Binary jumps in continuum. I. Equilibrium processes and their scaling limits

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Abstract

Let Γ denote the space of all locally finite subsets (configurations) in $\mathbb{R}^d$. A stochastic dynamics of binary jumps in continuum is a Markov process on Γ in which pairs of particles simultaneously hop over $\mathbb{R}^d$. In this paper, we study an equilibrium dynamics of binary jumps for which a Poisson measure is a symmetrizing (and hence invariant) measure. The existence and uniqueness of the corresponding stochastic dynamics are shown. We next prove the main result of this paper: a big class of dynamics of binary jumps converge, in a diffusive scaling limit, to a dynamics of interacting Brownian particles. We also study another scaling limit, which leads us to a spatial birth-and-death process in continuum. A remarkable property of the limiting dynamics is that its generator possesses a spectral gap, a property which is hopeless to expect from the initial dynamics of binary jumps.

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1 Introduction

Let $\Gamma = \Gamma_{\mathbb{R}^d}$ denote the space of all locally finite subsets (configurations) in $\mathbb{R}^d$, $d \in \mathbb{N}$. A stochastic dynamics of binary jumps in continuum is a Markov process on Γ in which pairs of particles simultaneously hop over $\mathbb{R}^d$, i.e., at each jump time two points of the configuration change their position. Thus, an (informal) generator of such a process has the form

$$(LF)(\gamma) = \sum_{\{x_1,x_2\}\subset\gamma} \int_{(\mathbb{R}^d)^2} Q(x_1,x_2,dh_1\times dh_2) \\ \times \big(F(\gamma\setminus\{x_1,x_2\}\cup\{x_1+h_1,x_2+h_2\}) - F(\gamma)\big). \quad (1)$$

Here, the measure $Q(x_1,x_2,dh_1\times dh_2)$ describes the rate at which two particles, x_1 and x_2, of configuration γ simultaneously hop to x_1+h_1 and x_2+h_2, respectively. Generally speaking, this rate may also depend on the rest of the configuration, $\gamma\setminus\{x_1,x_2\}$. However, in our current studies, we will restrict out attention to the case of the generator (1) only. As the reader will see below, already such dynamics lead, in a scaling limit, to interesting diffusion dynamics.

The stochastic dynamics of binary jumps may be compared with the Kawasaki dynamics in continuum. The latter is a Markov process on Γ in which particles hop over $\mathbb{R}^d$ so that, at each jump time, only one particle changes its position. For a study of equilibrium Kawasaki dynamics in continuum, we refer the reader to the papers [5, 7, 9, 12, 14, 16] and the references therein.

In this paper, we will study an equilibrium dynamics of binary jumps for which a Poisson measure is a symmetrizing (and hence invariant) measure. In several cases, an equilibrium stochastic dynamics on Γ with a Poisson symmetrizing measure is a free dynamics, i.e., there is no interaction between particles. For example, this is true for a Surgailis process (in particular, the Glauber dynamics without interaction) [25, 26] (see also [15]). Note that a Surgailis generator, in the symmetric Fock space realization of the L^2-space of Poisson measure, is the second quantization of the generator of a one-particle dynamics. Another example of a Surgalis dynamics is the free Kawasaki dynamics. There a Poisson measure is a symmetrizing one, and in the course of random evolution each particle of the configuration randomly hops over $\mathbb{R}^d$ without any interaction with other particles.

Let us stress at this point one essential difference between lattice and continuous systems. An important example of a Markov dynamics on lattice configurations is the so-called exclusion process. In this process particles jump over the lattice with only restriction to have no more than one particle at each point of the lattice. This process may have a Bernoulli measure as an invariant (and even symmetrizing) measure but the corresponding stochastic dynamics has non-trivial properties and possess interesting and reach scaling limit behaviors. A straightforward generalization of the exclusion process to the continuum gives just free Kawasaki dynamics because the exclusion restriction (and an interaction between particles) will obviously disappear for configurations in continuum. To introduce (in certain sense simplest) interaction we consider the generator above. The dynamics of binary jumps is not anymore a free particle process. In fact, in the mentioned Fock space realization, the generator of this dynamics has a Jacobi matrix (three-diagonal) form.

When this paper was nearing completion, the reference [2] came to our attention. There, at a rather heuristic level, the authors discuss a special kind of a stochastic dynamics of binary jumps, and derive a Boltzmann-type equation through a Vlasov-type scaling limit of this dynamics. In particular, the underlying space is $\mathbb{R}^3$, and points in $\mathbb{R}^3$ are treated as velocities of particles, rather than their positions. As a result of pair interaction, at random times, two particles change their velocities from v_i and v_j to $v_i' = v_i + h$ and $v_j' = v_j - h$, respectively. Thus, $v_i + v_j = v_i' + v_j'$ and hence the law of conservation of momentum is satisfied for this system. Hence, for such a dynamics, the measure $Q(x_1, x_2, dh_1 \times dh_2)$ in formula (1) is concentrated on the set

$$\{(h, -h) \mid h \in \mathbb{R}^3\} \subset (\mathbb{R}^3)^2.$$

In fact, further assumptions on Q appearing in [2] are almost identical to ours in this special case. Throughout the paper, we have added a series of statements and remarks regarding such a dynamics.

The paper is organized as follows. In Section 2, using the theory of Dirichlet forms [8, 17], we construct a rather general dynamics on the configuration space, whose generator has the form (1) on a set of test cylinder functions on Γ.

In Section 3, we show that the generator (1) with domain being the set of test cylinder functions uniquely identifies a Markov process on Γ. More exactly, this generator is essentially self-adjoint in the L^2-space of Poisson measure. This is done through an explicit formula for the form of the generator (1) realized as an operator acting in the symmetric Fock space. The reader may find these formulas to be of independent interest.

The central result of the paper is in Section 4, where we show that a big class of dynamics of binary jumps converge, in a diffusive scaling limit, to a dynamics of interacting Brownian particles. The form of the generator of the limiting diffusion resembles the generator of the gradient stochastic dynamics (e.g. [1, 20, 24, 27]), while staying symmetric with respect to the Poisson measure, rather than with respect to a Gibbs measure (as it is the case for the gradient stochastic dynamics). We prove the convergence of processes at the level of convergence, in the L^2-norm, of their generators applied to a test cylinder function on Γ.

Finally, in Section 5, we study another scaling limit of a class of dynamics of binary jumps which leads to a spatial birth-and-death process in continuum, in which pairs of particles, as well as single particles randomly appear (are born) and disappear (die). We prove the convergence of processes at the level of weak convergence of their finite-dimensional distributions. A remarkable property of the limiting dynamics is that its generator possesses a spectral gap, a property which is hopeless to expect from the generator of the initial dynamics of binary jumps. We also note that the result of the scaling essentially depends on the initial distribution of the dynamics.

In the second part of this paper [6] we discuss non-equilibrium dynamics of binary jumps. In particular, we show that a Vlasov-type mesoscopic scaling for such a dynamics leads to a generalized Boltzmann non-linear equation for the particle density.

2 Existence of dynamics

The configuration space over $\mathbb{R}^d$, $d \in \mathbb{N}$, is defined as the set of all subsets of $\mathbb{R}^d$ which are locally finite:

$$\Gamma := \left\{ \gamma \subset \mathbb{R}^d \mid |\gamma_\Lambda| < \infty \text{ for each compact } \Lambda \subset \mathbb{R}^d \right\}.$$

Here $|\cdot|$ denotes the cardinality of a set and $\gamma_\Lambda := \gamma \cap \Lambda$. One can identify any $\gamma \in \Gamma$ with the positive Radon measure $\sum_{x\in\gamma} \delta_x \in \mathcal{M}(\mathbb{R}^d)$, where δ_x is the Dirac measure with mass at x, and $\mathcal{M}(\mathbb{R}^d)$ stands for the set of all positive Radon measures on the Borel σ-algebra $\mathcal{B}(\mathbb{R}^d)$. The space Γ can be endowed with the relative topology as a subset of the space $\mathcal{M}(\mathbb{R}^d)$ with the vague topology, i.e., the weakest topology on Γ with respect to which all maps

$$\Gamma \ni \gamma \mapsto \langle f, \gamma \rangle := \int_{\mathbb{R}^d} f(x)\, \gamma(dx) = \sum_{x\in\gamma} f(x), \quad f \in C_0(\mathbb{R}^d),$$

are continuous. Here, $C_0(\mathbb{R}^d)$ is the space of all continuous functions on $\mathbb{R}^d$ with compact support. We will denote by $\mathcal{B}(\Gamma)$ the Borel σ-algebra on Γ.

We introduce the set $\mathcal{F}C_{\mathrm{b}}(C_0(\mathbb{R}^d), \Gamma)$ of all functions on Γ of the form

$$F(\gamma) = g(\langle \varphi_1, \gamma \rangle, \dots, \langle \varphi_N, \gamma \rangle), \tag{1}$$

where $N \in \mathbb{N}$, $\varphi_1, \dots, \varphi_N \in C_0(\mathbb{R}^d)$ and $g \in C_{\mathrm{b}}(\mathbb{R}^N)$, where $C_{\mathrm{b}}(\mathbb{R}^N)$ denotes the space of all continuous bounded functions on $\mathbb{R}^N$.

For any $x_1, x_2 \in \mathbb{R}^d$, $x_1 \neq x_2$, let $Q(x_1, x_2, dh_1 \times dh_2)$ be a measure on $\big((\mathbb{R}^d)^2, \mathcal{B}((\mathbb{R}^d)^2)\big)$. Some assumptions on Q will be discussed below. We are interested in a (formal) pre-generator of a Markov processes on Γ which has the form (1) on the set $\mathcal{F}C_{\mathrm{b}}(C_0(\mathbb{R}^d), \Gamma)$. We assume that, for any fixed $A, B \in \mathcal{B}(\mathbb{R}^d)$,

$$(x_1, x_2) \mapsto Q(x_1, x_2, A \times B)$$

is a measurable function. In order that the integration in (1) do not depend on the order of x_1, x_2, we also assume that

$$Q(x_1, x_2, A \times B) = Q(x_2, x_1, B \times A), \quad x_1, x_2 \in \mathbb{R}^d,\ x_1 \neq x_2,\ A, B \in \mathcal{B}(\mathbb{R}^d). \tag{2}$$

We would like $(L, \mathcal{F}C_{\mathrm{b}}(C_0(\mathbb{R}^d), \Gamma))$ to be a symmetric operator in the (real) L^2-space $L^2(\Gamma, \pi_z)$. Here π_z denotes the Poisson measure on $(\Gamma, \mathcal{B}(\Gamma))$ with intensity measure

$z\,dx$, $z>0$. We recall that π_z is uniquely characterized by the Mecke identity: for any measurable function $G:\Gamma\times\mathbb{R}^d\to[0,\infty]$,

$$\int_\Gamma \pi_z(d\gamma)\sum_{x\in\gamma}G(\gamma,x)=\int_\Gamma \pi_z(d\gamma)\int_{\mathbb{R}^d} z\,dx\,G(\gamma\cup\{x\},x). \tag{3}$$

Assume that, for each $\Lambda\in\mathcal{B}_0(\mathbb{R}^d)$ (a bounded Borel subset of $\mathbb{R}^d$),

$$\int_{\mathbb{R}^d}dx_1\int_{\mathbb{R}^d}dx_2\int_{(\mathbb{R}^d)^2}Q(x_1,x_2,dh_1\times dh_2)(\mathbf{1}_\Lambda(x_1)+\mathbf{1}_\Lambda(x_1+h_1))<\infty. \tag{4}$$

Here, $\mathbf{1}_\Lambda$ denotes the indicator function of Λ. Using (1) and (2)–(4), one easily concludes that the quadratic form

$$\mathcal{E}(F,G):=\int_\Gamma(-LF)(\gamma)G(\gamma)\,\pi_z(d\gamma),\quad F,G\in\mathcal{F}C_{\mathrm{b}}(C_0(\mathbb{R}^d),\Gamma),$$

is well defined.

In order to achieve the symmetry of $\mathcal{E}$, we will assume that there exists a measure m on $\big((\mathbb{R}^d)^2,\mathcal{B}((\mathbb{R}^d)^2)\big)$ such that

$$Q(x_1,x_2,dh_1\times dh_2)=m(dh_1\times dh_2)\,q(x_1,x_2,h_1,h_2), \tag{5}$$

where $q:(\mathbb{R}^d)^4\to[0,\infty]$ is a measurable function.

Lemma 1. *Assume that, for any $x_1,x_2\in\mathbb{R}^d$, $x_1\neq x_2$, we have the equality of measures*

$$m(dh_1\times dh_2)q(x_1,x_2,h_1,h_2)=m'(dh_1\times dh_2)q(x_1+h_1,x_2+h_2,-h_1,-h_2), \tag{6}$$

where m' denotes the pushforward of the measure m under the mapping $(h_1,h_2)\mapsto(-h_1,-h_2)$. Then, for any $F,G\in\mathcal{F}C_{\mathrm{b}}(C_0(\mathbb{R}^d),\Gamma)$,

$$\begin{aligned}\mathcal{E}(F,G)=\frac12\int_\Gamma\pi_z(d\gamma)\sum_{\{x_1,x_2\}\subset\gamma}\int_{(\mathbb{R}^d)^2}m(dh_1\times dh_2)q(x_1,x_2,h_1,h_2)\\ \times\big(F(\gamma\setminus\{x_1,x_2\}\cup\{x_1+h_1,x_2+h_2\})-F(\gamma)\big)\big(G(\gamma\setminus\{x_1,x_2\}\cup\{x_1+h_1,x_2+h_2\})-G(\gamma)\big).\end{aligned} \tag{7}$$

In particular, the quadratic form $(\mathcal{E},\mathcal{F}C_{\mathrm{b}}(C_0(\mathbb{R}^d),\Gamma))$ is symmetric.

Proof. Using the Mecke identity (3) and formula (6), we have

$$\int_\Gamma\pi_z(d\gamma)\sum_{\{x_1,x_2\}\subset\gamma}\int_{(\mathbb{R}^d)^2}m(dh_1\times dh_2)q(x_1,x_2,h_1,h_2)$$

$$
\begin{aligned}
&\times \big(F(\gamma \setminus \{x_1, x_2\} \cup \{x_1 + h_1, x_2 + h_2\}) - F(\gamma)\big)G(\gamma \setminus \{x_1, x_2\} \cup \{x_1 + h_1, x_2 + h_2\}) \\
&= \frac{1}{2} \int_\Gamma \pi_z(d\gamma) \int_{\mathbb{R}^d} z\, dx_1 \int_{\mathbb{R}^d} z\, dx_2 \int_{(\mathbb{R}^d)^2} m(dh_1 \times dh_2) q(x_1, x_2, h_1, h_2) \\
&\times \big(F(\gamma \cup \{x_1 + h_1, x_2 + h_2\}) - F(\gamma \cup \{x_1, x_2\})\big)G(\gamma \cup \{x_1 + h_1, x_2 + h_2\}) \\
&= \frac{1}{2} \int_\Gamma \pi_z(d\gamma) \int_{(\mathbb{R}^d)^2} m(dh_1 \times dh_2) \int_{\mathbb{R}^d} z\, dx_1 \int_{\mathbb{R}^d} z\, dx_2\, q(x_1 - h_1, x_2 - h_2, h_1, h_2) \\
&\times \big(F(\gamma \cup \{x_1, x_2\}) - F(\gamma \cup \{x_1 - h_1, x_2 - h_2\})\big)G(\gamma \cup \{x_1, x_2\}) \\
&= \frac{1}{2} \int_\Gamma \pi_z(d\gamma) \int_{\mathbb{R}^d} z\, dx_1 \int_{\mathbb{R}^d} z\, dx_2 \int_{(\mathbb{R}^d)^2} m'(dh_1 \times dh_2) q(x_1 + h_1, x_2 + h_2, -h_1, -h_2) \\
&\times \big(F(\gamma \cup \{x_1, x_2\}) - F(\gamma \cup \{x_1 + h_1, x_2 + h_2\})\big)G(\gamma \cup \{x_1, x_2\}) \\
&= \frac{1}{2} \int_\Gamma \pi_z(d\gamma) \int_{\mathbb{R}^d} z\, dx_1 \int_{\mathbb{R}^d} z\, dx_2 \int_{(\mathbb{R}^d)^2} m(dh_1 \times dh_2) q(x_1, x_2, h_1, h_2) \\
&\times \big(F(\gamma \cup \{x_1, x_2\}) - F(\gamma \cup \{x_1 + h_1, x_2 + h_2\})\big)G(\gamma \cup \{x_1, x_2\}) \\
&= - \int_\Gamma \pi_z(d\gamma) \sum_{\{x_1, x_2\} \subset \gamma} \int_{(\mathbb{R}^d)^2} m(dh_1 \times dh_2) q(x_1, x_2, h_1, h_2) \\
&\times \big(F(\gamma \setminus \{x_1, x_2\} \cup \{x_1 + h_1, x_2 + h_2\}) - F(\gamma)\big)G(\gamma).
\end{aligned}
$$

From here (7) follows. □

As easily seen, $(\mathcal{E}, \mathcal{F}C_{\mathrm{b}}(C_0(\mathbb{R}^d), \Gamma))$ is a pre-Dirichlet form, i.e., if this form is closable in $L^2(\Gamma, \pi_z)$, then it is a Dirichlet form, see e.g. [8, 17] for details on Dirichlet forms.

Lemma 2. *Assume that the following two conditions are satisfied:*

(C1) *For each* $\Lambda \in \mathcal{B}_0(\mathbb{R}^d)$

$$\int_{(\mathbb{R}^d)^2} Q(x_1, x_2, dh_1 \times dh_2)(\mathbf{1}_\Lambda(x_1) + \mathbf{1}_\Lambda(x_1 + h_1)) \in L^1((\mathbb{R}^d)^2, dx_1\, dx_2) \cap L^2((\mathbb{R}^d)^2, dx_1\, dx_2).$$

(C2) *We have*

$$\sup_{x_1 \in \mathbb{R}^d} \int_{\mathbb{R}^d} dx_2 \int_{(\mathbb{R}^d)^2} Q(x_1, x_2, dh_1 \times dh_2) < \infty.$$

Then, for each $F \in \mathcal{F}C_{\mathrm{b}}(C_0(\mathbb{R}^d), \Gamma)$, $LF \in L^2(\Gamma, \pi_z)$, *and so* $(-L, \mathcal{F}C_{\mathrm{b}}(C_0(\mathbb{R}^d), \Gamma))$ *is the generator of the quadratic form* $(\mathcal{E}, \mathcal{F}C_{\mathrm{b}}(C_0(\mathbb{R}^d), \Gamma))$ *on* $L^2(\Gamma, \pi_z)$.

Proof. Using the Mecke identity (3), we easily derive the following formula:

$$\int_\Gamma \pi_z(d\gamma)\left(\sum_{\{x_1,x_2\}\subset\gamma} f(x_1,x_2)\right)^2$$
$$= \frac14\left(\int_{\mathbb{R}^d} z\,dx_1 \int_{\mathbb{R}^d} z\,dx_2\, f(x_1,x_2)\right)^2 + \int_{\mathbb{R}^d} z\,dx_1 \int_{\mathbb{R}^d} z\,dx_2 \int_{\mathbb{R}^d} z\,dx_3\, f(x_1,x_2)f(x_2,x_3)$$
$$+\frac12 \int_{\mathbb{R}^d} z\,dx_1 \int_{\mathbb{R}^d} z\,dx_2\, f(x_1,x_2)^2 \quad (8)$$

for any measurable function $f : (\mathbb{R}^d)^2 \to [0,\infty]$ satisfying $f(x_1,x_2) = f(x_2,x_1)$ for all $x_1, x_2 \in \mathbb{R}^d$. For any $F \in \mathcal{F}C_\mathrm{b}(C_0(\mathbb{R}^d),\Gamma)$, there exists a $\Lambda \in \mathcal{B}_0(\mathbb{R}^d)$ such that

$$\big|F(\gamma\setminus\{x_1,x_2\}\cup\{x_1+h_1,x_2+h_2\}) - F(\gamma)\big|$$
$$\le C_1(\mathbf{1}_\Lambda(x_1)+\mathbf{1}_\Lambda(x_2)+\mathbf{1}_\Lambda(x_1+h_1)+\mathbf{1}_\Lambda(x_2+h_2)). \quad (9)$$

Here and below we denote by C_i, $i = 1,2,3,\dots$, strictly positive constants whose explicit value is not important for us. Now the statement of the lemma follows from (C1), (C2), (2), (8), and (9). □

Completely analogously to the proof of Theorem 3.1 in [14] (see also the proof of Theorem 3.1 in [16]), we easily conclude the following theorem from Lemmas 1 and 2.

Theorem 1. *Assume that conditions* (2), (6), (C1), *and* (C2) *are satisfied. Then the quadratic form* $(\mathcal{E}, \mathcal{F}C_\mathrm{b}(C_0(\mathbb{R}^d),\Gamma))$ *is closable in* $L^2(\Gamma,\pi_z)$ *and its closure will be denoted by* $(\mathcal{E}, D(\mathcal{E}))$. *Further there exists a conservative Hunt process*

$$M = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge0}, (\Theta_t)_{t\ge0}, (X(t))_{t\ge0}, (P_\gamma)_{\gamma\in\Gamma})$$

on Γ *which is properly associated with* $(\mathcal{E}, D(\mathcal{E}))$, *i.e., for each* ($\pi_z$*-version of*) $F \in L^2(\Gamma,\pi_z)$ *and* $t > 0$

$$\Gamma \ni \gamma \mapsto (p_tF)(\gamma) := \int_\Omega F(X(t))\,dP_\gamma$$

is an $\mathcal{E}$*-quasi continuous version of* $\exp(tL)F$. *Here* $(-L, D(L))$ *is the generator of the quadratic form* $(\mathcal{E}, D(\mathcal{E}))$*—the Friedrichs extension of the operator* $(-L, \mathcal{F}C_\mathrm{b}(C_0(\mathbb{R}^d),\Gamma))$. M *is up-to* π_z*-equivalence unique. In particular,* M *is* π_z*-symmetric and has* π_z *as invariant measure.*

Remark 1. We refer to [17] for an explanation of notations appearing in Theorem 1, see also a brief explanation of them in [16].

We will call a Markov process as in Theorem 1 a stochastic dynamics of binary jumps. Let us now consider two classes of such dynamics.

1) Let us assume that the measure $m(dh_1\times dh_2)$ in (5) is the Lebesgue measure $dh_1\,dh_2$, and let us assume that $q(x_1,x_2,h_1,h_2) = q(x_2-x_1,h_1,h_2)$ for some measurable function $q : (\mathbb{R}^d)^3 \to [0,\infty]$. (Here and below we are using an obvious abuse of notation.) Thus,

$$Q(x_1, x_2, dh_1 \times dh_2) = dh_1\, dh_2\, q(x_2 - x_1, h_1, h_2). \tag{10}$$

Proposition 1. *Assume that* (10) *holds and*

$$q(-x, h_1, h_2) = q(x, h_2, h_1), \tag{11}$$
$$q(x, h_1, h_2) = q(x + h_2 - h_1, -h_1, -h_2) \tag{12}$$

for all $x, h_1, h_2 \in \mathbb{R}^d$. Further assume that

$$q(x, h_1, h_2) \in L^1((\mathbb{R}^d)^3, dx\, dh_1\, dh_2), \tag{13}$$
$$\operatorname*{ess\,sup}_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} dh_1 \int_{\mathbb{R}^d} dh_2\, q(x, h_1, h_2) < \infty. \tag{14}$$

Then $Q(x_1, x_2, dh_1 \times dh_2)$ satisfies the conditions of Theorem 1.

Proof. By (10), conditions (2), (6) reduce to (11), (12). Condition (13) clearly implies (C2), so we only have to check (C1). For each $\Lambda \in \mathcal{B}_0(\mathbb{R}^d)$, (13) implies that

$$\int_{(\mathbb{R}^d)^2} Q(x_1, x_2, dh_1 \times dh_2)\mathbf{1}_\Lambda(x_1) \in L^1((\mathbb{R}^d)^2, dx_1\, dx_2).$$

By (12) and (13),

$$\begin{aligned}
&\int_{\mathbb{R}^d} dx_1 \int_{\mathbb{R}^d} dx_2 \int_{(\mathbb{R}^d)^2} Q(x_1, x_2, dh_1 \times dh_2)\mathbf{1}_\Lambda(x_1 + h_1)\\
&\quad = \int_{\mathbb{R}^d} dx_1 \int_{\mathbb{R}^d} dx_2 \int_{\mathbb{R}^d} dh_1 \int_{\mathbb{R}^d} dh_2\, q(x_2 - x_1 + h_2 - h_1, -h_1, -h_2)\mathbf{1}_\Lambda(x_1 + h_1)\\
&\quad = \int_{\Lambda} dx_1 \int_{\mathbb{R}^d} dx_2 \int_{\mathbb{R}^d} dh_1 \int_{\mathbb{R}^d} dh_2\, q(x_2 - x_1, -h_1, -h_2) < \infty.
\end{aligned}$$

Analogously, using additionally (14), we get

$$\begin{aligned}
&\int_{\mathbb{R}^d} dx_1 \int_{\mathbb{R}^d} dx_2 \left(\int_{(\mathbb{R}^d)^2} Q(x_1, x_2, dh_1 \times dh_2)\mathbf{1}_\Lambda(x_1) \right)^2\\
&\quad = \int_{\Lambda} dx_1 \int_{\mathbb{R}^d} dx_2 \int_{\mathbb{R}^d} dh_1 \int_{\mathbb{R}^d} dh_2\, q(x_2 - x_1, h_1, h_2) \int_{\mathbb{R}^d} dh_1' \int_{\mathbb{R}^d} dh_2'\, q(x_2 - x_1, h_1', h_2')\\
&\quad \le C_2 \int_{\Lambda} dx_1 \int_{\mathbb{R}^d} dx_2 \int_{\mathbb{R}^d} dh_1 \int_{\mathbb{R}^d} dh_2\, q(x_2 - x_1, h_1, h_2) < \infty,
\end{aligned}$$

and

$$\int_{\mathbb{R}^d} dx_1 \int_{\mathbb{R}^d} dx_2 \left(\int_{(\mathbb{R}^d)^2} Q(x_1, x_2, dh_1 \times dh_2) \mathbf{1}_\Lambda(x_1 + h_1) \right)^2$$
$$\le \int_{\mathbb{R}^d} dx_1 \int_{\mathbb{R}^d} dx_2 \int_{\mathbb{R}^d} dh_1 \int_{\mathbb{R}^d} dh_2\, q(x_2 - x_1, h_1, h_2) \mathbf{1}_\Lambda(x_1 + h_1)$$
$$\times \int_{\mathbb{R}^d} dh_1' \int_{\mathbb{R}^d} dh_2'\, q(x_2 - x_1, h_1', h_2')$$
$$\le C_2 \int_{\mathbb{R}^d} dx_1 \int_{\mathbb{R}^d} dx_2 \int_{\mathbb{R}^d} dh_1 \int_{\mathbb{R}^d} dh_2\, q(x_2 - x_1, h_1, h_2) \mathbf{1}_\Lambda(x_1 + h_1) < \infty.$$

Thus, (C1) is satisfied. □

The following proposition, whose proof is straightforward, presents a possible choice of a function $q(x, h_1, h_2)$.

Proposition 2. *Assume that functions $a : \mathbb{R}^d \to [0, \infty]$ and $b : \mathbb{R}^d \to [0, \infty]$ are measurable and even, i.e., $a(-x) = a(x)$, $b(-x) = b(x)$, $x \in \mathbb{R}^d$. Further assume that*

$$a, b \in L^1(\mathbb{R}^d, dx), \quad \operatorname*{ess\,sup}_{x \in \mathbb{R}^d} b(x) < \infty.$$

Set

$$q(x, h_1, h_2) := a(h_1) a(h_2) \big(b(x) + b(x + h_2 - h_1) \big).$$

Then the function $q(x, h_1, h_2)$ satisfies the conditions of Proposition 1, and so $Q(x_1, x_2, dh_1 \times dh_2)$ given by (10) satisfies the conditions of Theorem 1.

2) The following class of stochastic dynamics of binary jumps is inspired by the paper [2]. Let us assume that measure $m(dh_1 \times dh_2)$ is the pushforward of the Lebesgue measure dh on $\mathbb{R}^d$ under the mapping $\mathbb{R}^d \ni h \mapsto (h, -h) \in (\mathbb{R}^d)^2$. Let us further assume that $q(x_1, x_2, h_1, h_2) = q(x_2 - x_1, h_1)$ for some measurable function $q : (\mathbb{R}^d)^2 \to [0, \infty]$. Thus, for any measurable function $f : (\mathbb{R}^d)^4 \to [0, \infty]$,

$$\int_{(\mathbb{R}^d)^2} Q(x_1, x_2, dh_1 \times dh_2) f(x_1, x_2, h_1, h_2) = \int_{\mathbb{R}^d} dh\, q(x_2 - x_1, h) f(x_1, x_2, h, -h). \quad (15)$$

Hence, for any $F \in \mathcal{F}C_{\mathrm{b}}(C_0(\mathbb{R}^d), \Gamma)$,

$$(LF)(\gamma) = \sum_{\{x_1, x_2\} \subset \gamma} \int_{\mathbb{R}^d} dh\, q(x_2 - x_1, h) \big(F(\gamma \setminus \{x_1, x_2\} \cup \{x_1 + h, x_2 - h\}) - F(\gamma) \big).$$

Completely analogously to Propositions 1, 2, we get

Proposition 3. i) *Assume that* (15) *holds and*

$$q(-x,h)=q(x,-h),$$
$$q(x,h)=q(-x+2h,h)$$

for $x,h\in\mathbb{R}^d$. *Further assume that*

$$q(x,h)\in L^1((\mathbb{R}^d)^2,dx\,dh),$$
$$\operatorname*{ess\,sup}_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d}dh\,q(x,h)<\infty.$$

Then $Q(x_1,x_2,dh_1\times dh_2)$ *satisfies the conditions of Theorem* 1.

ii) *Let functions* a, b *be as in Proposition* 2. *Set*

$$q(x,h):=a(h)b(x-h).$$

Then the function $q(x,h)$ *satisfies the conditions of* i), *and so* $Q(x_1,x_2,dh_1\times dh_2)$ *given by* (15) *satisfies the conditions of Theorem* 1.

3 Uniqueness of dynamics

We will now show that the Markov pre-generator (1) defined on $\mathcal{F}C_{\mathrm{b}}(C_0(\mathbb{R}^d),\Gamma)$, with $Q(x_1,x_2,dh_1\times dh_2)$ being as in Proposition 1, or as in Proposition 3, i) uniquely identifies a Markov process on Γ.

Theorem 2. *Let* $Q(x_1,x_2,dh_1\times dh_2)$ *be either as in Proposition 1, or as in Proposition 3, i). Then the operator* $(-L,\mathcal{F}C_{\mathrm{b}}(C_0(\mathbb{R}^d),\Gamma))$ *is essentially selfadjoint in* $L^2(\Gamma,\pi_z)$, *so that* $(-L,D(L))$ *is the closure of* $(-L,\mathcal{F}C_{\mathrm{b}}(C_0(\mathbb{R}^d),\Gamma))$ *in* $L^2(\Gamma,\pi_z)$.

Proof. We will only prove the theorem in the case of the dynamics as in Proposition 1, which is the harder case. Denote by $(-\overline{L},D(\overline{L}))$ the closure of the symmetric operator $(-L,\mathcal{F}C_{\mathrm{b}}(C_0(\mathbb{R}^d),\Gamma))$ in $L^2(\Gamma,\pi_z)$. Thus, the operator $(-L,D(L))$ is an extension of $(-\overline{L},D(\overline{L})$. Analogously to the proof of Lemma 2 and Proposition 1, one can show that, for each $f\in C_0(\mathbb{R}^d)$ and $n\in\mathbb{N}$, $\langle f,\cdot\rangle^n\in D(\overline{L})$. Hence, by the polarization identity (e.g. [3, Chap. 2, formula (2.7)]), we have

$$\langle f_1,\cdot\rangle\cdots\langle f_n,\cdot\rangle\in D(\overline{L}),\quad f_1,\dots,f_n\in C_0(\mathbb{R}^d),\ n\in\mathbb{N}. \tag{1}$$

Let $\mathscr{P}$ denote the set of all functions on Γ which are finite sums of functions as in (1) and constants. Thus, $\mathscr{P}$ is a set of polynomials on Γ, and $\mathscr{P}\subset D(\overline{L})$.

For a real Hilbert space $\mathcal{H}$, denote by $\mathcal{F}(\mathcal{H})$ the symmetric Fock space over $\mathcal{H}$. Thus, $\mathcal{F}(\mathcal{H})$ is the Hilbert space

$$\mathcal{F}(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{F}^{(n)}(\mathcal{H}),$$

where $\mathcal{F}^{(0)}(\mathcal{H}) := \mathbb{R}$, and for $n \in \mathbb{N}$, $\mathcal{F}^{(n)}(\mathcal{H})$ coincides with $\mathcal{H}^{\odot n}$ as a set, and for any $f^{(n)}, g^{(n)} \in \mathcal{F}^{(n)}(\mathcal{H})$

$$(f^{(n)}, g^{(n)})_{\mathcal{F}(\mathcal{H})} := (f^{(n)}, g^{(n)})_{\mathcal{H}^{\odot n}} n!\,.$$

Here $\odot$ stands for symmetric tensor product.

Let

$$I : L^2(\Gamma, \pi_z) \to \mathcal{F}(L^2(\mathbb{R}^d, z\,dx))$$

denote the unitary isomorphism which is derived through multiple stochastic integrals with respect to the centered Poisson random measure on $\mathbb{R}^d$ with intensity measure $z\,dx$, see e.g. [25]. Denote by $\widetilde{\mathscr{P}}$ the subset of $\mathcal{F}(L^2(\mathbb{R}^d, z\,dx))$ which is the linear span of vectors of the form

$$f_1 \odot f_2 \odot \cdots \odot f_n, \quad f_1, \dots, f_n \in C_0(\mathbb{R}^d),\ n \in \mathbb{N},$$

and the vacuum vector $\Psi = (1, 0, 0, \dots)$. For any $f \in C_0(\mathbb{R}^d)$, denote by M_f the operator of multiplication by the function $\langle f, \cdot \rangle$ in $L^2(\Gamma, \pi_z)$. Using the representation of the operator IM_fI^{-1} as a sum of creation, neutral, and annihilation operators in the Fock space (see e.g. [25]), we easily conclude that $I\mathscr{P} = \widetilde{\mathscr{P}}$.

We define a quadratic form $(\widetilde{\mathcal{E}}, D(\widetilde{\mathcal{E}}))$ by

$$\widetilde{\mathcal{E}}(f, g) := \mathcal{E}(I^{-1}f, I^{-1}g), \quad f, g \in D(\widetilde{\mathcal{E}}) := ID(\mathcal{E}),$$

on $\mathcal{F}(L^2(\mathbb{R}^d, z\,dx))$, and let $(-\widetilde{L}, D(\widetilde{L}))$ denote the generator of this form. Thus, $D(\widetilde{L}) = ID(L)$ and $\widetilde{L} = ILI^{-1}$ on $D(\widetilde{L})$.

For each $x \in \mathbb{R}^d$, we define an annihilation operator at x as follows: $\partial_x : \widetilde{\mathscr{P}} \to \widetilde{\mathscr{P}}$ is a linear mapping given through

$$\partial_x \Psi := 0, \quad \partial_x f_1 \odot f_2 \odot \cdots \odot f_n := \sum_{i=1}^{n} f_i(x)\, f_1 \odot f_2 \odot \cdots \odot \check{f}_i \odot \cdots \odot f_n,$$

where $\check{f}_i$ denotes the absence of f_i. We will preserve the notation ∂_x for the operator $I\partial_x I^{-1} : \mathscr{P} \to \mathscr{P}$. This operator admits the following explicit representation

$$\partial_x F(\gamma) = F(\gamma \cup \{x\}) - F(\gamma)$$

for π_z-a.a. $\gamma \in \Gamma$, see e.g. [1, 11, 19]. By the Mecke formula, for any $F \in \mathscr{P}$,

$$\mathcal{E}(F,F) = \frac{1}{4}\int_{\mathbb{R}^d} z\,dx_1 \int_{\mathbb{R}^d} z\,dx_2 \int_{\mathbb{R}^d} dh_1 \int_{\mathbb{R}^d} dh_2\, q(x_2 - x_1, h_1, h_2) \\ \times \int_\Gamma \pi_z(d\gamma)\big(F(\gamma \cup \{x_1 + h_1, x_2 + h_2\}) - F(\gamma \cup \{x_1, x_2\})\big)^2.$$

Noting that

$$F(\gamma \cup \{x_1, x_2\}) - F(\gamma) = (\partial_{x_1}\partial_{x_2} - \partial_{x_1} - \partial_{x_2})F(\gamma),$$

we thus get, for any $f \in \widetilde{\mathscr{P}}$,

$$\widetilde{\mathcal{E}}(f,f) = \frac{1}{4}\int_{\mathbb{R}^d} z\,dx_1 \int_{\mathbb{R}^d} z\,dx_2 \int_{\mathbb{R}^d} dh_1 \int_{\mathbb{R}^d} dh_2\, q(x_2 - x_1, h_1, h_2) \\ \times \big\|(\partial_{x_1+h_1}\partial_{x_2+h_2} - \partial_{x_1+h_1} - \partial_{x_2+h_2} - \partial_{x_1}\partial_{x_2} + \partial_{x_1} + \partial_{x_2})f\big\|^2_{\mathcal{F}(L^2(\mathbb{R}^d, z\,dx))}. \quad (2)$$

Hence, at least heuristically, the generator of this form has representation

$$-\widetilde{L} = \frac{1}{4}\int_{\mathbb{R}^d} z\,dx_1 \int_{\mathbb{R}^d} z\,dx_2 \int_{\mathbb{R}^d} dh_1 \int_{\mathbb{R}^d} dh_2\, q(x_2 - x_1, h_1, h_2) \\ \times (\partial^\dagger_{x_1+h_1}\partial^\dagger_{x_2+h_2} - \partial^\dagger_{x_1+h_1} - \partial^\dagger_{x_2+h_2} - \partial^\dagger_{x_1}\partial^\dagger_{x_2} + \partial^\dagger_{x_1} + \partial^\dagger_{x_2}) \\ \times (\partial_{x_1+h_1}\partial_{x_2+h_2} - \partial_{x_1+h_1} - \partial_{x_2+h_2} - \partial_{x_1}\partial_{x_2} + \partial_{x_1} + \partial_{x_2}), \quad (3)$$

where $\partial^\dagger_x$ denotes a creation operator at point $x \in \mathbb{R}^d$ ($\partial^\dagger_x$ being rather an operator-valued distribution, see e.g. [10]) Noting that the operators ∂_x, $x \in \mathbb{R}^d$, commute, we get from (3) and (11), (12):

$$-\widetilde{L}_0 = J^+ + J^0 + J^-,$$

where

$$J^+ := \int_{\mathbb{R}^d} z\,dx_1 \int_{\mathbb{R}^d} z\,dx_2 \int_{\mathbb{R}^d} dh_1 \int_{\mathbb{R}^d} dh_2\, q(x_2 - x_1, h_1, h_2)\big(-\partial^\dagger_{x_1}\partial^\dagger_{x_2}\partial_{x_1} + \partial^\dagger_{x_1}\partial^\dagger_{x_2}\partial_{x_1+h_1}\big),$$

$$J^0 := \int_{\mathbb{R}^d} z\,dx_1 \int_{\mathbb{R}^d} z\,dx_2 \int_{\mathbb{R}^d} dh_1 \int_{\mathbb{R}^d} dh_2\, q(x_2 - x_1, h_1, h_2)\bigg(\frac{1}{2}\partial^\dagger_{x_1}\partial^\dagger_{x_2}\partial_{x_1}\partial_{x_2} \\ + \frac{1}{2}\partial^\dagger_{x_1}\partial^\dagger_{x_2}\partial_{x_1+h_1}\partial_{x_2+h_2} + \partial^\dagger_{x_1}\partial_{x_1} + \partial^\dagger_{x_1}\partial_{x_2} - \partial^\dagger_{x_1}\partial_{x_1+h_1} - \partial^\dagger_{x_1}\partial_{x_2+h_2}\bigg), \quad (4)$$

and J^- is the formal adjoint of J^+. Note that the operators of creation and annihilation in the above formulas are in Wick order, i.e., the creation operators act after the annihilation operators. Thus, one may hope to give a rigorous sense to the above integrals by using the corresponding quadratic forms, see e.g.[22, Chapter X.7].

Denote by $\mathcal{F}_{\mathrm{fin}}(L^2(\mathbb{R}^d, z\,dx))$ the subset of $\mathcal{F}(L^2(\mathbb{R}^d, z\,dx))$ consisting of all finite sequences $f = (f^{(0)}, f^{(1)}, \dots, f^{(n)}, 0, 0, \dots)$, $f^{(i)} \in \mathcal{F}^{(i)}(L^2(\mathbb{R}^d, z\,dx))$, $i = 1, \dots, n$, $n \in \mathbb{N}$. Clearly $\widetilde{\mathscr{P}} \subset \mathcal{F}_{\mathrm{fin}}(L^2(\mathbb{R}^d, z\,dx))$. Using (2), we conclude by approximation that $\mathcal{F}_{\mathrm{fin}}(L^2(\mathbb{R}^d, z\,dx)) \subset D(\widetilde{\mathcal{E}})$.

Using the corresponding quadratic form, we get, for each $f^{(n)} \in \mathcal{F}^{(n)}(L^2(\mathbb{R}^d, z\,dx))$,

$$\begin{aligned}&\bigg(\int_{\mathbb{R}^d} z\,dx_1 \int_{\mathbb{R}^d} z\,dx_2 \int_{\mathbb{R}^d} dh_1 \int_{\mathbb{R}^d} dh_2\, q(x_2 - x_1, h_1, h_2) \partial^\dagger_{x_1} \partial^\dagger_{x_2} \partial_{x_1} f^{(n)}\bigg)(y_1, \dots, y_{n+1})\\ &\quad = n\bigg(f^{(n)}(y_1, \dots, y_n) \int_{\mathbb{R}^d} dh_1 \int_{\mathbb{R}^d} dh_2\, q(y_{n+1} - y_n, h_1, h_2)\bigg)^{\sim},\end{aligned}$$

where $(\cdot)^{\sim}$ denotes symmetrization. We have, by (13) and (14):

$$\begin{aligned}&\int_{(\mathbb{R}^d)^{n+1}} \bigg(f^{(n)}(y_1, \dots, y_n) \int_{\mathbb{R}^d} dh_1 \int_{\mathbb{R}^d} dh_2\, q(y_{n+1} - y_n, h_1, h_2)\bigg)^2 z\,dy_1 \cdots z\,dy_{n+1}\\ &\quad \le C_3 \int_{(\mathbb{R}^d)^n} z\,dy_1 \cdots z\,dy_n\, f^{(n)}(y_1, \dots, y_n)^2 \int_{\mathbb{R}^d} z\,dy_{n+1} \int_{\mathbb{R}^d} dh_1 \int_{\mathbb{R}^d} dh_2\, q(y_{n+1} - y_n, h_1, h_2)\\ &\quad \le C_4 \|f^{(n)}\|^2_{L^2((\mathbb{R}^d)^n, z\,dy_1 \cdots z\,dy_n)}.\end{aligned}$$

Analogously,

$$\begin{aligned}&\bigg(\int_{\mathbb{R}^d} z\,dx_1 \int_{\mathbb{R}^d} z\,dx_2 \int_{\mathbb{R}^d} dh_1 \int_{\mathbb{R}^d} dh_2\, q(x_2 - x_1, h_1, h_2) \partial^\dagger_{x_1} \partial^\dagger_{x_2} \partial_{x_1 + h_1} f^{(n)}\bigg)(y_1, \dots, y_{n+1})\\ &\quad = n\bigg(\int_{\mathbb{R}^d} dh_1 \int_{\mathbb{R}^d} dh_2\, q(y_{n+1} - y_n, h_1, h_2) f^{(n)}(y_1, \dots, y_{n-1}, y_n + h_1)\bigg)^{\sim},\end{aligned}$$

and by the Cauchy inequality

$$\begin{aligned}&\int_{(\mathbb{R}^d)^{n+1}} \bigg(\int_{\mathbb{R}^d} dh_1 \int_{\mathbb{R}^d} dh_2\, q(y_{n+1} - y_n, h_1, h_2) f^{(n)}(y_1, \dots, y_{n-1}, y_n + h_1)\bigg)^2 z\,dy_1 \cdots z\,dy_{n+1}\\ &\quad \le \int_{(\mathbb{R}^d)^{n+1}} z\,dy_1 \cdots z\,dy_{n+1} \int_{\mathbb{R}^d} dh_1 \int_{\mathbb{R}^d} dh_2 \int_{\mathbb{R}^d} dh_1' \int_{\mathbb{R}^d} dh_2'\\ &\qquad \times q(y_{n+1} - y_n, h_1, h_2) q(y_{n+1} - y_n, h_1', h_2') f^{(n)}(y_1, \dots, y_{n-1}, y_n + h_1)^2\\ &\quad \le C_5 \int_{(\mathbb{R}^d)^{n+1}} z\,dy_1 \cdots z\,dy_{n+1} \int_{\mathbb{R}^d} dh_1 \int_{\mathbb{R}^d} dh_2\, q(y_{n+1} - y_n, h_1, h_2)\\ &\qquad \times f^{(n)}(y_1, \dots, y_{n-1}, y_n + h_1)^2\\ &\quad = C_5 \int_{(\mathbb{R}^d)^n} z\,dy_1 \cdots z\,dy_n\, f^{(n)}(y_1, \dots, y_n)^2 \int_{\mathbb{R}^d} dh_1 \int_{\mathbb{R}^d} dh_2 \int_{\mathbb{R}^d} z\,dy_{n+1}\\ &\qquad \times q(y_{n+1} - y_n + h_1, h_1, h_2)\end{aligned}$$

$$= C_6 \|f^{(n)}\|^2_{L^2((\mathbb{R}^d)^n, z\,dy_1 \cdots z\,dy_n)}.$$

Hence, J^+ can be realized as a linear operator on $\mathcal{F}_{\mathrm{fin}}(L^2(\mathbb{R}^d, z\,dx))$ and

$$\|J^+ \restriction \mathcal{F}^{(n)}(L^2(\mathbb{R}^d, z\,dx))\|_{\mathscr{L}(\mathcal{F}^{(n)}(L^2(\mathbb{R}^d,z\,dx)),\,\mathcal{F}^{(n+1)}(L^2(\mathbb{R}^d,z\,dx)))} \le C_7\, n\sqrt{n+1}, \qquad (5)$$

where the constant C_7 is independent of n. Furthermore, J^- can be realized as the restriction to $\mathcal{F}_{\mathrm{fin}}(L^2(\mathbb{R}^d, z\,dx))$ of the adjoint operator of J^+, and by (5)

$$\|J^- \restriction \mathcal{F}^{(n+1)}(L^2(\mathbb{R}^d, z\,dx))\|_{\mathscr{L}(\mathcal{F}^{(n+1)}(L^2(\mathbb{R}^d,z\,dx)),\,\mathcal{F}^{(n)}(L^2(\mathbb{R}^d,z\,dx)))} \le C_7\, n\sqrt{n+1}. \qquad (6)$$

Using again the corresponding quadratic form, we get

$$\begin{aligned}
&(J_1^0 f^{(n)})(y_1, \dots, y_n) := \\
&\quad = \bigg(\int_{\mathbb{R}^d} z\,dx_1 \int_{\mathbb{R}^d} z\,dx_2 \int_{\mathbb{R}^d} dh_1 \int_{\mathbb{R}^d} dh_2\, q(x_2 - x_1, h_1, h_2) \partial^\dagger_{x_1} \partial^\dagger_{x_2} \partial_{x_1} \partial_{x_2} f^{(n)} \bigg)(y_1, \dots, y_n) \\
&\quad = n(n-1) \bigg(f^{(n)}(y_1, \dots, y_n) \int_{\mathbb{R}^d} dh_1 \int_{\mathbb{R}^d} dh_2\, q(y_2 - y_1, h_1, h_2) \bigg)^{\sim},
\end{aligned}$$

and hence

$$\|J_1^0 \restriction \mathcal{F}^{(n)}(L^2(\mathbb{R}^d, z\,dx))\|_{\mathscr{L}(\mathcal{F}^{(n)}(L^2(\mathbb{R}^d,z\,dx)),\,\mathcal{F}^{(n)}(L^2(\mathbb{R}^d,z\,dx)))} \le C_8\, n(n-1). \qquad (7)$$

Analogously,

$$\begin{aligned}
&(J_2^0 f^{(n)})(y_1, \dots, y_n) := \\
&\quad = \bigg(\int_{\mathbb{R}^d} z\,dx_1 \int_{\mathbb{R}^d} z\,dx_2 \int_{\mathbb{R}^d} dh_1 \int_{\mathbb{R}^d} dh_2\, q(x_2 - x_1, h_1, h_2) \partial^\dagger_{x_1} \partial^\dagger_{x_2} \partial_{x_1+h_1} \partial_{x_2+h_2} f^{(n)} \bigg)(y_1, \dots, y_n) \\
&\quad = n(n-1) \bigg(\int_{\mathbb{R}^d} dh_1 \int_{\mathbb{R}^d} dh_2\, q(y_2 - y_1, h_1, h_2) f^{(n)}(y_1 + h_1, y_2 + h_2, y_3 \dots, y_n) \bigg)^{\sim}.
\end{aligned}$$

We have, by the Cauchy inequality,

$$\begin{aligned}
&\int_{(\mathbb{R}^d)^n} \bigg(\int_{\mathbb{R}^d} dh_1 \int_{\mathbb{R}^d} dh_2\, q(y_2 - y_1, h_1, h_2) f^{(n)}(y_1 + h_1, y_2 + h_2, y_3 \dots, y_n) \bigg)^2 z\,dy_1 \cdots z\,dy_n \\
&\quad \le \int_{(\mathbb{R}^d)^n} z\,dy_1 \cdots z\,dy_n \int_{\mathbb{R}^d} dh_1 \int_{\mathbb{R}^d} dh_2 \int_{\mathbb{R}^d} dh_1' \int_{\mathbb{R}^d} dh_2' \\
&\qquad \times q(y_2 - y_1, h_1, h_2) q(y_2 - y_1, h_1', h_2') f^{(n)}(y_1 + h_1, y_2 + h_2, y_3 \dots, y_n)^2 \\
&\quad \le C_9 \int_{(\mathbb{R}^d)^n} z\,dy_1 \cdots z\,dy_n \int_{\mathbb{R}^d} dh_1 \int_{\mathbb{R}^d} dh_2\, q(y_2 - h_2 - y_1 + h_1, h_1, h_2) f^{(n)}(y_1, \dots, y_n)^2 \\
&\quad = C_9 \int_{(\mathbb{R}^d)^n} z\,dy_1 \cdots z\,dy_n \int_{\mathbb{R}^d} dh_1 \int_{\mathbb{R}^d} dh_2\, q(y_2 + h_2 - y_1 - h_1, -h_1, -h_2) f^{(n)}(y_1, \dots, y_n)^2
\end{aligned}$$

$$
\begin{aligned}
&= C_9 \int_{(\mathbb{R}^d)^n} z\,dy_1 \cdots z\,dy_n \int_{\mathbb{R}^d} dh_1 \int_{\mathbb{R}^d} dh_2\, q(y_2 - y_1, h_1, h_2) f^{(n)}(y_1, \dots, y_n)^2 \\
&\le C_{10} \| f^{(n)} \|^2_{L^2((\mathbb{R}^d)^n, z\,dy_1 \cdots z\,dy_n)}.
\end{aligned}
$$

Therefore, an estimate similar to (7) holds for J_2^0.

We next have:

$$
\begin{aligned}
&(J_3^0 f^{(n)})(y_1, \dots, y_n) := \\
&\quad = \left(\int_{\mathbb{R}^d} z\,dx_1 \int_{\mathbb{R}^d} z\,dx_2 \int_{\mathbb{R}^d} dh_1 \int_{\mathbb{R}^d} dh_2\, q(x_2 - x_1, h_1, h_2) \partial^\dagger_{x_1} \partial_{x_1} f^{(n)} \right)(y_1, \dots, y_n) \\
&\quad = n \left(\int_{\mathbb{R}^d} z\,dy \int_{\mathbb{R}^d} dh_1 \int_{\mathbb{R}^d} dh_2\, q(y, h_1, h_2) \right) f^{(n)}(y_1, \dots, y_n).
\end{aligned}
$$

Hence

$$
\| J_3^0 \restriction \mathcal{F}^{(n)}(L^2(\mathbb{R}^d, z\,dx)) \|_{\mathscr{L}(\mathcal{F}^{(n)}(L^2(\mathbb{R}^d, z\,dx)), \mathcal{F}^{(n)}(L^2(\mathbb{R}^d, z\,dx)))} \le C_{11}\, n. \tag{8}
$$

(In fact, in this case we have equality, rather than inequality.) Next

$$
\begin{aligned}
&(J_4^0 f^{(n)})(y_1, \dots, y_n) := \\
&\quad = \left(\int_{\mathbb{R}^d} z\,dx_1 \int_{\mathbb{R}^d} z\,dx_2 \int_{\mathbb{R}^d} dh_1 \int_{\mathbb{R}^d} dh_2\, q(x_2 - x_1, h_1, h_2) \partial^\dagger_{x_1} \partial_{x_2} f^{(n)} \right)(y_1, \dots, y_n) \\
&\quad = n \left(\int_{\mathbb{R}^d} z\,dx \int_{\mathbb{R}^d} dh_1 \int_{\mathbb{R}^d} dh_2\, q(x - y_1, h_1, h_2) f^{(n)}(x, y_2, \dots, y_n) \right)^{\sim}.
\end{aligned}
$$

Hence, by the Cauchy inequality, we easily conclude that J_4^0 satisfies an estimate similar to (8). The two remaining terms with $\partial^\dagger_{x_1} \partial_{x_1 + h_1}$ and $\partial^\dagger_{x_1} \partial_{x_2 + h_2}$ can be treated similarly. Thus, J^0 can be realized as a linear operator on $\mathcal{F}_{\mathrm{fin}}(L^2(\mathbb{R}^d, z\,dx))$ and

$$
\| J^0 \restriction \mathcal{F}^{(n)}(L^2(\mathbb{R}^d, z\,dx)) \|_{\mathscr{L}(\mathcal{F}^{(n)}(L^2(\mathbb{R}^d, z\,dx)), \mathcal{F}^{(n)}(L^2(\mathbb{R}^d, z\,dx)))} \le C_{12}\, n(n-1). \tag{9}
$$

Denote by $(-\mathcal{L}, D(\mathcal{L}))$ the closure of the operator $(-\widetilde{L}, \widetilde{\mathscr{P}})$ in $\mathcal{F}(L^2(\mathbb{R}^d, z\,dx))$. Thus, $(-\widetilde{L}, D(\widetilde{L}))$ is an extension of $(-\mathcal{L}, D(\mathcal{L}))$. We now easily see that

$$
\mathcal{F}_{\mathrm{fin}}(L^2(\mathbb{R}^d, z\,dx)) \subset D(\mathcal{L})
$$

and the action of $-\mathcal{L}$ on $\mathcal{F}_{\mathrm{fin}}(L^2(\mathbb{R}^d, z\,dx))$ is indeed given by the above formulas. By (5), (6) and (9), for each $f^{(n)} \in \mathcal{F}^{(n)}(L^2(\mathbb{R}^d, z\,dx))$, there exists $t > 0$ such that

$$
\sum_{k=1}^{\infty} \frac{t^k}{(2k)!} \| (-\mathcal{L})^k f^{(n)} \|_{\mathcal{F}(L^2(\mathbb{R}^d, z\,dx))} < \infty.
$$

Since $\mathscr{P} \subset D(\overline{L})$, we therefore conclude that $I^{-1}\mathcal{F}_{\mathrm{fin}}(L^2(\mathbb{R}^d, z\,dx)) \subset D(\overline{L})$ and for each $F \in I^{-1}\mathcal{F}_{\mathrm{fin}}(L^2(\mathbb{R}^d, z\,dx))$ there exists $t > 0$ such that

$$\sum_{k=1}^{\infty} \frac{t^k}{(2k)!} \|(-\overline{L})^k F\|_{L^2(\Gamma,\pi_z)} < \infty. \tag{10}$$

Hence, by the Nussbaum theorem (see e.g. [22, Theorem X.40]), the operator $(-\overline{L}, D(\overline{L}))$ is essentially selfadjoint on $I^{-1}\mathcal{F}_{\mathrm{fin}}(L^2(\mathbb{R}^d, z\,dx))$. From here the statement of the theorem follows. □

4 Diffusion approximation

We will now consider a diffusion approximation for the stochastic dynamics as in Proposition 2.

Denote by $\mathcal{F}C_{\mathrm{b}}^{\infty}(C_0^{\infty}(\mathbb{R}^d), \Gamma)$ the space of all functions of the form (1), where $N \in \mathbb{N}$, $\varphi_1, \dots, \varphi_N \in C_0^{\infty}(\mathbb{R}^d)$ and $g_F \in C_{\mathrm{b}}^{\infty}(\mathbb{R}^N)$, where $C_0^{\infty}(\mathbb{R}^d)$ and $C_{\mathrm{b}}^{\infty}(\mathbb{R}^N)$ denote the space of all smooth functions on $\mathbb{R}^d$ with compact support and the space of all smooth, bounded functions on $\mathbb{R}^N$ whose all derivatives are bounded, respectively. Clearly $\mathcal{F}C_{\mathrm{b}}^{\infty}(C_0^{\infty}(\mathbb{R}^d), \Gamma) \subset \mathcal{F}C_{\mathrm{b}}(C_0(\mathbb{R}^d), \Gamma)$ and $\mathcal{F}C_{\mathrm{b}}^{\infty}(C_0^{\infty}(\mathbb{R}^d), \Gamma)$ is a dense subset of $L^2(\Gamma, \pi_z)$.

For a function $F \in \mathcal{F}C_{\mathrm{b}}^{\infty}(C_0^{\infty}(\mathbb{R}^d), \Gamma)$, $\gamma \in \Gamma$, and $x \in \gamma$, we denote

$$\nabla_x F(\gamma) := \nabla_y F(\gamma \setminus \{x\} \cup \{y\})\Big|_{y=x}, \tag{1}$$

where ∇_y stands for the gradient in the y variable. Analogously, we define a Laplacian $\Delta_x F(\gamma)$.

We now scale the dynamics as follows. For each $\varepsilon > 0$, we denote

$$q_\varepsilon(x, h_1, h_2) := \varepsilon^{-2d-2} a(h_1/\varepsilon) a(h_2/\varepsilon)\big(b(x) + b(x + h_2 - h_1)\big), \tag{2}$$

and let L_ε denote the corresponding L operator.

Theorem 3. *Assume that functions $a : \mathbb{R}^d \to [0, \infty]$ and $b : \mathbb{R}^d \to [0, \infty)$ satisfy the following conditions:*

a) *$a(x) = \tilde{a}(|x|)$, where $\tilde{a} : [0, \infty) \to [0, \infty]$ is a measurable function, and b is an even function;*

b) *$a, b \in L^1(\mathbb{R}^d, dx)$;*

c) *The function a has a compact support;*

d) *$b \in C_{\mathrm{b}}^1(\mathbb{R}^d)$, where $C_{\mathrm{b}}^1(\mathbb{R}^d)$ denotes the space of all bounded, continuously differentiable functions on $\mathbb{R}^d$ whose gradient is bounded;*

e) *There exists $R>0$ such that*

$$\int_{\mathbb{R}^d} dx \sup_{y\in B(x,R)} |\nabla b(y)| < \infty.$$

Here $B(x,R)$ denotes the closed ball in $\mathbb{R}^d$ centered at x and of radius R.

Then, for each $F\in \mathcal{F}C_{\mathrm{b}}^\infty(C_0^\infty(\mathbb{R}^d),\Gamma)$,

$$L_\varepsilon F \to L_0 F \text{ in } L^2(\Gamma,\pi_z) \text{ as } \varepsilon\to 0.$$

Here

$$(L_0F)(\gamma) := c\sum_{x\in\gamma}\left[\Delta_x F(\gamma)\sum_{y\in\gamma\setminus\{x\}} b(x-y) + \left\langle \nabla_x F(\gamma), \sum_{y\in\gamma\setminus\{x\}} \nabla b(x-y)\right\rangle\right], \tag{3}$$

where

$$c := \int_{\mathbb{R}^d} a(h)(h^1)^2\, dh, \tag{4}$$

h^i denoting the i-th coordinate of $h\in\mathbb{R}^d$, $i=1,\dots,d$, and $\langle\cdot,\cdot\rangle$ stands for the scalar product in $\mathbb{R}^d$.

Remark 2. As will be seen from the proof of Theorem 3, all the series on the right hand side of formula (3) converge absolutely for π_z-a.a. $\gamma\in\Gamma$.

Remark 3. For each $\gamma\in\Gamma$ and $x\in\gamma$, denote $A_x(\gamma) := c\sum_{y\in\gamma\setminus\{x\}} b(x-y)$. Also let $T\Gamma_\gamma := L^2(\mathbb{R}^d\to\mathbb{R}^d, d\gamma)$ be a tangent space to Γ at γ (cf. [1]). Then $\nabla F(\gamma) = (\nabla_x F(\gamma))_{x\in\gamma}\in T\Gamma_\gamma$ and set $B(\gamma) := \left(\sum_{y\in\gamma\setminus\{x\}} c\nabla b(x-y)\right)_{x\in\gamma}$. Then formula (3) can be written in the form

$$(L_0F)(\gamma) = \sum_{x\in\gamma} A_x(\gamma)\Delta_x F(\gamma) + \left\langle \nabla F(\gamma), B(\gamma)\right\rangle_{T\Gamma_\gamma}.$$

Remark 4. Note that condition e) of Theorem 3 is slightly stronger than the condition $|\nabla b|\in L^1(\mathbb{R}^d,dx)$.

Remark 5. Recall that the generator of the gradient stochastic dynamics has the form

$$(L_0'F)(\gamma) := \frac{1}{2}\sum_{x\in\gamma}\left[\Delta_x F(\gamma) - \beta\left\langle \nabla_x F(\gamma), \sum_{y\in\gamma\setminus\{x\}} \nabla\phi(x-y)\right\rangle\right],$$

where β is the inverse temperature and ϕ is the potential of pair interaction, see [1, 20, 24, 27] for further details. The difference between the generators L_0 and L_0' is in the non-trivial coefficient $A_x(\gamma)$ by $\Delta_x F(\gamma)$ in the operator L_0. This coefficient allows L_0 to be symmetric with respect to the Poisson measure, while L_0' is symmetric with respect to the Gibbs measure corresponding to the inverse temperature β and the potential of pair interaction ϕ.

Proof of Theorem 3. Denote by $\ddot{\Gamma}$ the space of all multiple configurations in $\mathbb{R}^d$, i.e., $\ddot{\Gamma}$ consists of all $\{0,1,2,3,\dots,\infty\}$-valued Radon measures on $(\mathbb{R}^d,\mathcal{B}(\mathbb{R}^d))$, this space being also equipped with the vague topology. Evidently $\Gamma\subset\ddot{\Gamma}$. For any $\gamma\in\ddot{\Gamma}$ and $f\in C_0(\mathbb{R}^d)$, we set $\langle f,\gamma\rangle:=\int_{\mathbb{R}^d} f(x)\,\gamma(dx)$. Hence, we can extend each function $F\in\mathcal{F}C_{\mathrm{b}}^\infty(C_0^\infty(\mathbb{R}^d),\Gamma)$ to $\ddot{\Gamma}$ by using formula (1). (As easily seen, such an extension does not depend on the choice of the function's representation in the form (1).) For each $\gamma\in\Gamma$ and $x\in\gamma$, the function

$$\mathbb{R}^d\ni y\mapsto u(y):=F(\gamma-\delta_x+\delta_y)\in\mathbb{R}$$

is clearly smooth. Note that, while $\gamma-\delta_x\in\Gamma$, the measure $\gamma-\delta_x+\delta_y$ belongs to $\ddot{\Gamma}$ and not necessarily to Γ. For a fixed $y\in\mathbb{R}^d$, denote $\tilde{\gamma}:=\gamma-\delta_x+\delta_y$ and set $\nabla_x F(\tilde{\gamma}):=\nabla u(y)$. In the case where $y=x$ and so $\tilde{\gamma}=\gamma$, the just given definition of $\nabla_x F(\gamma)$ coincides with (1).

By (10) and (2), for $F\in\mathcal{F}C_{\mathrm{b}}^\infty(C_0^\infty(\mathbb{R}^d),\Gamma)$,

$$\begin{aligned}
&(L_\varepsilon F)(\gamma)\\
&=\sum_{\{x_1,x_2\}\subset\gamma}\varepsilon^{-2}\int_{\mathbb{R}^d}dh_1\int_{\mathbb{R}^d}dh_2\,a(h_1)a(h_2)\big(b(x_2-x_1)+b(x_2-x_1+\varepsilon(h_2-h_1))\big)\\
&\qquad\times\big(F(\gamma\setminus\{x_1,x_2\}\cup\{x_1+\varepsilon h_1,x_2+\varepsilon h_2\})-F(\gamma)\big)\\
&=\sum_{\{x_1,x_2\}\subset\gamma}\varepsilon^{-2}\int_{\mathbb{R}^d}dh_1\int_{\mathbb{R}^d}dh_2\,a(h_1)a(h_2)\big(b(x_2-x_1)+b(x_2-x_1+\varepsilon(h_2-h_1))\big)\\
&\qquad\times\big(F(\gamma-\delta_{x_1}-\delta_{x_2}+\delta_{x_1+\varepsilon h_1}+\delta_{x_2+\varepsilon h_2})-F(\gamma)\big).
\end{aligned}\tag{5}$$

We have used the fact that, for any $\{x_1,x_2\}\subset\gamma$ and a.a. $(h_1,h_2)\in(\mathbb{R}^d)^2$, we have

$$\{x_1+\varepsilon h_1,x_2+\varepsilon h_2\}\cap\gamma=\varnothing.$$

For any $x,y\in\mathbb{R}^N$, $x\ne y$, $N\in\mathbb{N}$, denote by $[x,y]$ the line segment connecting points x and y. By (5), condition d), and Taylor's formula, we get

$$\begin{aligned}
(L_\varepsilon F)(\gamma)&=\sum_{\{x_1,x_2\}\subset\gamma}\varepsilon^{-2}\int_{\mathbb{R}^d}dh_1\int_{\mathbb{R}^d}dh_2\,a(h_1)a(h_2)\\
&\qquad\times\big[2b(x_2-x_1)+\langle\nabla b(\tilde{y}(x_2-x_1,\varepsilon(h_2-h_1))),\varepsilon(h_2-h_1)\rangle\big]\\
\times\big[\langle\nabla_{(x_1,x_2)}F(\gamma),(\varepsilon h_1,\varepsilon h_2)\rangle&+(1/2)\langle\nabla^2_{(x_1,x_2)}F(\tilde{\gamma}(\gamma,x_1,x_2,\varepsilon h_1,\varepsilon h_2)),(\varepsilon h_1,\varepsilon h_2)^{\otimes 2}\rangle\big].
\end{aligned}\tag{6}$$

Here $\nabla_{(x_1,x_2)}F(\gamma)$ is defined analogously to $\nabla_x F(\gamma)$,

$$\tilde{y}(x_2-x_1,\varepsilon(h_2-h_1))\in[x_2-x_1,x_2-x_1+\varepsilon(h_2-h_1)]\tag{7}$$

and

$$\tilde{\gamma}(\gamma, x_1, x_2, \varepsilon h_1, \varepsilon h_2) = \gamma - \delta_{x_1} - \delta_{x_2} + \delta_{y_1} + \delta_{y_2}$$

with

$$(y_1, y_2) \in [(x_1, x_2), (x_1 + \varepsilon h_1, x_2 + \varepsilon h_2)].$$

By condition a),

$$\int_{\mathbb{R}^d} dh\, a(h) h^i = 0, \quad i = 1, \ldots, d. \tag{8}$$

Hence, for any $\{x_1,\, x_2\} \subset \gamma$,

$$\begin{aligned}
&\varepsilon^{-2} \int_{\mathbb{R}^d} dh_1 \int_{\mathbb{R}^d} dh_2\, a(h_1) a(h_2) 2b(x_2 - x_1) \langle \nabla_{(x_1,x_2)} F(\gamma), (\varepsilon h_1, \varepsilon h_2) \rangle \\
&\quad = \varepsilon^{-1} 2b(x_2 - x_1) \left[\left\langle \nabla_{x_1} F(\gamma), \int_{\mathbb{R}^d} dh_1\, a(h_1) h_1 \int_{\mathbb{R}^d} dh_2\, a(h_2) \right\rangle \right. \\
&\qquad \left. + \left\langle \nabla_{x_2} F(\gamma), \int_{\mathbb{R}^d} dh_1\, a(h_1) \int_{\mathbb{R}^d} dh_2\, a(h_2) h_2 \right\rangle \right] = 0.
\end{aligned} \tag{9}$$

By condition a),

$$\int_{\mathbb{R}^d} dh\, a(h) h^i h^j = 0, \quad i, j = 1, \ldots, d,\ i \neq j. \tag{10}$$

Now, by d), (4), (7), (8), (10), and the dominated convergence theorem, we get, as $\varepsilon \to 0$,

$$\begin{aligned}
&\varepsilon^{-2} \int_{\mathbb{R}^d} dh_1 \int_{\mathbb{R}^d} dh_2\, a(h_1) a(h_2) \langle \nabla b(\tilde{y}(x_2 - x_1, \varepsilon(h_2 - h_1))), \varepsilon(h_2 - h_1) \rangle \\
&\quad \times \langle \nabla_{(x_1,x_2)} F(\gamma), (\varepsilon h_1, \varepsilon h_2) \rangle \\
&= \int_{\mathbb{R}^d} dh_1 \int_{\mathbb{R}^d} dh_2\, a(h_1) a(h_2) \langle \nabla b(\tilde{y}(x_2 - x_1, \varepsilon(h_2 - h_1))), (h_2 - h_1) \rangle \\
&\quad \times \langle \nabla_{(x_1,x_2)} F(\gamma), (h_1, h_2) \rangle \\
&\to \int_{\mathbb{R}^d} dh_1 \int_{\mathbb{R}^d} dh_2\, a(h_1) a(h_2) \langle \nabla b(x_2 - x_1), (h_2 - h_1) \rangle \langle \nabla_{(x_1,x_2)} F(\gamma), (h_1, h_2) \rangle \\
&= \int_{\mathbb{R}^d} dh_1 \int_{\mathbb{R}^d} dh_2\, a(h_1) a(h_2) \sum_{i=1}^d \frac{\partial}{\partial y_i} b(y) \Big|_{y = x_2 - x_1} (h_2^i - h_1^i) \\
&\quad \times \sum_{j=1}^d \left(\frac{\partial}{\partial x_1^j} F(\gamma) h_1^j + \frac{\partial}{\partial x_2^j} F(\gamma) h_2^j \right) \\
&= c\big(\langle \nabla_{x_1} F(\gamma), -\nabla b(x_2 - x_1) \rangle + \langle \nabla_{x_2} F(\gamma), \nabla b(x_2 - x_1) \rangle \big) \\
&= c\big(\langle \nabla_{x_1} F(\gamma), \nabla b(x_1 - x_2) \rangle + \langle \nabla_{x_2} F(\gamma), \nabla b(x_2 - x_1) \rangle \big).
\end{aligned}$$

(We have used obvious notation.) Let $R > 0$ be as in condition e). Further, let $r > 0$ be such that the function $a(h)$ vanishes outside the ball $B(0, r)$ and $\nabla_x F(\gamma) = 0$ for all $\gamma \in \Gamma$ and $x \in \gamma$, $x \notin B(0, r)$. Then, for all $\varepsilon < R/(2r)$,

$$\begin{aligned}
&\int_{\mathbb{R}^d} dh_1 \int_{\mathbb{R}^d} dh_2\, a(h_1)a(h_2)\big|\langle \nabla b(\tilde y(x_2 - x_1, \varepsilon(h_2 - h_1))), (h_2 - h_1)\rangle\big| \\
&\qquad \times \big|\langle \nabla_{(x_1,x_2)} F(\gamma), (h_1, h_2)\rangle\big| \\
&\quad \le C_3 \int_{B(0,r)} dh_1 \int_{B(0,r)} dh_2 (\mathbf{1}_{B(0,r)}(x_1) + \mathbf{1}_{B(0,r)}(x_2)) \sup_{y \in B(x_2 - x_1, R)} |\nabla b(y)|(|h_1|^2 + |h_2|^2) \\
&\quad = C_4(\mathbf{1}_{B(0,r)}(x_1) + \mathbf{1}_{B(0,r)}(x_2)) \sup_{y \in B(x_2 - x_1, R)} |\nabla b(y)|. \qquad (11)
\end{aligned}$$

By (8), (11), d) and e),

$$\sum_{\{x_1, x_2\} \subset \gamma} (\mathbf{1}_{B(0,r)}(x_1) + \mathbf{1}_{B(0,r)}(x_2)) \sup_{y \in B(x_2 - x_1, R)} |\nabla b(y)| \in L^2(\Gamma, \pi_z).$$

Therefore, by the dominated convergence theorem,

$$\begin{aligned}
&\sum_{\{x_1,x_2\} \subset \gamma} \varepsilon^{-2} \int_{\mathbb{R}^d} dh_1 \int_{\mathbb{R}^d} dh_2\, a(h_1)a(h_2)\langle \nabla b(\tilde y(x_2 - x_1, \varepsilon(h_2 - h_1))), \varepsilon(h_2 - h_1)\rangle \\
&\qquad \times \langle \nabla_{(x_1,x_2)} F(\gamma), (\varepsilon h_1, \varepsilon h_2)\rangle \\
&\quad \to \sum_{\{x_1,x_2\} \subset \gamma} c\big(\langle \nabla_{x_1} F(\gamma), \nabla b(x_1 - x_2)\rangle + \langle \nabla_{x_2} F(\gamma), \nabla b(x_2 - x_1)\rangle\big) \\
&\quad = c \sum_{x \in \gamma} \left\langle \nabla_x F(\gamma), \sum_{y \in \gamma \setminus \{x\}} b(x - y) \right\rangle \qquad (12)
\end{aligned}$$

in $L^2(\Gamma, \pi_z)$ as $\varepsilon \to 0$.

Analogously,

$$\begin{aligned}
&\varepsilon^{-2} \int_{\mathbb{R}^d} dh_1 \int_{\mathbb{R}^d} dh_2\, a(h_1)a(h_2) 2b(x_2 - x_1) \\
&\qquad \times (1/2)\langle \nabla^2_{(x_1,x_2)} F(\tilde\gamma(\gamma, x_1, x_2, \varepsilon h_1, \varepsilon h_2)), (\varepsilon h_1, \varepsilon h_2)^{\otimes 2}\rangle \\
&\quad = \int_{\mathbb{R}^d} dh_1 \int_{\mathbb{R}^d} dh_2\, a(h_1)a(h_2)b(x_2 - x_1)\langle \nabla^2_{(x_1,x_2)} F(\tilde\gamma(\gamma, x_1, x_2, \varepsilon h_1, \varepsilon h_2)), (h_1, h_2)^{\otimes 2}\rangle \\
&\quad \to cb(x_2 - x_1)(\Delta_{x_1} F(\gamma) + \Delta_{x_2} F(\gamma))
\end{aligned}$$

as $\varepsilon \to 0$, and for $0 < \varepsilon \le 1$,

$$\int_{\mathbb{R}^d} dh_1 \int_{\mathbb{R}^d} dh_2\, a(h_1)a(h_2)b(x_2 - x_1)\langle \nabla^2_{(x_1,x_2)} F(\tilde\gamma(\gamma, x_1, x_2, \varepsilon h_1, \varepsilon h_2)), (h_1, h_2)^{\otimes 2}\rangle$$

$$\le C_5\big(\mathbf{1}_{B(0,2r)}(x_1)+\mathbf{1}_{B(0,2r)}(x_2)\big)b(x_2-x_1).$$

From here, by the dominated convergence,

$$\begin{aligned}
&\sum_{\{x_1,x_2\}\subset\gamma}\varepsilon^{-2}\int_{\mathbb{R}^d}dh_1\int_{\mathbb{R}^d}dh_2\,a(h_1)a(h_2)2b(x_2-x_1)\\
&\quad\times(1/2)\langle\nabla^2_{(x_1,x_2)}F(\tilde\gamma(\gamma,x_1,x_2,\varepsilon h_1,\varepsilon h_2)),(\varepsilon h_1,\varepsilon h_2)^{\otimes 2}\rangle\\
&\quad\to c\sum_{\{x_1,x_2\}\subset\gamma}b(x_2-x_1)(\Delta_{x_1}F(\gamma)+\Delta_{x_2}F(\gamma))\\
&\quad=\sum_{x\in\gamma}\Delta_xF(\gamma)\sum_{y\in\gamma\setminus\{x\}}b(x-y).
\end{aligned}\tag{13}$$

in $L^2(\Gamma,\pi_z)$ as $\varepsilon\to0$.

Finally, we easily conclude that

$$\begin{aligned}
&\sum_{\{x_1,x_2\}\subset\gamma}\varepsilon^{-2}\int_{\mathbb{R}^d}dh_1\int_{\mathbb{R}^d}dh_2\,a(h_1)a(h_2)\langle\nabla b(\tilde y(x_2-x_1,\varepsilon(h_2-h_1))),\varepsilon(h_2-h_1)\rangle\\
&\quad\times(1/2)\langle\nabla^2_{(x_1,x_2)}F(\tilde\gamma(\gamma,x_1,x_2,\varepsilon h_1,\varepsilon h_2)),(\varepsilon h_1,\varepsilon h_2)^{\otimes 2}\rangle\\
&=\varepsilon\sum_{\{x_1,x_2\}\subset\gamma}\int_{\mathbb{R}^d}dh_1\int_{\mathbb{R}^d}dh_2\,a(h_1)a(h_2)\langle\nabla b(\tilde y(x_2-x_1,\varepsilon(h_2-h_1))),h_2-h_1\rangle\\
&\quad\times(1/2)\langle\nabla^2_{(x_1,x_2)}F(\tilde\gamma(\gamma,x_1,x_2,\varepsilon h_1,\varepsilon h_2)),(h_1,h_2)^{\otimes 2}\rangle\to0
\end{aligned}\tag{14}$$

in $L^2(\Gamma,\pi_z)$ as $\varepsilon\to0$. Now, the statement of the theorem follows from (6), (9), (12)–(14). □

We will now show that the operator $(L_0,\mathcal{F}C_{\mathrm{b}}^\infty(C_0^\infty(\mathbb{R}^d),\Gamma))$ in $L^2(\Gamma,\pi_z)$ from Theorem 3 is a pre-generator of a diffusion dynamics.

Proposition 4. *Let the conditions of Theorem* 3 *be satisfied.*

i) *Define a quadratic form*

$$\mathcal{E}_0(F,G):=\int_\Gamma(-L_0F)(\gamma)G(\gamma)\pi_z(d\gamma),\quad F,G\in\mathcal{F}C_{\mathrm{b}}^\infty(C_0^\infty(\mathbb{R}^d),\Gamma).$$

Then, for all $F,G\in\mathcal{F}C_{\mathrm{b}}^\infty(C_0^\infty(\mathbb{R}^d),\Gamma)$,

$$\begin{aligned}
\mathcal{E}_0(F,G)&=c\int_\Gamma\pi_z(d\gamma)\sum_{\{x_1,x_2\}\subset\gamma}b(x_1-x_2)\big(\langle\nabla_{x_1}F(\gamma),\nabla_{x_1}G(\gamma)\rangle+\langle\nabla_{x_2}F(\gamma),\nabla_{x_2}G(\gamma)\rangle\big)\\
&=c\int_\Gamma\pi_z(d\gamma)\sum_{x\in\gamma}\langle\nabla_xF(\gamma),\nabla_xG(\gamma)\rangle\sum_{y\in\gamma\setminus\{x\}}b(x-y).
\end{aligned}\tag{15}$$

Hence, the quadratic form $(\mathcal{E}_0, \mathcal{F}C_{\mathrm{b}}^\infty(C_0^\infty(\mathbb{R}^d), \Gamma))$ *is symmetric and closable in* $L^2(\Gamma, \pi_z)$*, and its closure will be denoted by* $(\mathcal{E}_0, D(\mathcal{E}_0))$.

ii) *For* $d \geq 2$*, there exists a conservative diffusion process*

$$M = \big(\Omega^0, \mathcal{F}^0, (\mathcal{F}_t^0)_{t\geq 0}, (\Theta_t^0)_{t\geq 0}, (X^0(t))_{t\geq 0}, (P_\gamma^0)_{\gamma\in\Gamma}\big)$$

on Γ *which is properly associated with* $(\mathcal{E}_0, D(\mathcal{E}_0))$*, i.e., for each (*$\pi_z$*-version of)* $F \in L^2(\Gamma, \pi_z)$ *and* $t > 0$

$$\Gamma \ni \gamma \mapsto (p_t^0 F)(\gamma) := \int_\Omega F(X^0(t))\, dP_\gamma^0$$

is an $\mathcal{E}_0$*-quasi continuous version of* $\exp(tL_0)F$. *Here* $(-L_0, D(L_0))$ *is the generator of the quadratic form* $(\mathcal{E}_0, D(\mathcal{E}_0))$*—the Friedrichs extension of the operator* $(-L_0, \mathcal{F}C_{\mathrm{b}}^\infty(C_0^\infty(\mathbb{R}^d), \Gamma))$.

iii) *If* $d = 1$*, then the result of* ii) *remains true if we replace* Γ *with the bigger space* $\ddot{\Gamma}$ *of all multiple configurations in* $\mathbb{R}^d$.

Proof. Analogously to the proof of Theorem 3, we easily see that the quadratic form $(\mathcal{E}_0', \mathcal{F}C_{\mathrm{b}}^\infty(C_0^\infty(\mathbb{R}^d), \Gamma))$ given by the right-hand side of formula (15) is well defined. By the Mecke identity (3),

$$\begin{aligned}
\mathcal{E}_0'(F, G) &= c \int_\Gamma \pi_z(d\gamma) \int_{\mathbb{R}^d} z\, dx \int_{\mathbb{R}^d} z\, dy\, \langle \nabla_x F(\gamma + \delta_x + \delta_y), \nabla_x G(\gamma + \delta_x + \delta_y)\rangle b(x - y) \\
&= c \int_\Gamma \pi_z(d\gamma) \int_{\mathbb{R}^d} z\, dx \int_{\mathbb{R}^d} z\, dy\, \big(- \Delta_x F(\gamma + \delta_x + \delta_y) b(x - y) \\
&\quad - \langle \nabla_x F(\gamma + \delta_x + \delta_y), \nabla b(x - y)\rangle\big) G(\gamma + \delta_x + \delta_y) \\
&= c \int_\Gamma \pi_z(d\gamma) \sum_{x\in\gamma} \sum_{y\in\gamma\setminus\{x\}} \big(- \Delta_x F(\gamma) b(x - y) - \langle \nabla_x F(\gamma), \nabla b(x - y)\rangle\big) G(\gamma) \\
&= \int_\Gamma (-L_0 F)(\gamma) G(\gamma) \pi_z(d\gamma) = \mathcal{E}_0(F, G), \quad F, G \in \mathcal{F}C_{\mathrm{b}}^\infty(C_0^\infty(\mathbb{R}^d), \Gamma).
\end{aligned}$$

Thus, $(-L_0, \mathcal{F}C_{\mathrm{b}}^\infty(C_0^\infty(\mathbb{R}^d), \Gamma))$ is the generator of the quadratic symmetric form $(\mathcal{E}_0, \mathcal{F}C_{\mathrm{b}}^\infty(C_0^\infty(\mathbb{R}^d), \Gamma))$ in $L^2(\Gamma, \pi_z)$. Hence, this form is closable in $L^2(\Gamma, \pi_z)$, and so statement i) is proven. Statements ii) and iii) can be shown analogously to Theorems 6.1 and 6.3 in [13], see also [18] and [23]. □

A result similar to Theorem 3 and Proposition 4 can be obtained for the stochastic dynamics from Proposition 3, ii). Let us briefly outline it. The scaled q function is given by

$$q_\varepsilon(x, h) := \varepsilon^{-d-2} a(h/\varepsilon) b(x - h),$$

and let L_ε denote the corresponding L operator. Hence,

$$(L_\varepsilon F)(\gamma) = \sum_{\{x_1,x_2\}\subset\gamma} \varepsilon^{-2}\int_{\mathbb{R}^d} dh\, a(h)b(x_2-x_1-\varepsilon h)$$
$$\times \big(F(\gamma\setminus\{x_1,x_2\}\cup\{x_1+\varepsilon h, x_2-\varepsilon h\})-F(\gamma)\big).$$

Under the conditions of Theorem 3, for each $F\in\mathcal{F}C_{\mathrm{b}}^\infty(C_0^\infty(\mathbb{R}^d),\Gamma)$, $L_\varepsilon F\to L_0F$ in $L^2(\Gamma,\pi_z)$ as $\varepsilon\to 0$, where

$$(L_0F)(\gamma) := c\sum_{x\in\gamma}\left[\frac{1}{2}\,\Delta_x F(\gamma)\sum_{y\in\gamma\setminus\{x\}} b(x-y) + \left\langle \nabla_x F(\gamma), \sum_{y\in\gamma\setminus\{x\}}\nabla b(x-y)\right\rangle\right]$$
$$- c\sum_{\{x_1,x_2\}\subset\gamma} b(x_2-x_1)\sum_{i=1}^d \frac{\partial}{\partial x_1^i}\,\frac{\partial}{\partial x_2^i}\,F(\gamma),$$

c being given by (4). The corresponding Dirichlet form $\mathcal{E}_0$ has the following representation on $\mathcal{F}C_{\mathrm{b}}^\infty(C_0^\infty(\mathbb{R}^d),\Gamma)$:

$$\mathcal{E}_0(F,G) = \frac{c}{2}\int_\Gamma \pi_z(d\gamma)\sum_{\{x_1,x_2\}\subset\gamma} b(x_2-x_1)\big\langle(\nabla_{x_1}-\nabla_{x_2})F(\gamma),(\nabla_{x_1}-\nabla_{x_2})G(\gamma)\big\rangle.$$

5 Convergence to a birth-and-death process in continuum

We will now consider another scaling limit of the stochastic dynamics as in Proposition 2, which will lead us to a birth-and-death process in continuum. So, we now scale the dynamics as follows. For any $\varepsilon>0$, we denote

$$q_\varepsilon(x,h_1,h_2) := \varepsilon^{2d}a(\varepsilon h_1)a(\varepsilon h_2)\big(b(x)+b(x+h_2-h_1)\big),$$

and let L_ε denote the corresponding L generator. Hence, for each $F\in\mathcal{F}C_{\mathrm{b}}(C_0(\mathbb{R}^d),\Gamma)$,

$$(L_\varepsilon F)(\gamma) = \sum_{\{x_1,x_2\}\subset\gamma}\int_{\mathbb{R}^d} dh_1\int_{\mathbb{R}^d} dh_2\, a(h_1)a(h_2)\big(b(x_2-x_1)+b(x_2-x_1+(h_2-h_1)/\varepsilon)\big)$$
$$\times\big(F(\gamma\setminus\{x_1,x_2\}\cup\{x_1+(h_1/\varepsilon),x_2+(h_2/\varepsilon)\})-F(\gamma)\big). \quad (1)$$

It is not hard to show by approximation that, for any $\varphi\in C_0(\mathbb{R}^d)$, we have $e^{\langle\varphi,\cdot\rangle}\in D(L_\varepsilon)$ and the action of L_ε on $F=e^{\langle\varphi,\cdot\rangle}$ is given by (1) (compare with the beginning of the proof of Theorem 2).

Below, for a function $f\in L^1(\mathbb{R}^d,dx)$, we denote $\langle f\rangle := \int_{\mathbb{R}^d} f(x)\,dx$.

Theorem 4. *Let the conditions of Proposition* 2 *be satisfied. Additionally assume that*

$$b(x) \to 0 \quad as\ |x| \to \infty. \tag{2}$$

Then, for each $\varphi \in C_0(\mathbb{R}^d)$*,*

$$L_\varepsilon F \to L_0 F \ \ in\ L^2(\Gamma, \pi_z)\ \ as\ \varepsilon \to 0,$$

where $F = e^{\langle \varphi, \cdot \rangle}$ *and*

$$\begin{aligned}(L_0F)(\gamma) := \langle a\rangle^2 \Bigg[&\sum_{\{x_1,\,x_2\}\subset\gamma} b(x_2-x_1)\big(F(\gamma\setminus\{x_1,x_2\})-F(\gamma)\big)\\ &+\frac{1}{2}\int_{\mathbb{R}^d} z\,dx_1 \int_{\mathbb{R}^d} z\,dx_2\, b(x_2-x_1)\big(F(\gamma\cup\{x_1,x_2\})-F(\gamma)\big)\\ &+z\langle b\rangle \sum_{x\in\gamma}\big(F(\gamma\setminus\{x\})-F(\gamma)\big)+z\langle b\rangle\int_{\mathbb{R}^d} z\,dx\big(F(\gamma\cup\{x\})-F(\gamma)\big)\Bigg].\end{aligned} \tag{3}$$

Proof. We represent $L_\varepsilon F$ as follows:

$$(L_\varepsilon F)(\gamma) = (L_0^{(1)}F)(\gamma) + \sum_{i=2}^{4}(L_\varepsilon^{(i)}F)(\gamma),$$

where

$$\begin{aligned}(L_0^{(1)}F)(\gamma) &:= \langle a\rangle^2 \sum_{\{x_1,\,x_2\}\subset\gamma} b(x_2-x_1)\big(F(\gamma\setminus\{x_1,x_2\})-F(\gamma)\big),\\ (L_\varepsilon^{(2)}F)(\gamma) &:= \sum_{\{x_1,\,x_2\}\subset\gamma}\int_{\mathbb{R}^d} dh_1 \int_{\mathbb{R}^d} dh_2\, a(h_1)a(h_2)b(x_2-x_1+(h_2-h_1)/\varepsilon))\\ &\quad\times\big(F(\gamma\setminus\{x_1,x_2\})-F(\gamma)\big),\\ (L_\varepsilon^{(3)}F)(\gamma) &:= \sum_{\{x_1,\,x_2\}\subset\gamma}\int_{\mathbb{R}^d} dh_1 \int_{\mathbb{R}^d} dh_2\, a(h_1)a(h_2)b(x_2-x_1)\\ &\quad\times\big(F(\gamma\setminus\{x_1,x_2\}\cup\{x_1+(h_1/\varepsilon),\,x_2+(h_2/\varepsilon)\})-F(\gamma\setminus\{x_1,x_2\})\big),\\ (L_\varepsilon^{(4)}F)(\gamma) &:= \sum_{\{x_1,\,x_2\}\subset\gamma}\int_{\mathbb{R}^d} dh_1 \int_{\mathbb{R}^d} dh_2\, a(h_1)a(h_2)b(x_2-x_1+(h_2-h_1)/\varepsilon)\\ &\quad\times\big(F(\gamma\setminus\{x_1,x_2\}\cup\{x_1+(h_1/\varepsilon),\,x_2+(h_2/\varepsilon)\})-F(\gamma\setminus\{x_1,x_2\})\big).\end{aligned}$$

The statement of the theorem will follow if we show that, for each $F = e^{\langle\varphi,\cdot\rangle}$, $\varphi \in C_0(\mathbb{R}^d)$, and $i = 2, 3, 4$,

$$\int_\Gamma \pi_z(d\gamma)(L_\varepsilon^{(i)}F)^2(\gamma) \to \int_\Gamma \pi_z(d\gamma)(L_0^{(i)}F)^2(\gamma) \quad \text{as } \varepsilon \to 0, \tag{4}$$

$$\int_\Gamma \pi_z(d\gamma)(L_\varepsilon^{(i)}F)(\gamma)(L_0^{(i)}F)(\gamma) \to \int_\Gamma \pi_z(d\gamma)(L_0^{(i)}F)^2(\gamma) \quad \text{as } \varepsilon\to 0, \tag{5}$$

where

$$\begin{aligned}
(L_0^{(2)}F(\gamma) &:= \langle a\rangle^2 z\langle b\rangle \sum_{x\in\gamma}\big(F(\gamma\setminus\{x\})-F(\gamma)\big),\\
(L_0^{(3)}F)(\gamma) &:= \langle a\rangle^2 z\langle b\rangle \int_{\mathbb{R}^d} z\,dx\big(F(\gamma\cup\{x\})-F(\gamma)\big),\\
(L_0^{(4)}F)(\gamma) &:= \langle a\rangle^2\,\frac{1}{2}\int_{\mathbb{R}^d} z\,dx_1\int_{\mathbb{R}^d} z\,dx_2\, b(x_2-x_1)\big(F(\gamma\cup\{x_1,x_2\})-F(\gamma)\big).
\end{aligned}$$

In fact, we have, for $\varepsilon>0$,

$$\begin{aligned}
(L_\varepsilon^{(2)}F)(\gamma) &:= e^{\langle\varphi,\gamma\rangle}\sum_{\{x_1,x_2\}\subset\gamma}\int_{\mathbb{R}^d} dh_1\int_{\mathbb{R}^d} dh_2\, a(h_1)a(h_2)b(x_2-x_1+(h_2-h_1)/\varepsilon))\\
&\quad\times\big(e^{-\varphi(x_1)-\varphi(x_2)}-1\big),\\
(L_\varepsilon^{(3)}F)(\gamma) &:= e^{\langle\varphi,\gamma\rangle}\sum_{\{x_1,x_2\}\subset\gamma}\int_{\mathbb{R}^d} dh_1\int_{\mathbb{R}^d} dh_2\, a(h_1)a(h_2)b(x_2-x_1)\\
&\quad\times e^{-\varphi(x_1)-\varphi(x_2)}\big(e^{\varphi(x_1+(h_1/\varepsilon))+\varphi(x_2+(h_2/\varepsilon))}-1\big),\\
(L_\varepsilon^{(4)}F)(\gamma) &:= e^{\langle\varphi,\gamma\rangle}\sum_{\{x_1,x_2\}\subset\gamma}\int_{\mathbb{R}^d} dh_1\int_{\mathbb{R}^d} dh_2\, a(h_1)a(h_2)b(x_2-x_1+(h_2-h_1)/\varepsilon)\\
&\quad\times e^{-\varphi(x_1)-\varphi(x_2)}\big(e^{\varphi(x_1+(h_1/\varepsilon))+\varphi(x_2+(h_2/\varepsilon))}-1\big).
\end{aligned}$$

and

$$\begin{aligned}
(L_0^{(2)}F)(\gamma) &:= \langle a\rangle^2 z\langle b\rangle\, e^{\langle\varphi,\gamma\rangle}\sum_{x\in\gamma}(e^{-\varphi(x)}-1),\\
(L_0^{(3)}F)(\gamma) &:= \langle a\rangle^2 z\langle b\rangle\, e^{\langle\varphi,\gamma\rangle}\int_{\mathbb{R}^d} z\,dx\,(e^{\varphi(x)}-1),\\
(L_0^{(4)}F)(\gamma) &:= \frac{1}{2}\,\langle a\rangle^2 e^{\langle\varphi,\gamma\rangle}\int_{\mathbb{R}^d} z\,dx_1\int_{\mathbb{R}^d} z\,dx_2\, b(x_2-x_1)(e^{\varphi(x_1)+\varphi(x_2)}-1).
\end{aligned}$$

Analogously to (8), we get

$$\begin{aligned}
&\int_\Gamma \pi_z(d\gamma)\left(e^{\langle\varphi,\gamma\rangle}\sum_{\{x_1,x_2\}\subset\gamma} f(x_1,x_2)\right)^2\\
&= \int_\Gamma \pi_z(d\gamma)e^{\langle 2\varphi,\gamma\rangle}\left[\frac{1}{4}\left(\int_{\mathbb{R}^d} z\,dx_1\int_{\mathbb{R}^d} z\,dx_2\, e^{2\varphi(x_1)+2\varphi(x_2)}f(x_1,x_2)\right)^2\right.
\end{aligned}$$

$$+\int_{\mathbb{R}^d} z\,dx_1 \int_{\mathbb{R}^d} z\,dx_2 \int_{\mathbb{R}^d} z\,dx_3\, e^{2\varphi(x_1)+2\varphi(x_2)+2\varphi(x_3)} f(x_1,x_2)f(x_2,x_3)$$

$$+\frac{1}{2}\int_{\mathbb{R}^d} z\,dx_1 \int_{\mathbb{R}^d} z\,dx_2\, e^{2\varphi(x_1)+2\varphi(x_2)} f(x_1,x_2)^2\Bigg] \qquad (6)$$

for any measurable function $f:(\mathbb{R}^d)^2\to[0,\infty]$ and any $\varphi\in C_0(\mathbb{R}^d)$.

Let us show that (4) holds for $i=2$. Since

$$e^{-\varphi(x_1)-\varphi(x_2)}-1 = e^{-\varphi(x_1)}(e^{-\varphi(x_2)}-1)+(e^{-\varphi(x_1)}-1),$$

by the dominated convergence theorem, we have

$$\frac{1}{4}\int_\Gamma \pi_z(d\gamma)e^{\langle 2\varphi,\gamma\rangle}\Bigg(\int_{\mathbb{R}^d} z\,dx_1\int_{\mathbb{R}^d} z\,dx_2\int_{\mathbb{R}^d} dh_1\int_{\mathbb{R}^d} dh_2\, a(h_1)a(h_2)e^{2\varphi(x_1)+2\varphi(x_2)}$$
$$\times\, b(x_2-x_1+(h_2-h_1)/\varepsilon)(e^{-\varphi(x_1)-\varphi(x_2)}-1)\Bigg)^2$$
$$=\frac{1}{4}\int_\Gamma \pi_z(d\gamma)e^{\langle 2\varphi,\gamma\rangle}\Bigg(\int_{\mathbb{R}^d} z\,dx_1\int_{\mathbb{R}^d} z\,dx_2\int_{\mathbb{R}^d} dh_1\int_{\mathbb{R}^d} dh_2\, a(h_1)a(h_2)b(x_2-x_1)$$
$$\times\, e^{\varphi(x_1-(h_1/\varepsilon)+(h_2/\varepsilon))+2\varphi(x_2)}(e^{-\varphi(x_2)}-1)$$
$$+\int_{\mathbb{R}^d} z\,dx_1\int_{\mathbb{R}^d} z\,dx_2\int_{\mathbb{R}^d} dh_1\int_{\mathbb{R}^d} dh_2\, a(h_1)a(h_2)b(x_2-x_1)$$
$$\times\, e^{2\varphi(x_1)+2\varphi(x_2-(h_2/\varepsilon)+(h_1/\varepsilon))}(e^{-\varphi(x_1)}-1)\Bigg)^2$$
$$\to\int_\Gamma \pi_z(d\gamma)e^{\langle 2\varphi,\gamma\rangle}\left(\langle a\rangle^2 z\,\langle b\rangle\int_{\mathbb{R}^d} z\,dx\, e^{2\varphi(x)}(e^{-\varphi(x)}-1)\right)^2 \quad \text{as } \varepsilon\to 0. \qquad (7)$$

Analogously,

$$\int_\Gamma \pi_z(d\gamma)e^{\langle 2\varphi,\gamma\rangle}\int_{\mathbb{R}^d} z\,dx_1\int_{\mathbb{R}^d} z\,dx_2\int_{\mathbb{R}^d} z\,dx_3\int_{\mathbb{R}^d} dh_1\int_{\mathbb{R}^d} dh_2\int_{\mathbb{R}^d} dh_1'\int_{\mathbb{R}^d} dh_2'$$
$$\times\, a(h_1)a(h_2)a(h_1')a(h_2')e^{2\varphi(x_1)+2\varphi(x_2)+2\varphi(x_3)}$$
$$\times\, b(x_2-x_1+(h_2-h_1)/\varepsilon)b(x_3-x_2+(h_2'-h_1')/\varepsilon)$$
$$\times\, \big(e^{-\varphi(x_1)}(e^{-\varphi(x_2)}-1)+(e^{-\varphi(x_1)}-1)\big)\big(e^{-\varphi(x_2)}(e^{-\varphi(x_3)}-1)+(e^{-\varphi(x_2)}-1)\big)$$
$$\to\int_\Gamma \pi_z(d\gamma)e^{\langle 2\varphi,\gamma\rangle}\int_{\mathbb{R}^d} z\,dx_1\int_{\mathbb{R}^d} z\,dx_2\int_{\mathbb{R}^d} z\,dx_3\int_{\mathbb{R}^d} dh_1\int_{\mathbb{R}^d} dh_2\int_{\mathbb{R}^d} dh_1'\int_{\mathbb{R}^d} dh_2'$$
$$\times\, a(h_1)a(h_2)a(h_1')a(h_2')e^{2\varphi(x_2)}b(x_2-x_1)b(x_3-x_2)(e^{-\varphi(x_2)}-1)^2$$
$$=\langle a\rangle^4 z^2\langle b\rangle^2\int_\Gamma \pi_z(d\gamma)e^{\langle 2\varphi,\gamma\rangle}\int_{\mathbb{R}^d} z\,dx\, e^{2\varphi(x)}(e^{-\varphi(x)}-1)^2, \qquad (8)$$

and using additionally (2),

$$\frac{1}{2}\int_\Gamma \pi_z(d\gamma)e^{\langle 2\varphi,\gamma\rangle}\int_{\mathbb{R}^d} z\,dx_1\int_{\mathbb{R}^d} z\,dx_2\int_{\mathbb{R}^d} dh_1\int_{\mathbb{R}^d} dh_2\int_{\mathbb{R}^d} dh_1'\int_{\mathbb{R}^d} dh_2'$$
$$\times a(h_1)a(h_2)a(h_1')a(h_2')e^{2\varphi(x_1)+2\varphi(x_2)}$$
$$\times b(x_2-x_1+(h_2-h_1)/\varepsilon))b(x_2-x_1+(h_2'-h_1')/\varepsilon))$$
$$\times\left[e^{-\varphi(x_1)}(e^{-\varphi(x_2)}-1)+(e^{-\varphi(x_1)}-1)\right]^2\to 0\quad\text{as }\varepsilon\to 0. \tag{9}$$

By (7)–(9), formula (4) for $i=2$ follows.

Next, we show (4) for $i=3$. Analogously to the above, we get, as $\varepsilon\to 0$,

$$\frac{1}{4}\int_\Gamma \pi_z(d\gamma)e^{\langle 2\varphi,\gamma\rangle}\bigg(\int_{\mathbb{R}^d} z\,dx_1\int_{\mathbb{R}^d} z\,dx_2\int_{\mathbb{R}^d} dh_1\int_{\mathbb{R}^d} dh_2\, a(h_1)a(h_2)b(x_2-x_1)$$
$$\times e^{2\varphi(x_1)+2\varphi(x_2)-\varphi(x_1)-\varphi(x_2)}(e^{\varphi(x_1+(h_1/\varepsilon))+\varphi(x_2+(h_2/\varepsilon))}-1)\bigg)^2$$
$$=\frac{1}{4}\int_\Gamma \pi_z(d\gamma)e^{\langle 2\varphi,\gamma\rangle}\bigg(\int_{\mathbb{R}^d} z\,dx_1\int_{\mathbb{R}^d} z\,dx_2\int_{\mathbb{R}^d} dh_1\int_{\mathbb{R}^d} dh_2\, a(h_1)a(h_2)$$
$$\times b(x_2-(h_2/\varepsilon)-x_1+(h_1/\varepsilon))e^{\varphi(x_1-(h_1/\varepsilon))+\varphi(x_2-(h_2/\varepsilon))}(e^{\varphi(x_1)+\varphi(x_2)}-1)\bigg)^2$$
$$\to\frac{1}{4}\int_\Gamma \pi_z(d\gamma)e^{\langle 2\varphi,\gamma\rangle}\bigg(\int_{\mathbb{R}^d} z\,dx_1\int_{\mathbb{R}^d} z\,dx_2\int_{\mathbb{R}^d} dh_1\int_{\mathbb{R}^d} dh_2\, a(h_1)a(h_2)b(x_2-x_1)$$
$$\times\big((e^{\varphi(x_1)}-1)+(e^{\varphi(x_2)}-1)\big)\bigg)^2$$
$$=\langle a\rangle^4 z^2\langle b\rangle^2\int_\Gamma \pi_z(d\gamma)e^{\langle 2\varphi,\gamma\rangle}\bigg(\int_{\mathbb{R}^d} dx\,(e^{\varphi(x)}-1)\bigg)^2, \tag{10}$$

$$\int_\Gamma \pi_z(d\gamma)e^{\langle 2\varphi,\gamma\rangle}\int_{\mathbb{R}^d} z\,dx_1\int_{\mathbb{R}^d} z\,dx_2\int_{\mathbb{R}^d} z\,dx_3\int_{\mathbb{R}^d} dh_1\int_{\mathbb{R}^d} dh_2\int_{\mathbb{R}^d} dh_1'\int_{\mathbb{R}^d} dh_2'$$
$$\times a(h_1)a(h_2)a(h_1')a(h_2')b(x_2-x_1)b(x_3-x_2)e^{2\varphi(x_1)+2\varphi(x_2)+2\varphi(x_3)-\varphi(x_1)-2\varphi(x_2)-\varphi(x_3)}$$
$$\times(e^{\varphi(x_1+(h_1/\varepsilon))+\varphi(x_2+(h_2/\varepsilon))}-1)(e^{\varphi(x_2+(h_1'/\varepsilon))+\varphi(x_3+(h_2'/\varepsilon))}-1)$$
$$=\int_\Gamma \pi_z(d\gamma)e^{\langle 2\varphi,\gamma\rangle}\int_{\mathbb{R}^d} z\,dx_1\int_{\mathbb{R}^d} z\,dx_2\int_{\mathbb{R}^d} z\,dx_3\int_{\mathbb{R}^d} dh_1\int_{\mathbb{R}^d} dh_2\int_{\mathbb{R}^d} dh_1'\int_{\mathbb{R}^d} dh_2'$$
$$\times a(h_1)a(h_2)a(h_1')a(h_2')b(x_2-x_1)b(x_3-x_2)e^{\varphi(x_1)+\varphi(x_3)}$$
$$\times(e^{\varphi(x_1+(h_1/\varepsilon))+\varphi(x_2+(h_2/\varepsilon))}-1)\left[e^{\varphi(x_2+(h_1'/\varepsilon))}(e^{\varphi(x_3+(h_2'/\varepsilon))}-1)+(e^{\varphi(x_2+(h_1'/\varepsilon))}-1)\right]$$
$$=\int_\Gamma \pi_z(d\gamma)e^{\langle 2\varphi,\gamma\rangle}\int_{\mathbb{R}^d} z\,dx_1\int_{\mathbb{R}^d} z\,dx_2\int_{\mathbb{R}^d} z\,dx_3\int_{\mathbb{R}^d} dh_1\int_{\mathbb{R}^d} dh_2\int_{\mathbb{R}^d} dh_1'\int_{\mathbb{R}^d} dh_2'$$
$$\times a(h_1)a(h_2)a(h_1')a(h_2')b(x_2-x_1)b(x_3-x_2)$$
$$\times\big[e^{\varphi(x_1)+\varphi(x_3-(h_2'/\varepsilon))}(e^{\varphi(x_1+(h_1/\varepsilon))+\varphi(x_2+(h_2/\varepsilon))}-1)e^{\varphi(x_2+(h_1'/\varepsilon))}(e^{\varphi(x_3)}-1)$$

$$+ e^{\varphi(x_1)+\varphi(x_3)}\big(e^{\varphi(x_1+(h_1/\varepsilon))+\varphi(x_2+(h_2/\varepsilon)-(h_1'/\varepsilon))} - 1)(e^{\varphi(x_2)} - 1)\big] \to 0, \tag{11}$$

$$\frac{1}{2}\int_\Gamma \pi_z(d\gamma)e^{\langle 2\varphi,\gamma\rangle}\int_{\mathbb{R}^d} z\,dx_1\int_{\mathbb{R}^d} z\,dx_2\int_{\mathbb{R}^d} dh_1\int_{\mathbb{R}^d} dh_2\int_{\mathbb{R}^d} dh_1'\int_{\mathbb{R}^d} dh_2'$$
$$\times a(h_1)a(h_2)a(h_1')a(h_2')b(x_2-x_1)^2 e^{2\varphi(x_1)+2\varphi(x_2)-2\varphi(x_1)-2\varphi(x_2)}$$
$$\times (e^{\varphi(x_1+(h_1/\varepsilon))+\varphi(x_2+(h_2/\varepsilon))} - 1)(e^{\varphi(x_1+(h_1'/\varepsilon))+\varphi(x_2+(h_2'/\varepsilon))} - 1) \to 0. \tag{12}$$

By (10)–(12), formula (4) for $i = 3$ follows.

Now, we show (4) for $i = 4$. Similarly to the above, we get:

$$\frac{1}{4}\int_\Gamma \pi_z(d\gamma)e^{\langle 2\varphi,\gamma\rangle}\bigg(\int_{\mathbb{R}^d} z\,dx_1\int_{\mathbb{R}^d} z\,dx_2\int_{\mathbb{R}^d} dh_1\int_{\mathbb{R}^d} dh_2\,a(h_1)a(h_2)$$
$$\times b(x_2-x_1+(h_2-h_1)/\varepsilon))e^{\varphi(x_1)+\varphi(x_2)}(e^{\varphi(x_1+(h_1/\varepsilon))+\varphi(x_2+(h_2/\varepsilon))} - 1)\bigg)^2$$
$$= \frac{1}{4}\int_\Gamma \pi_z(d\gamma)e^{\langle 2\varphi,\gamma\rangle}\bigg(\int_{\mathbb{R}^d} z\,dx_1\int_{\mathbb{R}^d} z\,dx_2\int_{\mathbb{R}^d} dh_1\int_{\mathbb{R}^d} dh_2\,a(h_1)a(h_2)$$
$$\times b(x_2-x_1)e^{\varphi(x_1-(h_1/\varepsilon))+\varphi(x_2-(h_2/\varepsilon))}(e^{\varphi(x_1)+\varphi(x_2))} - 1)\bigg)^2$$
$$\to \frac{1}{4}\int_\Gamma \pi_z(d\gamma)e^{\langle 2\varphi,\gamma\rangle}\bigg(\int_{\mathbb{R}^d} z\,dx_1\int_{\mathbb{R}^d} z\,dx_2\int_{\mathbb{R}^d} dh_1\int_{\mathbb{R}^d} dh_2\,a(h_1)a(h_2)$$
$$\times b(x_2-x_1)(e^{\varphi(x_1)+\varphi(x_2))} - 1)\bigg)^2$$
$$= \frac{1}{4}\int_\Gamma \pi_z(d\gamma)e^{\langle 2\varphi,\gamma\rangle}\langle a\rangle^4\bigg(\int_{\mathbb{R}^d} z\,dx_1\int_{\mathbb{R}^d} z\,dx_2 b(x_2-x_1)(e^{\varphi(x_1)+\varphi(x_2))} - 1)\bigg)^2, \tag{13}$$

$$\int_\Gamma \pi_z(d\gamma)e^{\langle 2\varphi,\gamma\rangle}\int_{\mathbb{R}^d} z\,dx_1\int_{\mathbb{R}^d} z\,dx_2\int_{\mathbb{R}^d} z\,dx_3\int_{\mathbb{R}^d} dh_1\int_{\mathbb{R}^d} dh_2\int_{\mathbb{R}^d} dh_1'\int_{\mathbb{R}^d} dh_2'$$
$$\times a(h_1)a(h_2)a(h_1')a(h_2')b(x_2+(h_2/\varepsilon)-x_1-(h_1/\varepsilon))b(x_3+(h_2'/\varepsilon)-x_2-(h_1'/\varepsilon))$$
$$\times e^{\varphi(x_1)+\varphi(x_3)}(e^{\varphi(x_1+(h_1/\varepsilon))+\varphi(x_2+(h_2/\varepsilon))} - 1)(e^{\varphi(x_2+(h_1'/\varepsilon))+\varphi(x_3+(h_2'/\varepsilon))} - 1) \to 0, \tag{14}$$

$$\frac{1}{2}\int_\Gamma \pi_z(d\gamma)e^{\langle 2\varphi,\gamma\rangle}\int_{\mathbb{R}^d} z\,dx_1\int_{\mathbb{R}^d} z\,dx_2\int_{\mathbb{R}^d} dh_1\int_{\mathbb{R}^d} dh_2\int_{\mathbb{R}^d} dh_1'\int_{\mathbb{R}^d} dh_2'$$
$$\times a(h_1)a(h_2)a(h_1')a(h_2')b(x_2+(h_2/\varepsilon)-x_1-(h_1/\varepsilon))b(x_2-x_1+(h_2'-h_1')/\varepsilon))$$
$$\times (e^{\varphi(x_1+(h_1/\varepsilon))+\varphi(x_2+(h_2/\varepsilon))} - 1)(e^{\varphi(x_1+(h_1'/\varepsilon))+\varphi(x_2+(h_2'/\varepsilon))} - 1) \to 0. \tag{15}$$

By (13)–(15), formula (4) for $i = 4$ follows. Thus, (4) is proven. Formula (5) follows analogously. □

Denote by $\mathscr{F}_{\mathrm{exp}}$ the linear span of $\{e^{\langle\varphi,\cdot\rangle},\ \varphi \in C_0(\mathbb{R}^d)\}$. Consider a linear operator $(L_0, \mathscr{F}_{\mathrm{exp}})$ where, for each $F \in \mathscr{F}_{\mathrm{exp}}$, L_0F is given by (3). Analogously to Theorem 1, Proposition 2, and Theorem 2, we get

Proposition 5. i) *Let the conditions of Proposition* 2 *be satisfied. Define a quadratic form*

$$\mathcal{E}_0(F,G) := \int_\Gamma (-L_0 F)(\gamma)G(\gamma)\pi_z(d\gamma), \quad F,G \in \mathscr{F}_{\mathrm{exp}}.$$

Then, for all $F,G \in \mathscr{F}_{\mathrm{exp}}$,

$$\begin{aligned}\mathcal{E}_0(F,G) = \langle a\rangle^2 \int_\Gamma \pi_z(d\gamma) \Bigg[&\sum_{\{x_1,x_2\}\subset\gamma} b(x_2 - x_1) \\ &\times \big(F(\gamma\setminus\{x_1,x_2\}) - F(\gamma)\big)\big(G(\gamma\setminus\{x_1,x_2\}) - G(\gamma)\big) \\ &+ z\langle b\rangle \sum_{x\in\gamma} \big(F(\gamma\setminus\{x\}) - F(\gamma)\big)\big(G(\gamma\setminus\{x\}) - G(\gamma)\big)\Bigg]. \end{aligned} \tag{16}$$

Hence, the quadratic form $(\mathcal{E}_0, \mathscr{F}_{\mathrm{exp}})$ is symmetric and closable in $L^2(\Gamma,\pi_z)$, and its closure will be denoted by $(\mathcal{E}_0, D(\mathcal{E}_0))$. Further there exists a conservative Hunt process

$$M_0 = \big(\Omega^0, \mathcal{F}^0, (\mathcal{F}^0_t)_{t\ge0}, (\Theta^0_t)_{t\ge0}, (X^0(t))_{t\ge0}, (P^0_\gamma)_{\gamma\in\Gamma}\big)$$

on Γ which is properly associated with $(\mathcal{E}_0, D(\mathcal{E}_0))$.

ii) *The operator $(-L_0, \mathscr{F}_{\mathrm{exp}})$ is essentially selfadjoint in $L^2(\Gamma,\pi_z)$.*

Denote by

$$M_\varepsilon = \big(\Omega^\varepsilon, \mathcal{F}^\varepsilon, (\mathcal{F}^\varepsilon_t)_{t\ge0}, (\Theta^\varepsilon_t)_{t\ge0}, (X^\varepsilon(t))_{t\ge0}, (P^\varepsilon_\gamma)_{\gamma\in\Gamma}\big)$$

the Markov processes from Theorem 1 which corresponds to the L_ε generator. By the theory of Dirichlet forms [17], the Markov processes M_ε, as well as the Markov process M_0 from Proposition 5 can be chosen in the canonical form, i.e., for each $\varepsilon \ge 0$, Ω^ε is the set $D([0,+\infty),\Gamma)$ of all *cádlág* functions $\omega : [0,+\infty) \to \Gamma$ (i.e., ω is right continuous on $[0,+\infty)$ and has left limits on $(0,+\infty)$), $X^\varepsilon(t)(\omega) = \omega(t)$, $t \ge 0$, $\omega \in \Omega^\varepsilon$, $(\mathcal{F}^\varepsilon_t)_{t\ge0}$ together with $\mathcal{F}^\varepsilon$ is the corresponding minimum completed admissible family (cf. [8, Section 4.1]) and Θ^ε_t, $t \ge 0$, are the corresponding natural time shifts. So, for each $\varepsilon \ge 0$, we choose the canonical version of the M^ε process and define a stochastic process $Y_\varepsilon = (Y_\varepsilon(t))_{t\ge0}$ whose law is the probability measure on $D([0,+\infty),\Gamma)$ given by $Q_\varepsilon := \int_\Gamma \pi_z(d\gamma)P^\varepsilon_\gamma$. Note that π_z is an invariant measure for Y_ε.

Corollary 1. *Let the conditions of Theorem* 4 *be satisfied. Then the finite-dimensional distributions of the process Y_ε weakly converge to the finite-dimensional distributions of Y_0 as $\varepsilon \to 0$.*

Proof. The statement follows analogously to [5, Theorem 5.1], however, since the argumentation is rather short, we present it. By Theorem 4, Proposition 5, ii) and [4, Chapter 3, Theorem 3.17], we have, for each $t \geq 0$, $e^{tL_\varepsilon} \to e^{tL_0}$ strongly in $L^2(\Gamma, \pi_z)$ as $\varepsilon \to 0$. We now fix any $0 \leq t_1 < t_2 < \cdots < t_n$, $n \in \mathbb{N}$. For $\varepsilon \geq 0$, denote by $\mu^\varepsilon_{t_1,\dots,t_n}$ the finite-dimensional distribution of the process Y_ε at times $t_1, \dots, t_n$, which is a probability measure on Γ^n. Since Γ is a Polish space, by [21, Chapter II, Theorem 3.2], the measure π_z is tight on Γ. Since all the marginal distributions of the measure $\mu^\varepsilon_{t_1,\dots,t_n}$ are π_z, we therefore conclude that the set $\{\mu^\varepsilon_{t_1,\dots,t_n} \mid \varepsilon > 0\}$ is pre-compact in the space $\mathcal{M}(\Gamma^n)$ of the probability measures on Γ^n with respect to the weak topology, see e.g. [21, Chapter II, Section 6]. Hence, the weak convergence of finite-dimensional distributions follows from the strong convergence of the semigroups. □

Remark 6. The dynamics as in Proposition 3, ii) can be scaled as follows:

$$q_\varepsilon(x, h) := \varepsilon^d a(\varepsilon h) b(x - h).$$

By analogy, one can show that the corresponding dynamics converge, as $\varepsilon \to 0$, to a birth-and-death process in continuum with generator

$$(L_0 F)(\gamma) = \langle a \rangle \langle b \rangle \left(\sum_{x \in \gamma} \big(F(\gamma \setminus \{x\}) - F(\gamma)\big) + \int_{\mathbb{R}^d} z\, dx \big(F(\gamma \cup \{x\}) - F(\gamma)\big) \right).$$

The operator L_0, realized in the Fock space $\mathcal{F}(L^2(\mathbb{R}^d, z\, dx))$, is the differential second quantization of the operator $\langle a \rangle \langle b \rangle \mathbf{1}$, so the corresponding dynamics is 'free', i.e., without interaction between particles, see [15, 25, 26] for further detail.

Corollary 2. *The quadratic form $(\mathcal{E}_0, D(\mathcal{E}_0))$ from Proposition 5, i) satisfies the Poincaré inequality:*

$$\mathcal{E}_0(F, F) \geq \langle a \rangle^2 z \langle b \rangle \int_\Gamma (F(\gamma) - \langle F \rangle_{\pi_z})^2 \pi_z(d\gamma), \quad F \in D(\mathcal{E}_0), \tag{17}$$

where $\langle F \rangle_{\pi_z} := \int_\Gamma F(\gamma) \pi_z(d\gamma)$.

Remark 7. The Poincaré inequality means that the operator $(-L_0, D(L_0))$ has a spectral gap, the set $\big(0, \langle a \rangle^2 z \langle b \rangle\big)$, and that the kernel of $(-L_0, D(L_0))$ consists only of the constants.

Proof. Recall the set $\mathscr{P}$ from the proof of Theorem 2. Clearly, $\mathscr{P}$ is a core for the quadratic form $(\mathcal{E}_0, D(\mathcal{E}_0))$, so it suffices to prove (17) only for any $F \in \mathscr{P}$. By (16),

$$\mathcal{E}_0(F, F) \geq \langle a \rangle^2 z \langle b \rangle\, \mathcal{E}_0'(F, F), \quad F \in \mathscr{P}, \tag{18}$$

where

$$\mathcal{E}_0'(F, G) := \int_\Gamma \big(F(\gamma \setminus \{x\}) - F(\gamma)\big)\big(G(\gamma \setminus \{x\}) - G(\gamma)\big) \pi_z(d\gamma), \quad F, G \in \mathscr{P}.$$

The generator of the quadratic form $(\mathcal{E}_0', D(\mathcal{E}_0'))$, realized in the Fock space $\mathcal{F}(L^2(\mathbb{R}^d, z\,dx))$, has a representation $\int_{\mathbb{R}^d} z\,dx\,\partial_x^\dagger \partial_x$, i.e., it is the differential second quantization of the identity operator $\mathbf{1}$, i.e., for any $f^{(n)} \in \mathcal{F}^{(n)}(L^2(\mathbb{R}^d.z\,dx))$,

$$\left(\int_{\mathbb{R}^d} z\,dx\,\partial_x^\dagger \partial_x f^{(n)}\right)(y_1,\dots,y_n) = nf^{(n)}(y_1,\dots,y_n).$$

Hence,

$$\mathcal{E}_0'(F,F) \geq \int_\Gamma (F(\gamma) - \langle F\rangle_{\pi_z})^2 \pi_z(d\gamma), \quad F \in \mathscr{P}. \tag{19}$$

The statement now follows from (18) and (19). □

Remark 8. We note that the initial dynamics of binary jumps is translation invariant and conservative. So it is hopeless to expect that its generator has a spectral gap. So the spectral gap of the generator L_0 appears as a result of the scaling limit.

Note also that the generator of the dynamics of binary jumps is independent of the intensity parameter $z > 0$. Hence, at least heuristically, the initial dynamics has a continuum of symmetrizing Poisson measures, indexed by the intensity $z > 0$. On the other hand, the limiting birth-and-death dynamics has only one of these measures as the symmetrizing one. Thus, the result of the scaling essentially depends on the initial distribution of the dynamics.

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