t)dt = \frac{a_0}{2} + (f^{\overline{\psi}} * \Psi)(x) \ ,$$

where $\Psi(x)$ is a certain function that has the Fourier series

$$\sum_{k=1}^{\infty}(\psi_1(k)\cos kx + \psi_2(k)\sin kx),$$

$\overline{\psi} = (\psi_1, \psi_2)$ is a pair of arbitrary fixed systems of numbers $\psi_1(k)$ and $\psi_2(k)$, $k = 1, 2, \ldots$. Here, the function φ is called $\overline{\psi}$-derivative of function f, and is denoted by $f^{\overline{\psi}}(\cdot)$, $\operatorname{ess\,sup}_t |f^{\overline{\psi}}(t)| \le 1$, $\int_{-\pi}^{\pi} f^{\overline{\psi}}(t)dt = 0$.

In [1], if $\psi_1(v) = \psi(v)\cos\frac{\beta\pi}{2}$ and $\psi_2(v) = \psi(v)\sin\frac{\beta\pi}{2}$, then the classes $C^{\overline{\psi}}_{\infty}$ coincide with the classes $C^{\psi}_{\beta,\infty}$. Moreover, if $\psi(v) = v^{-r}$, then the classes $C^{\overline{\psi}}_{\infty}$ coincide with the well known the classes Weil–Nagy $W^r_{\beta,\infty}$.

The value

$$\mathcal{E}_n(C^{\overline{\psi}}_{\infty}, Z^s_n)_C = \sup_{f\in C^{\overline{\psi}}_{\infty}} \|f(x) - Z^s_n(f;x)\|_C \qquad (1)$$

is the main subject of our studying aimed at obtained asymptotic equalities under various conditions on functions $\psi_1(\cdot)$ and $\psi_2(\cdot)$, where $\|\varphi\|_C = \max_x |\varphi(x)|$.

The value $\mathcal{E}_n(\mathfrak{N}, Z^s_n)_C$ was investigated by many mathematicians. Some of whom are A. Zygmund [2] that investigated in case of $\mathfrak{N} = W^r_{\infty}$, $r > 0$; B. Nagy, S.A. Teljakovskií [3, 4] that investigated in case of $\mathfrak{N} = W^r_{\beta,\infty}$ under various conditions on β, s, r; A.I. Stepanets, D.N. Busev [1, 5] that investigated in case of $\mathfrak{N} = C^{\psi}_{\beta,\infty}$ under the condition on function $\psi(\cdot)$; A.S. Federenko [6, 7] that investigated in case of $\mathfrak{N} = C^{\overline{\psi}}_{\infty}$ under the various conditions on functions $\psi_1(\cdot)$ and $\psi_2(\cdot)$.

The value of (1) is depend on the functions $g_i(v) = v^s\psi_i(v)$, $i = 1, 2$, which are convex upwards or convex downwards. There are five possible cases for functions $g_i(v)$, $i = 1, 2$:

a) $g_i(v)$ are convex downwards with $\lim\limits_{v\to\infty} g_i(v) = \infty$,
b) $g_i(v)$ are convex downwards with $\lim\limits_{v\to\infty} g_i(v) = C > 0$,
c) $g_i(v)$ are convex downwards with $\lim\limits_{v\to\infty} g_i(v) = 0$,
d) $g_i(v)$ are convex upwards with $\lim\limits_{v\to\infty} g_i(v) = c > 0$,
e) $g_i(v)$ are convex upwards with $\lim\limits_{v\to\infty} g_i(v) = \infty$.

It is noted that the case of (a) is possible only if $s > 1$. A. S. Fedorenko, [7], investigated the cases of b)-d) for $\psi_1 \in \mathfrak{M}$ (or $-\psi_1 \in \mathfrak{M}$), and $\psi_2 \in \mathfrak{M}'$ (or $-\psi_2 \in \mathfrak{M}'$) and got the exact asymptotic equalities in these cases for value (1). In this study, we investigated the cases of d) and e) for $\psi_1 \in \mathfrak{M}$ (or $-\psi_1 \in \mathfrak{M}$), and $\psi_2 \in \mathfrak{M}'$ (or $-\psi_2 \in \mathfrak{M}'$) for value (1). Here, [1, Chpt.IV], $\mathfrak{M}$ denotes the set of continuous positive functions $\psi(\cdot)$ which are convex downwards for all $v \geq 1$ and with $\lim\limits_{v\to\infty} \psi(v) = 0$ and $\mathfrak{M}'$ denotes the subset of functions $\psi(\cdot)$ from $\mathfrak{M}$ that satisfy in addition the following condition:

$$\int_1^\infty \frac{\psi(t)}{t} dt < \infty \ .$$

2. Main Results.

Theorem. *Let* $\psi_1 \in \mathfrak{M}$, $\psi_2 \in \mathfrak{M}'$ *and* $g_i(v) = v^s\psi_i(v)$, $s > 0$, $i = 1, 2$, *be convex upwards on* $v \geq b \geq 1$ *with* $\lim\limits_{v\to\infty} g_i(v) = \infty$ *or* $\lim\limits_{v\to\infty} g_i(v) = c_i > 0$. *Then as* $n \to \infty$, *we have*

$$\mathcal{E}_n(C^{\overline{\psi}}_{\infty}, Z^s_n)_C = \frac{2}{\pi n^s}\int\limits_1^n v^{s-1}\psi_2(v)dv + \frac{2}{\pi}\int\limits_n^{\infty}\frac{\psi_2(v)}{v}dv + O(1)\overline{\psi}(n), \quad (2)$$

where $\overline{\psi}(n) = (\psi_1^2(n) + \psi_2^2(n))^{1/2}$ *and* $O(1)$ *is a quantity uniformly bounded in* n.

Let $\psi \in \mathfrak{M}$ and $\alpha(t) = \dfrac{\psi(t)}{t|\psi'(t)|}$ for $t \geq 1$. If there exist $\lim\limits_{t\to\infty} \alpha(t)$, then let us denote value of this limit by $\alpha_0(\psi) \stackrel{\text{df}}{=} \lim\limits_{t\to\infty} \alpha(t)$. Therefore we get the following corollaries:

Corollary 1. *Let* $\psi_1 \in \mathfrak{M}$, $\psi_2 \in \mathfrak{M}'$ *and* $g_i(v) = v^s\psi_i(v)$, $s > 0$, $i = 1, 2$, *be convex upwards on* $v \geq b \geq 1$ *with* $\lim\limits_{v\to\infty} g_i(v) = \infty$. *If* $\alpha_0(\psi_2) = \infty$, *then as* $n \to \infty$, *we have the following asymptotic equality:*

$$\mathcal{E}_n(C^{\overline{\psi}}_{\infty}, Z^s_n)_C = \frac{2}{\pi}\int\limits_n^{\infty}\frac{\psi_2(v)}{v}dv + O(1)\overline{\psi}(n),$$

where $\overline{\psi}(n) = (\psi_1^2(n) + \psi_2^2(n))^{1/2}$ *and* $O(1)$ *is a quantity uniformly bounded in* n.

Remark 1. In [6], A.S. Fedorenko investigated the same problem and also D.N. Busev, [5], investigated same problem in case of $\psi_1(v) = \psi(v)\cos\dfrac{\beta\pi}{2}$ and $\psi_2(v) = \psi(v)\sin\dfrac{\beta\pi}{2}$, but they didn't find exact asymptotic equalities for value (1).

If we take functions $\psi_i(t) = \ln^{-\alpha_i}(t + e)$, $\alpha_i \geq 1$, $i = 1, 2$, then this function satisfies the conditions of Corollary 1.

Corollary 2. *Let* $\psi_1 \in \mathfrak{M}$, $\psi_2 \in \mathfrak{M}'$ *and* $g_i(v) = v^s\psi_i(v)$, $s > 0$, $i = 1, 2$, *be convex upwards on* $v \ge b \ge 1$ *with* $\lim\limits_{v\to\infty} g_i(v) = \infty$ *or* $\lim\limits_{v\to\infty} g_i(v) = c_i \ge 0$. *If* $\alpha_0(\psi_2) = 1/s$ *then as* $n \to \infty$, *we have the following asymptotic equality:*

$$\mathcal{E}_n(C^{\overline{\psi}}_\infty, Z^s_n)_C = \frac{2}{\pi n^s}\int\limits_1^n v^{s-1}\psi_2(v)dv + O(1)\overline{\psi}(n),$$

where $\overline{\psi}(n) = (\psi_1^2(n) + \psi_2^2(n))^{1/2}$ *and* $O(1)$ *is a quantity uniformly bounded in* n.

Remark 2. This result coincides with Theorem 1 in [7] in case of that $g_i(v) = v^s\psi_i(v)$, $s > 0$, $i = 1, 2$, is convex upwards on $v \ge b \ge 1$ with $\lim\limits_{v\to\infty} g_i(v) = c_i \ge 0$.

If we take functions $\psi_i(t) = \dfrac{\ln(t^{r_i} + e)}{t^s}$, $o < r_i < s$, $0 < r_i < 1$, $i = 1, 2$, and $\psi_i(t) = \dfrac{t^{\varepsilon_i} - c_i}{t^{\varepsilon_i + s}}$, $\varepsilon_i > 0$, $s > 0$, $c_i \ge 0$, $i = 1, 2$, then these functions satisfy the conditions of Corollary 2.

Corollary 3. *Let* $\psi_1 \in \mathfrak{M}$, $\psi_2 \in \mathfrak{M}'$ *and* $g_i(v) = v^s\psi_i(v)$, $s > 0$, $i = 1, 2$, *be convex upwards on* $v \ge b \ge 1$ *with* $\lim\limits_{v\to\infty} g_i(v) = \infty$. *If* $\alpha_0(\psi_2) \in (1/s, \infty)$, *then as* $n \to \infty$, *we have*

$$\mathcal{E}_n(C^{\overline{\psi}}_\infty, Z^s_n)_C = O(1)\overline{\psi}(n),$$

where $\overline{\psi}(n) = (\psi_1^2(n) + \psi_2^2(n))^{1/2}$ *and* $O(1)$ *is a quantity uniformly bounded in* n.

If we take functions $\psi_i(t) = \dfrac{1}{t^{\varepsilon_i}}$, $0 < \varepsilon_i < s < \varepsilon_i + 1$, $i = 1, 2$, then this function satisfies the conditions of Corollary 3.

3. Some Auxiliary Results. In this section, we shall give some auxiliary results which used for the proof of the Theorem.

Proposition 1. *Let* $\psi_1(\cdot) \in \mathfrak{M}$ *and let* $g_1(v) = v^s\psi_1(v)$, $s > 0$, *be convex upwards on* $v \ge b \ge 1$ *with* $\lim\limits_{v\to\infty} g_1(v) = \infty$

or $\lim_{v\to\infty} g_1(v) = c_1 > 0$. *Then as* $n \to \infty$, *we have*

$$\int_{-\infty}^{\infty} |\frac{1}{\pi} \int_{0}^{\infty} \tau_1(v) \cos vt \, dv| dt = O(1)\psi_1(n), \tag{3}$$

where

$$\tau_1(v) = \begin{cases} \dfrac{v\psi_1(1)}{n^s} & , 0 \le v \le 1 \\ \dfrac{v^s\psi_1(v)}{n^s} & , 1 \le v \le n \\ \psi_1(v) & , v \ge n \end{cases} \tag{4}$$

where $O(1)$ *is a quantity uniformly bounded in* n.

Proposition 2. *Let* $\psi_2(\cdot) \in \mathfrak{M}'$ *and let* $g_2(v) = v^s\psi_2(v)$, $s > 0$, *be convex upwards on* $v \ge b \ge 1$ *with* $\lim_{v\to\infty} g_2(v) = \infty$ *or* $\lim_{v\to\infty} g_2(v) = c_2 > 0$. *Then as* $n \to \infty$, *we have*

$$\int_{-\infty}^{\infty} |\frac{1}{\pi} \int_{0}^{\infty} \tau_2(v) \sin vt \, dv| dt = \frac{2}{\pi n^s} \int_{1}^{n} v^{s-1}\psi_2(v) dv + \frac{2}{\pi} \int_{n}^{\infty} \frac{\psi_2(v)}{v} dv + +O(1)\psi_2(n), \tag{5}$$

where

$$\tau_2(v) = \begin{cases} \dfrac{v\psi_2(1)}{n^s} & , 0 \le v \le 1 \\ \dfrac{v^s\psi_2(v)}{n^s} & , 1 \le v \le n \\ \psi_2(v) & , v \ge n \end{cases} \tag{6}$$

and $O(1)$ *is a quantity uniformly bounded in* n.

4. Proofs. First of all, we will began with proof of the Proposition 1-2, after give proof of the Theorem and finally the proofs of Corollaries.

Proof of Proposition 1. By partial integration, we have

$$\int_0^\infty \tau_1(v)\cos vt dv = -\frac{1}{t}\int_0^n \tau_1'(v)\sin vt dv - \frac{1}{t}\int_n^\infty \tau_1'(v)\sin vt dv.$$

Hence

$$\int_{-\infty}^\infty |\frac{1}{\pi}\int_0^\infty \tau_1(v)\cos vt\ dv|dt \le \int_{-\infty}^\infty |\frac{1}{\pi t}\int_0^n \tau_1'(v)\sin vt dv|dt +$$

$$+\int_{-\infty}^\infty |\frac{1}{\pi t}\int_n^\infty \psi_1'(v)\sin vt dv|dt := I_{n,1} + I_{n,2}. \tag{7}$$

Now let us estimate that

$$I_{n,1} = O(1)\psi_1(n). \tag{8}$$

For any fixed $n \in \mathbb{N}$ the function $\tau_n'(v)$ is a nonincreasing function on $[0, n]$, therefore for all $t > 0$

$$\frac{1}{t}\int_0^n \tau_1'(v)\sin vt dv \ge 0. \tag{9}$$

Taking account to (9), we obtain

$$I_{n,1} = 2\int_0^\infty \frac{1}{\pi t}\int_0^n \tau_1'(v)\sin vt dv dt = \frac{2}{\pi}\int_0^n \tau_1'(v)\int_0^\infty \frac{\sin vt}{t} dt dv =$$

$$= \int_0^n \tau_1'(v) dv = \tau_1(n) - \tau_1(0) = \psi_1(n).$$

Then we get (8). From proof of lemma 4.3.1 [1], we know that

$$I_{n,2} = O(1)\psi_1(n). \tag{10}$$

Therefore according to (8) and (10), we have (3). The proof of Proposition 1 is completed.

Proof of Proposition 2. The function $\tau_2(v)$ is a nonnegative continuous function on interval $[0,\infty)$ that is increasing on intervals [0,n] and

$$\lim_{v\to\infty} \tau_2(v) = \lim_{v\to\infty} \tau_2'(v) = 0.$$

Hence, by applying two times partial integration, we have

$$\frac{1}{\pi}\int_0^\infty \tau_2(v)\sin vt dv = \frac{1}{\pi t^2}[(\tau_2'(1-0) - \tau_2'(1+0))\sin t +$$

$$+(\tau_2'(n-0) - \tau_2'(n+0))\sin nt -$$

$$(\int_1^n \tau_2''(v)\sin vt dv + \int_n^\infty \tau_2''(v)\sin vt dv)]. \tag{11}$$

From (11), since $g_2(v) = v^s\psi_2(v)$ is increasing, we get

$$\begin{aligned} |\frac{1}{\pi}\int_0^\infty \tau_2(v)\sin vt dv| &\le \frac{1}{\pi t^2}(2s\frac{\psi_2(n)}{n} + \frac{\psi_2(1)}{n^s}) \le \\ &\le \frac{(2s+1)\psi_2(n)}{\pi t^2}. \end{aligned} \tag{12}$$

Therefore, accordingly (12) we have

$$\int_{|t|\ge\pi/2} |\frac{1}{\pi}\int_0^\infty \tau_2(v)\sin vt dv| dt = 2\int_{\pi/2}^\infty \frac{1}{\pi}|\int_0^\infty \tau_2(v)\sin vt dv| dt \le$$

$$\leq \frac{2(2s+1)}{\pi^2}\psi_2(n) = O(1)\psi_2(n). \tag{13}$$

By partial integration, we have

$$\frac{1}{\pi}\int_0^\infty \tau_2(v)\sin vt dv = \frac{1}{\pi t}\Big(\int_0^n \tau_2'(v)\cos vt dv + \int_n^\infty \tau_2'(v)\cos vt dv\Big). \tag{14}$$

To estimate the integral on right hand of (14), first of all we will consider integrals on [0,n] and show that

$$\int_{\pi/2n}^{\pi/2} \Big|\frac{1}{t}\int_0^n \tau_2'(v)\cos vt dv\Big| dt = \frac{1}{n^s}\int_1^n v^{s-1}\psi_2(v)dv + O(\psi_2(n)). \tag{15}$$

In this case, let represent the function under the integral sign in left part of (15) in such form:

$$\frac{1}{t}\int_0^n \tau_2'(v)\cos vt dv = \frac{1}{t}\int_0^{\pi/2t} \tau_2'(v)\cos vt dv + \frac{1}{t}\int_{\pi/2t}^n \tau_2'(v)\cos vt dv =$$

$$\stackrel{\text{df}}{=} I_1(t) + I_2(t). \tag{16}$$

In order to prove of (15) it will suffice to prove following equalities

$$\int_{\pi/2n}^{\pi/2} |I_1(t)| dt = \frac{1}{n^s}\int_1^n v^{s-1}\psi_2(v)dv + O(\psi_2(n)) \tag{17}$$

and

$$\int_{\pi/2n}^{\pi/2} |I_2(t)| dt \leq O(\psi_2(n)). \tag{18}$$

Since the function $\tau_2'(v)$ is nonnegative and nonincreasing on $[0, n]$, then $I_1(t) \geq 0$, $t \in [\pi/2n, \pi/2]$, and $I_2(t) \leq 0$, $t \in [\pi/2n, \pi/2]$. Therefore changing the order of integration, we obtain

$$\int_{\pi/2n}^{\pi/2} |I_1(t)| dt = \int_{\pi/2n}^{\pi/2} \frac{1}{t} \int_0^{\pi/2t} \tau_2'(v) \cos vt \, dv \, dt =$$

$$= \int_0^1 \int_{\pi/2n}^{\pi/2} \frac{\cos vt}{t} \, dt \, d\tau_2(v) + \int_1^n \int_{\pi/2n}^{\pi/2v} \frac{\cos vt}{t} \, dt \, d\tau_2(v) =$$

$$= \int_0^1 \int_{\pi v/2n}^{\pi v/2} \frac{\cos z}{z} \, dz \, d\tau_2(v) + \int_1^n \int_{\pi v/2n}^{\pi/2} \frac{\cos z}{z} \, dz \, d\tau_2(v) =$$

$$= \left(\tau_2(v) \int_{\pi v/2n}^{\pi v/2} \frac{\cos z}{z} \, dz \right)\Bigg|_0^1 - \int_0^1 \frac{\tau_2(v)}{v} (\cos \frac{\pi v}{2} - \cos \frac{\pi v}{2n}) \, dv +$$

$$+ \left(\tau_2(v) \int_{\pi v/2n}^{\pi/2} \frac{\cos z}{z} \, dz \right)\Bigg|_1^n + \int_1^n \frac{\tau_2(v)}{v} \cos \frac{\pi v}{2n} \, dv =$$

$$= \frac{1}{n^s} \int_1^n v^{s-1} \psi_2(v) \cos \frac{\pi v}{2n} \, dv + \frac{\psi_2(1)}{n^s} \int_0^1 (\cos \frac{\pi v}{2n} - \cos \frac{\pi v}{2}) \, dv =$$

$$= \frac{1}{n^s} \int_1^n v^{s-1} \psi_2(v) \cos \frac{\pi v}{2n} \, dv + O(\frac{1}{n^s}). \tag{19}$$

Now let's show that

$$\frac{1}{n^s} \int_1^n v^{s-1} \psi_2(v) \cos \frac{\pi v}{2n} dv = \frac{1}{n^s} \int_1^n v^{s-1} \psi_2(v) dv + O(\psi_2(n)). \tag{20}$$

For proof of (20) we will obtain necessary estimation of following difference

$$\frac{1}{n^s}\int_1^n v^{s-1}\psi_2(v)(1-\cos\frac{\pi v}{2n})\,dv=$$

$$=\frac{2}{n^s}\int_1^n v^{s-1}\psi_2(v)v\frac{\sin\pi v/4n}{\pi v/4n}\frac{\pi}{4n}\sin\frac{\pi v}{4n}\,dv\le$$

$$\le\frac{2}{n^s}\psi_2(n)n^s\frac{\pi}{4n}\int_1^n\sin\frac{\pi v}{4n}\,dv\le 2\psi_2(n).$$

Hence by combining (19) and (20), we get (17). Now we will obtain (18). Since $I_2(t)\le 0,\ t\in[\frac{\pi}{2n},\frac{\pi}{2}]$, then we have

$$\int_{\pi/2n}^{\pi/2}|I_2(t)|dt=-\int_{\pi/2n}^{\pi/2}\frac{1}{t}\int_{\pi/2t}^{n}\tau_2'(v)\cos vt\,dv\,dt=$$

$$=-\int_1^n\tau_2'(v)\int_{\pi/2v}^{\pi/2}\frac{\cos vt}{t}\,dt\,dv=-\int_1^n\tau_2'(v)\int_{\pi/2}^{\pi v/2}\frac{\cos z}{z}\,dz\,dv\le$$

$$\le 2\mathrm{ci}(\frac{\pi}{2})\int_1^n\tau_2'(v)dv=2\mathrm{ci}(\frac{\pi}{2})(\psi_2(n)-\frac{\psi_2(1)}{n^s})\le O(\psi_2(n)).\qquad(21)$$

Therefore take into account (17) and (21), we get (15).
Now we will estimate integral on interval $[n,\infty)$ on right hand of (14). Taking into account that $\tau_2(v)$ is convex downwards with $\lim_{v\to\infty}\tau_2(v)=0$ on interval $[n,\infty)$ and that $-\frac{n\psi_2'(n)}{\psi_2(n)}<s$, by partial

integration, we get

$$\frac{1}{\pi t}|\int_n^\infty \tau_2'(v)\cos vtdv| = \frac{1}{\pi t^2}|(\tau_2'(v)\sin vt|_n^\infty) - \int_n^\infty \tau_2''(v)\sin vtdv| \le$$

$$\le \frac{2}{\pi n t^2}\left(-\frac{n\psi_2'(n)}{\psi_2(n)}\right)\psi_2(n) < \frac{2s\psi_2(n)}{\pi n t^2}. \tag{22}$$

According to (22), we obtain

$$\int_{\pi/2n}^{\pi/2} |\frac{1}{\pi t}\int_n^\infty \tau_2'(v)\cos vtdv|dt \le \frac{2s}{\pi^2}\psi_2(n) = O(1)\psi_2(n). \tag{23}$$

Taking account to (15) and (22), we have

$$\int_{\frac{\pi}{2n}\le|t|\le\frac{\pi}{2}} |\frac{1}{\pi}\int_0^\infty \tau_2(v)\sin vtdv|dt = \frac{2}{\pi n^s}\int_1^n v^{s-1}\psi_2(v)dv + O(1)\psi_2(n). \tag{24}$$

Now let's investigate in neighborhood of origin: Since $\tau_2(v) = \psi_2(v)$ on $[n,\infty)$, in [1, p. 204], there exist $a > 0$ for $\forall\ n \ge 1$, such that we have

$$\int_{|t|\le a/n} |\frac{1}{\pi}\int_n^\infty \tau_2(v)\sin vtdv|dt = \frac{2}{\pi}\int_n^\infty \frac{\psi_2(v)}{v}dv + O(1)\overline{\psi}(n). \tag{25}$$

After that we obtain that

$$2\frac{1}{\pi}|\int_{a/n}^{\pi/2n}\int_n^\infty \tau_2(v)\sin vtdv|dt \le \frac{1}{\pi}|\int_{a/n}^{\pi/2n} \frac{2\psi_2(n)}{nt^2}dt \le O(1)\psi_2(n). \tag{26}$$

Finally we will estimate integral

$$\int_{-\pi/2n}^{\pi/2n} |\frac{1}{\pi}\int_{0}^{n} \tau_2(v)\sin vtdv|dt = 2\int_{0}^{\pi/2n} |\frac{1}{\pi}\int_{0}^{n} \tau_2(v)\sin vtdv|dt.$$

By considering that $\tau_2(v)$ is a continuous increasing function on interval [0,n], we have $|\tau_2(v)| \le \psi_2(n)$. Hence, we obtain

$$\int_{-\pi/2n}^{\pi/2n} |\frac{1}{\pi}\int_{0}^{n} \tau_2(v)\sin vtdv|dt \le 2\psi_2(n) = O(1)\psi_2(n). \qquad (27)$$

Therefore, by using (15) and (23)-(27), for $n \ge 1$, we get (5). The proof of Proposition 2 is completed.

Proof of Theorem. For any $f \in C_{\infty}^{\overline{\psi}}$

$$f(x) - Z_n^s(f;x) = \int_{-\infty}^{\infty} f^{\overline{\psi}}(x-t)\hat{\tau}_n(t)dt, \qquad (28)$$

equality is true, [1, Chp. IV], where

$$\hat{\tau}_n(t)=\hat{\tau}_{1+}(t)+\hat{\tau}_{2-}(t) \overset{\text{df}}{=} \frac{1}{\pi}\int_{0}^{\infty}\tau_1(v)\cos vtdv+\frac{1}{\pi}\int_{0}^{\infty}\tau_2(v)\sin vtdv. \quad (29)$$

$\tau_i(v)$ which is determined by (4) and (6), respectively for $i = 1, 2$, are functions continuous for all $v \ge 0$, and their transformations $\hat{\tau}_{1+}(t)$ and $\hat{\tau}_{2-}(t)$ are absolutely summable on the entire number axis.
According to the classes $C_{\infty}^{\overline{\psi}}$ are invariant under the shift of an argument, that is, if $f \in C_{\infty}^{\overline{\psi}}$, then the function $f_1(x) = f(x+h)$

also belongs to $C^{\overline{\psi}}_{\infty}$ for any fixed $h \in \mathbb{R}$, from (28) we have

$$\mathcal{E}_n(C^{\overline{\psi}}_{\infty}, Z^s_n)_C = \sup_{f \in C^{\overline{\psi}}_{\infty}} \left| \int_{-\infty}^{\infty} f^{\overline{\psi}}(-t)\hat{\tau}_n(t)dt \right| \leq \int_{-\infty}^{\infty} |\hat{\tau}_n(t)|dt. \quad (30)$$

On the other hand, since the inequality $|\varphi(t)| \leq 1$ *a.e.* is true, there exist a function $f(x) = f(\varphi; x)$ for which $f^{\overline{\psi}}(x) = \varphi(x)$, in classes $C^{\overline{\psi}}_{\infty}$. Hence, there exist a function $f_*(t)$ in $C^{\overline{\psi}}_{\infty}$ such that it has the form

$$f^{\overline{\psi}}_*(t) = \begin{cases} \operatorname{sign}(\hat{\tau}_n(t)), & |t| \leq \frac{\pi}{2} \\ \textit{Periodicly}, & |t| \geq \frac{\pi}{2} \end{cases} \quad (31)$$

on the set $I_1 \cup I_2$, where $I_1 = \{t : |t| \leq \dfrac{\pi}{2}\}$, $I_2 = \{t : |t| \geq \dfrac{\pi}{2}\}$. Then, owing to (31), we have

$$f_*(0) - Z^s_n(f_*; 0) = \int_{-\infty}^{\infty} |\hat{\tau}_n(t)|dt - \int_{|t| \geq \frac{\pi}{2}} |\hat{\tau}_n(t)|dt + \int_{|t| \geq \frac{\pi}{2}} f^{\overline{\psi}}_*(-t)\hat{\tau}_n(t)dt \geq$$

$$\geq \int_{-\infty}^{\infty} |\hat{\tau}_n(t)|dt - 2 \int_{|t| \geq \frac{\pi}{2}} |\hat{\tau}_n(t)|dt. \quad (32)$$

Taking into account (30) and (32), as $n \to \infty$, we obtain

$$\mathcal{E}_n(C^{\overline{\psi}}_{\infty}, Z^s_n)_C = \int_{-\infty}^{\infty} |\hat{\tau}_n(t)|dt + \gamma(n), \quad s > 0 \quad (33)$$

where $\gamma(n) \le 0$ and

$$|\gamma(n)| = O(\int\limits_{|t|\ge\frac{\pi}{2}} |\hat{\tau}_n(t)|dt) \quad . \tag{34}$$

By using (33) and Proposition 1-2, we will proof the Theorem. Firstly, let us estimate $\gamma(n)$:

$$|\gamma(n)| \le O(1) \int\limits_{|t|\ge\frac{\pi}{2}} |\hat{\tau}_n(t)|dt \le O(1) \int\limits_{|t|\ge\frac{\pi}{2}} |\frac{1}{\pi}\int\limits_0^\infty \tau_1(v)\cos vt\, dv|dt+$$

$$+O(1) \int\limits_{|t|\ge\frac{\pi}{2}} |\frac{1}{\pi}\int\limits_0^\infty \tau_2(v)\sin vt\, dv|dt := \gamma_1 + \gamma_2.$$

We obtain owing to the estimations of (3) and (13) that $\gamma_1 = O(1)\psi_1(n)$ and $\gamma_2 = O(1)\psi_2(n)$, respectively. Hence we have $|\gamma(n)| \le O(1)\overline{\psi}(n)$. Finally, according to Proposition 1-2, we get (2). Therefore, the proof of the Theorem is completed.

Proof of Corollary 1-3. By L'Hopital's and Leibniz rules we obtain the following correlations:

$$\lim_{x\to\infty} \frac{\frac{1}{x^s}\int_1^x v^{s-1}\psi_2(v)dv}{\int_x^\infty \frac{\psi_2(v)}{v}dv} = -1 + \lim_{x\to\infty} \frac{s\int_1^x v^{s-1}\psi_2(v)dv}{x^s\psi_2(x)} =$$

$$= -1 + \lim_{x\to\infty} \frac{sx^{s-1}\psi_2(x)}{sx^{s-1}\psi_2(x) - x^s|\psi_2'(x)|} = -1 + \lim_{x\to\infty} \frac{1}{1 - \frac{x|\psi_2'(x)|}{s\psi_2(x)}}, \tag{35}$$

$$\lim_{x\to\infty} \frac{\psi_2(x)}{\dfrac{1}{x^s}\displaystyle\int_1^x v^{s-1}\psi_2(v)dv} = \lim_{x\to\infty} s - \frac{x|\psi_2'(x)|}{\psi_2(x)} \tag{36}$$

and

$$\lim_{x\to\infty} \frac{\displaystyle\int_x^\infty \frac{\psi_2(v)}{v}dv}{\psi_2(x)} = \lim_{x\to\infty} \frac{-\dfrac{\psi_2(x)}{x}}{\psi_2'(x)} = \lim_{x\to\infty} \frac{\psi_2(x)}{x|\psi_2'(x)|} \tag{37}$$

Therefore, the proofs of the Corollary 1–3 are easily get by correlations (2) and (35)–(37).

1. *Stepanets A.I.* Methods of Approximation Theory // Boston.,VSP.— 2005.— 880 p.
2. *Zygmund A.* The approximation of functions by typical means of their Fourier series // Duke Math.J. — 1945. — V.12. — P. 695–704.
3. *Nagy B.* Sur une class generall de procedes de sommation pour les de Fourier // Hung. Acta. Math. — 1948. —V. 1. — P. 14–62.
4. *Teljakovskiı́ S.A.* On norms of trigonometric polynomials and approximation of differentiable functions by linear averages of their Fourier series I // Trudy Mat. Inst. Steklov.— 1961. — V. 62. — P. 61–97; English transl., Amer. Math. Soc. Transl. // — 1963. — V. 28. — P. 283–322.
5. *Bushev D. N.* Approximation of classes of continuous periodic functions Zygmund sums [in Russian] // Institute of Mathematics, Ukrainian Academy of Sciences,Kiev. —1984. — Preprint no. **84.56**. — 64 p.
6. *Fedorenko A.S.* Approximation by Zygmund sums in the classes $C^{\overline{\psi}}_{\infty}$ [in Ukrainian] // Ukrainian Math. Jour. —2000. — V. 52. — P. 856–860.
7. *Fedorenko A.S.* The speed of convergence of Zygmund sums on the classes $C^{\overline{\psi}}_{\infty}$[in Ukrainian] // Approximation and its applications: — 2000.— V. 31. — Proc. of Institute of Math. NAS of Ukrainian. Kiev. — P. 122–127.