Markov evolutions and hierarchical equations in the continuum I. One-component systems

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Abstract

General birth-and-death as well as hopping stochastic dynamics of infinite particle systems in the continuum are considered. We derive corresponding evolution equations for correlation functions and generating functionals. General considerations are illustrated in a number of concrete examples of Markov evolutions appearing in applications.

Keywords: Birth-and-death process; Hopping particles; Continuous system; Glauber dynamics; Contact model; Voter model; Kawasaki dynamics; Configuration spaces; Generating functional; Markov generator; Markov process; Gibbs measure; Stochastic dynamics

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1 Introduction

The theory of stochastic lattice gases on the cubic lattice $\mathbb{Z}^d$, $d \in \mathbb{N}$, is one of the most well developed areas in the interacting particle systems theory. In the lattice gas models with spin space $S = \{0, 1\}$, the configuration space is defined as $\mathcal{X} = \{0, 1\}^{\mathbb{Z}^d}$. Given a configuration $\sigma = \{\sigma(x) : x \in \mathbb{Z}^d\} \in \mathcal{X}$, we say that a lattice site $x \in \mathbb{Z}^d$ is free or occupied by a particle depending on $\sigma(x) = 0$ or $\sigma(x) = 1$, respectively. The spin-flip dynamics of such a system means that, at each site x of the lattice, a particle randomly appears (if the site x is free) or disappears from that site. The generator of this dynamics is given by

$$(Lf)(\sigma) = \sum_{x\in\mathbb{Z}^d} a(x, \sigma)(f(\sigma^x) - f(\sigma)),$$

where σ^x denotes the configuration σ in which a particle located at x has disappeared or a new particle has appeared at x. Hence, this dynamics may be interpreted as a birth-and-death process on $\mathbb{Z}^d$. An example of such a type of process is given by the classical contact model, which describes the spread of an infectious disease. In this model an individual at $x \in \mathbb{Z}^d$ is infected if $\sigma(x) = 1$ and healthy if $\sigma(x) = 0$. Healthy individuals become infected at a rate which is proportional to the number of infected neighbors ($\lambda \sum_{y:|y-x|=1} \sigma(y)$, for some $\lambda \geq 0$), while infected individuals recover at a rate identically equal to 1. An additional example is the linear voter model, in which an individual located at a $x \in \mathbb{Z}^d$ has one of two possible positions on an issue. He reassesses his view by the influence of surrounding people. Further examples of such a type may be found e.g. in [27], [28].

In all these examples clearly there is no conservation on the number of particles involved. In contrast to them, in the spin-exchange dynamics there is conservation on the number of particles. In this case, particles randomly hop from one site in $\mathbb{Z}^d$ to another one. The generator of such a dynamics is given by

$$(Lf)(\sigma) = \sum_{x\in\mathbb{Z}^d} \sum_{y\in\mathbb{Z}^d:|y-x|=1} c(x, y, \sigma)(f(\sigma^{xy}) - f(\sigma)),$$

where σ^{xy} denotes the configuration σ in which a particle located at x hops to a site y.

In this work we consider continuous particle systems, i.e., systems of particles which can be located at any site in the Euclidean space $\mathbb{R}^d$, $d \in \mathbb{N}$. In this case, the configuration space of such systems is the space Γ of all locally finite subsets of $\mathbb{R}^d$. Thus, an analog of the above mentioned spin-flip dynamics should be a process in which particles randomly appear or disappear from the space $\mathbb{R}^d$, i.e., a spatial birth-and-death process. The generator of such a process is informally given by

$$\begin{aligned}(LF)(\gamma) = &\sum_{x\in\gamma} d(x, \gamma \setminus \{x\}) \left(F(\gamma \setminus \{x\}) - F(\gamma)\right)\\ &+ \int_{\mathbb{R}^d} dx\, b(x, \gamma) \left(F(\gamma \cup \{x\}) - F(\gamma)\right),\end{aligned}$$

where the coefficient $d(x, \gamma)$ indicates the rate at which a particle located at x in a configuration γ dies or disappears, while $b(x, \gamma)$ indicates the rate at which, given a configuration γ, a new particle is born or appears at a site x.

By analogy, one may also consider a continuous version of the contact and voter models above presented. Both continuous versions yield a similar informal expression for the corresponding generators.

Moreover, one may also consider the analog of the spin-exchange dynamics. We consider a general case of hopping particle systems, in which particles randomly hop over the space $\mathbb{R}^d$. In terms of generators, this means that the dynamics is informally given by

$$(LF)(\gamma) = \sum_{x \in \gamma} \int_{\mathbb{R}^d} dy\, c(x, y, \gamma) \left(F(\gamma \setminus \{x\} \cup \{y\}) - F(\gamma) \right),$$

where the coefficient $c(x, y, \gamma)$ indicates the rate at which a particle located at x in a configuration γ hops to a site y.

Spatial birth-and-death processes in the continuum were first discussed by C. Preston in [33]. Under some conditions on the birth and death rates, b and d, the author has proved the existence of such processes in a bounded volume on $\mathbb{R}^d$. In this case, although the number of particles can be arbitrarily large, at each moment of time the total number of particles is always finite. Later on, the problem of convergence of these processes to an equilibrium one was analyzed in [29], [30]. Problems of existence, construction, and uniqueness of spatial birth-and-death processes in an infinite volume were initiated by R. A. Holley and D. W. Stroock in [9] for a special case of neighbor birth-and-death processes on the real line. An extension of the uniqueness result stated therein may be found in [3].

E. Glötzl analyzed in [7], [8] the birth-and-death and the hopping dynamics of continuous particle systems for which a Gibbs measure μ is reversible. Although he could not prove the existence of such processes, he has identified the conditions on the coefficients, b, d and c under which the corresponding generators are symmetric operators on the space $L^2(\mu)$. For the particular case of the Glauber stochastic dynamics, such a process was effectively constructed in [19]. The procedure used therein was extended in [20] to a general case of birth-and-death dynamics and to the hopping dynamics. Recently, in [22] the authors have proved the existence of a contact process.

In this work we propose an approach to the study of a dynamics based on combinatorial harmonic analysis techniques on configuration spaces. This particular standpoint of configuration space analysis was introduced and developed in [10], [23] (Subsection 2.1). For this purpose, we assume that the coefficients b, d and c are of the type

$$a(x, \gamma) = \sum_{\substack{\eta \subset \gamma \\ |\eta| < \infty}} A_x(\eta),\ a = b, d, \quad c(x, y, \gamma) = \sum_{\substack{\eta \subset \gamma \\ |\eta| < \infty}} C_{x,y}(\eta), \tag{1.1}$$

respectively. This special form of the coefficients allows the use of harmonic analysis techniques, namely, the specific ones yielding from the natural relations between states, observables, correlation measures, and correlation functions (Subsection 2.2). Usually, the starting point for the construction of a dynamics is the Markov generator L related to the Kolmogorov equation

$$\frac{\partial}{\partial t} F_t = LF_t.$$

Given an initial distribution μ of the system (from a set of admissible initial distributions on Γ), the generator L determines a Markov process on Γ with initial distribution μ. In alternative to this approach, the natural relations between observables (i.e., functions defined on Γ), states, correlation measures, and correlation functions yield a description of the underlying dynamics in terms of those elements (Subsection 2.2), through corresponding Kolmogorov equations. Such equations are presented under quite general assumptions, sufficient to define these equations. However, let us observe that on each concrete application the explicit form of the rates determines specific assumptions, which only hold for that concrete application. Such an analysis is discussed separately. In Subsection 2.3 we widen the dynamical description towards the Bogoliubov functionals [2], cf. [14].

Let us pointing out that assumptions (1.1) are natural and quite general. As a matter of fact, the birth and death rates on the Glauber, the contact model and the linear and polynomial voter models dynamics, are both of this type (Subsections 3.2.1–3.2.4), as well the coefficient c for the Kawasaki dynamics (Subsection 4.2.1).

From the technical point of view, the approach that is presented here turns out to be an effective method for the study of Markov evolution for infinite particle systems in the continuum, see Subsection 2.2 for details. This has been recently emphasized in the construction of a non-equilibrium Glauber dynamics done in [18], cf. considerations at the end of Subsection 3.2.1 (see also [4], [16]). In our forthcoming publication [6] we present an extension of this technique towards multicomponent systems.

2 Markov evolutions in configuration spaces

2.1 Harmonic analysis on configuration spaces

The configuration space $\Gamma := \Gamma_{\mathbb{R}^d}$ over $\mathbb{R}^d$, $d \in \mathbb{N}$, is defined as the set of all locally finite subsets of $\mathbb{R}^d$,

$$\Gamma := \left\{ \gamma \subset \mathbb{R}^d : |\gamma_\Lambda| < \infty \text{ for every compact } \Lambda \subset \mathbb{R}^d \right\},$$

where $|\cdot|$ denotes the cardinality of a set and $\gamma_\Lambda := \gamma \cap \Lambda$. As usual we identify each $\gamma \in \Gamma$ with the non-negative Radon measure $\sum_{x\in\gamma} \delta_x \in \mathcal{M}(\mathbb{R}^d)$, where δ_x is the Dirac measure with unit mass at x, $\sum_{x\in\emptyset} \delta_x$ is, by definition, the zero measure, and $\mathcal{M}(\mathbb{R}^d)$ denotes the space of all non-negative Radon measures on the Borel σ-algebra $\mathcal{B}(\mathbb{R}^d)$. This identification allows to endow Γ with the topology induced by the vague topology on $\mathcal{M}(\mathbb{R}^d)$, i.e., the weakest topology on Γ with respect to which all mappings

$$\Gamma \ni \gamma \longmapsto \langle f,\gamma\rangle := \int_{\mathbb{R}^d} d\gamma(x)\, f(x) = \sum_{x\in\gamma} f(x), \quad f \in C_c(\mathbb{R}^d),$$

are continuous. Here $C_c(\mathbb{R}^d)$ denotes the set of all continuous functions on $\mathbb{R}^d$ with compact support. We denote by $\mathcal{B}(\Gamma)$ the corresponding Borel σ-algebra on Γ.

Let us now consider the space of finite configurations

$$\Gamma_0 := \bigsqcup_{n=0}^{\infty} \Gamma^{(n)},$$

where $\Gamma^{(n)} := \Gamma^{(n)}_{\mathbb{R}^d} := \{\gamma \in \Gamma : |\gamma| = n\}$ for $n \in \mathbb{N}$ and $\Gamma^{(0)} := \{\emptyset\}$. For $n \in \mathbb{N}$, there is a natural bijection between the space $\Gamma^{(n)}$ and the symmetrization $\widetilde{(\mathbb{R}^d)^n}\big/S_n$ of the set $\widetilde{(\mathbb{R}^d)^n} := \{(x_1,...,x_n) \in (\mathbb{R}^d)^n : x_i \neq x_j \text{ if } i \neq j\}$ under the permutation group S_n over $\{1,...,n\}$ acting on $\widetilde{(\mathbb{R}^d)^n}$ by permuting the coordinate indexes. This bijection induces a metrizable topology on $\Gamma^{(n)}$, and we endow Γ_0 with the topology of disjoint union of topological spaces (for more details see [10]). By $\mathcal{B}(\Gamma^{(n)})$ and $\mathcal{B}(\Gamma_0)$ we denote the corresponding Borel σ-algebras on $\Gamma^{(n)}$ and Γ_0, respectively.

We proceed to consider the K-transform [24], [25], [26], [10], that is, a mapping which maps functions defined on Γ_0 into functions defined on the space Γ. Let $\mathcal{B}_c(\mathbb{R}^d)$ denote the set of all bounded Borel sets in $\mathbb{R}^d$, and for any $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ let $\Gamma_\Lambda := \{\eta \in \Gamma : \eta \subset \Lambda\}$. Evidently $\Gamma_\Lambda = \bigsqcup_{n=0}^{\infty} \Gamma_\Lambda^{(n)}$, where $\Gamma_\Lambda^{(n)} := \Gamma_\Lambda \cap \Gamma^{(n)}$ for each $n \in \mathbb{N}_0$, leading to a situation similar to the one for Γ_0, described above. We endow Γ_Λ with the topology of the disjoint union of topological spaces and with the corresponding Borel σ-algebra $\mathcal{B}(\Gamma_\Lambda)$.

Given a $\mathcal{B}(\Gamma_0)$-measurable function G with local support, that is, $G\upharpoonright_{\Gamma\setminus\Gamma_\Lambda} \equiv 0$ for some $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, the K-transform of G is a mapping $KG : \Gamma \to \mathbb{R}$ defined at each $\gamma \in \Gamma$ by

$$(KG)(\gamma) := \sum_{\substack{\eta\subset\gamma \\ |\eta|<\infty}} G(\eta). \tag{2.1}$$

Note that for every such function G the sum in (2.1) has only a finite number of summands different from zero, and thus KG is a well-defined function on Γ. Moreover, if G has support described as before, then the restriction $(KG)\upharpoonright_{\Gamma_\Lambda}$ is a $\mathcal{B}(\Gamma_\Lambda)$-measurable function and $(KG)(\gamma) = (KG)\upharpoonright_{\Gamma_\Lambda}(\gamma_\Lambda)$ for all $\gamma \in \Gamma$, i.e., KG is a cylinder function.

Let now G be a bounded $\mathcal{B}(\Gamma_0)$-measurable function with bounded support, that is, $G\upharpoonright_{\Gamma_0\setminus\left(\bigsqcup_{n=0}^{N}\Gamma_\Lambda^{(n)}\right)} \equiv 0$ for some $N \in \mathbb{N}_0, \Lambda \in \mathcal{B}_c(\mathbb{R}^d)$. In this situation, for each $C \geq |G|$ one finds $|(KG)(\gamma)| \leq C(1 + |\gamma_\Lambda|)^N$ for all $\gamma \in \Gamma$. As a result, besides the cylindricity property, KG is also polynomially bounded. In the sequel we denote the space of all bounded $\mathcal{B}(\Gamma_0)$-measurable functions with bounded support by $B_{bs}(\Gamma_0)$. It has been shown in [10] that the K-transform is a linear isomorphism which inverse mapping is defined on cylinder functions by

$$\left(K^{-1}F\right)(\eta) := \sum_{\xi\subset\eta}(-1)^{|\eta\backslash\xi|}F(\xi), \quad \eta\in\Gamma_0.$$

As a side remark, we observe that this property of the K-transform yields a full complete description of the elements in $\mathcal{FP}(\Gamma) := K\left(B_{bs}(\Gamma_0)\right)$ which may be found in [10], [13]. However, throughout this work we shall only make use of the above described cylindricity and polynomial boundedness properties of the functions in $\mathcal{FP}(\Gamma)$.

Among the elements in the domain of the K-transform are also the so-called Lebesgue–Poisson coherent states $e_\lambda(f)$ corresponding to $\mathcal{B}(\mathbb{R}^d)$-measurable functions f. By definition, for any $\mathcal{B}(\mathbb{R}^d)$-measurable function f,

$$e_\lambda(f,\eta) := \prod_{x\in\eta} f(x),\ \ \eta\in\Gamma_0\backslash\{\emptyset\}, \quad e_\lambda(f,\emptyset) := 1.$$

If f has compact support, then the image of $e_\lambda(f)$ under the K-transform is a function on Γ given by

$$\left(Ke_\lambda(f)\right)(\gamma) = \prod_{x\in\gamma}(1+f(x)), \quad \gamma\in\Gamma.$$

As well as the K-transform, its dual operator K^* will also play an essential role in our setting. Let $\mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ denote the set of all probability measures μ on $(\Gamma,\mathcal{B}(\Gamma))$ with finite local moments of all orders, i.e.,

$$\int_\Gamma d\mu(\gamma)\,|\gamma_\Lambda|^n < \infty \quad \text{for all } n\in\mathbb{N} \text{ and all } \Lambda\in\mathcal{B}_c(\mathbb{R}^d). \tag{2.2}$$

By the definition of a dual operator, given a $\mu\in\mathcal{M}^1_{\mathrm{fm}}(\Gamma)$, the so-called correlation measure $\rho_\mu := K^*\mu$ corresponding to μ is a measure on $(\Gamma_0,\mathcal{B}(\Gamma_0))$ defined for each $G\in B_{bs}(\Gamma_0)$ by

$$\int_{\Gamma_0} d\rho_\mu(\eta)\,G(\eta) = \int_\Gamma d\mu(\gamma)\ (KG)(\gamma). \tag{2.3}$$

Observe that under the above conditions $K|G|$ is μ-integrable. In terms of correlation measures this means that $B_{bs}(\Gamma_0)\subset L^1(\Gamma_0,\rho_\mu)$.

Actually, $B_{bs}(\Gamma_0)$ is dense in $L^1(\Gamma_0,\rho_\mu)$. Moreover, still by (2.3), on $B_{bs}(\Gamma_0)$ the inequality $\|KG\|_{L^1(\mu)}\le\|G\|_{L^1(\rho_\mu)}$ holds, allowing then an extension of the K-transform to a bounded operator $K: L^1(\Gamma_0,\rho_\mu)\to L^1(\Gamma,\mu)$ in such a way that equality (2.3) still holds for any $G\in L^1(\Gamma_0,\rho_\mu)$. For the extended operator the explicit form (2.1) still holds, now μ-a.e. This means, in particular,

$$\left(Ke_\lambda(f)\right)(\gamma) = \prod_{x\in\gamma}(1+f(x)), \quad \mu\text{–a.a. }\gamma\in\Gamma, \tag{2.4}$$

for all $\mathcal{B}(\mathbb{R}^d)$-measurable functions f such that $e_\lambda(f) \in L^1(\Gamma_0, \rho_\mu)$, cf. e.g. [10].

We also note that in terms of correlation measures ρ_μ property (2.2) means that ρ_μ is locally finite, that is, $\rho_\mu(\Gamma_\Lambda^{(n)}) < \infty$ for all $n \in \mathbb{N}_0$ and all $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$. By $\mathcal{M}_{\mathrm{lf}}(\Gamma_0)$ we denote the class of all locally finite measures on Γ_0.

Example 2.1. *Given a constant $z > 0$, let π_z be the Poisson measure with intensity zdx, that is, the probability measure on $(\Gamma, \mathcal{B}(\Gamma))$ with Laplace transform given by*

$$\int_\Gamma d\pi_z(\gamma) \exp\left(\sum_{x\in\gamma} \varphi(x)\right) = \exp\left(z \int_{\mathbb{R}^d} dx \left(e^{\varphi(x)} - 1\right)\right)$$

for all $\varphi \in \mathcal{D}$. Here $\mathcal{D}$ denotes the Schwartz space of all infinitely differentiable real-valued functions on $\mathbb{R}^d$ with compact support. The correlation measure corresponding to π_z is the so-called Lebesgue–Poisson measure

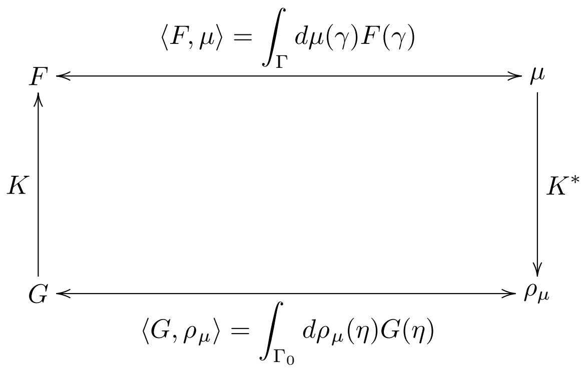
$$\lambda_z := \sum_{n=0}^{\infty} \frac{z^n}{n!} m^{(n)},$$

where each $m^{(n)}$, $n \in \mathbb{N}$, is the image measure on $\Gamma^{(n)}$ of the product measure $dx_1...dx_n$ under the mapping $\widetilde{(\mathbb{R}^d)^n} \ni (x_1, ..., x_n) \mapsto \{x_1, ..., x_n\} \in \Gamma^{(n)}$. For $n = 0$ we set $m^{(0)}(\{\emptyset\}) := 1$. This special case emphasizes the technical role of the coherent states in our setting. First, $e_\lambda(f) \in L^p(\Gamma_0, \lambda_z)$ whenever $f \in L^p(\mathbb{R}^d, dx)$ for some $p \geq 1$, and, moreover, $\|e_\lambda(f)\|^p_{L^p(\lambda_z)} = \exp(z\|f\|^p_{L^p(dx)})$. Second, given a dense subspace $\mathcal{L} \subset L^2(\mathbb{R}^d, dx)$, the set $\{e_\lambda(f) : f \in \mathcal{L}\}$ is total in $L^2(\Gamma_0, \lambda_z)$.

Given a probability measure μ on Γ, let $\mu \circ p_\Lambda^{-1}$ be the image measure on the space Γ_Λ, $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, under the mapping $p_\Lambda : \Gamma \to \Gamma_\Lambda$ defined by $p_\Lambda(\gamma) := \gamma_\Lambda$, $\gamma \in \Gamma$, i.e., the projection of μ onto Γ_Λ. A measure $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ is called locally absolutely continuous with respect to $\pi := \pi_1$ whenever for each $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ the measure $\mu \circ p_\Lambda^{-1}$ is absolutely continuous with respect to $\pi \circ p_\Lambda^{-1}$. In this case, the correlation measure ρ_μ is absolutely continuous with respect to the Lebesgue–Poisson measure $\lambda := \lambda_1$. The Radon-Nikodym derivative $k_\mu := \frac{d\rho_\mu}{d\lambda}$ is the so-called correlation function corresponding to μ. For more details see e.g. [10].

2.2 Markov generators and related evolution equations

Before proceeding further, let us first summarize graphically all the above described notions as well as their relations (see the diagram below). Having in mind concrete applications, let us also mention the natural meaning of this diagram in the context of a given infinite particle system.



The state of such a system is described by a probability measure μ on Γ and the functions F on Γ are considered as observables of the system. They represent physical quantities which can be measured. The expected values of the measured observables correspond to the expectation values $\langle F, \mu \rangle := \int_\Gamma d\mu(\gamma)\, F(\gamma)$.

In this interpretation we call the functions G on Γ_0 quasi-observables, because they are not observables themselves, but they can be used to construct observables via the K-transform. In this way we obtain all observables which are additive in the particles, namely, energy.

The description of the underlying dynamics of such a system is an essentially interesting and often a difficult question. The number of particles involved, which imposes a natural complexity to the study, on the one hand, and the infinite dimensional analysis methods and tools available, once in a while either limited or insufficient, on the other hand, are physical and mathematical reasons for the difficulties, and failures, pointed out. However, it arises from the previous diagram an alternative approach to the construction of the dynamics, overcoming some of those difficulties.

As usual the starting point for this approach is the Markov generator of the dynamics, in the sequel denoted by L, related to the Kolmogorov equation for observables

$$\frac{\partial}{\partial t} F_t = L F_t. \tag{KE}$$

Given an initial distribution μ of the system (from a set of admissible initial distributions on Γ), the generator L determines a Markov process on Γ which initial distribution is μ. Within the diagram context, the distribution μ_t of the Markov process at each time t is then a solution of the dual Kolmogorov equation

$$\frac{d}{dt} \mu_t = L^* \mu_t, \tag{KE*}$$

L^* being the dual operator of L.

The use of the K-transform allows us to proceed further. As a matter of fact, if L is well-defined for instance on $\mathcal{FP}(\Gamma)$, then its image under the K-transform

$\hat{L} := K^{-1}LK$ yields a Kolmogorov equation for quasi-observables

$$\frac{\partial}{\partial t} G_t = \hat{L} G_t. \tag{QKE}$$

Through the dual relation between quasi-observables and correlation measures this leads naturally to a time evolution description of the correlation function k_μ corresponding to the initial distribution μ given above. Of course, in order to obtain such a description we must assume that at each time t the correlation measure corresponding to the distribution μ_t is absolutely continuous with respect to the Lebesgue–Poisson measure λ. Then, denoting by $\hat{L}^*$ the dual operator of $\hat{L}$ in the sense

$$\int_{\Gamma_0} d\lambda(\eta)\, (\hat{L}G)(\eta) k(\eta) = \int_{\Gamma_0} d\lambda(\eta)\, G(\eta) (\hat{L}^* k)(\eta),$$

one derives from (QKE) its dual equation,

$$\frac{\partial}{\partial t} k_t = \hat{L}^* k_t. \tag{QKE*}$$

Clearly, the correlation function k_t corresponding to μ_t, $t \geq 0$, is a solution of (QKE)*. At this point it is opportune to underline that a solution of (QKE)* does not have to be a correlation function (corresponding to some measure on Γ), a fact which is frequently not taken into account in theoretical physics discussions. An additional analysis is needed in order to distinguish the correlation functions from the set of solutions of the (QKE)* equation. Within our setting, some criteria were developed in [1], [26], [10], [23].

In this way we have derived four equations related to the dynamics of an infinite particle system in the continuum. Starting with (KE), one had derived (QKE)*, both equations being well-known in physics. Concerning the latter equation, let us mention its Bogoliubov hierarchical structure, which in the Hamiltonian dynamics case yields the well-known BBGKY-hierarchy (see e.g. [2]). In our case, the hierarchical structure is given by a countable infinite system of equations

$$\frac{\partial}{\partial t} k_t^{(n)} = (\hat{L}^* k_t)^{(n)}, \quad k_t^{(n)} := k_t\restriction_{\Gamma^{(n)}},\ (\hat{L}^* k_t)^{(n)} := (\hat{L}^* k_t)\restriction_{\Gamma^{(n)}},\ n \in \mathbb{N}_0. \tag{2.5}$$

In contrast to (KE), note that each equation in (2.5) only depends on a finite number of coordinates. This explains the technical efficacy of equation (QKE)* in concrete applications.

Although equations (QKE) and (KE)* being also known in physics, their studied is not so developed and usually they are not exploit in concrete applications. However, in such applications those equations often turn out to be an effective method.

Before proceeding to concrete applications, let us observe that for some concrete models it is possible to widen the dynamical description towards Bogoliubov functionals [2].

2.3 Generating functionals

Given a probability measure μ on $(\Gamma, \mathcal{B}(\Gamma))$ the so-called Bogoliubov or generating functional B_μ corresponding to μ is the functional defined at each $\mathcal{B}(\mathbb{R}^d)$-measurable function θ by

$$B_\mu(\theta) := \int_\Gamma d\mu(\gamma) \prod_{x\in\gamma}(1+\theta(x)), \tag{2.6}$$

provided the right-hand side exists for $|\theta|$. In the same way one cannot define the Laplace transform for all measures on Γ, it is clear from (2.6) that one cannot define the Bogoliubov functional for all probability measures on Γ as well. Actually, for each $\theta > -1$ so that the right-hand side of (2.6) exists, one may equivalently rewrite (2.6) as

$$B_\mu(\theta) := \int_\Gamma d\mu(\gamma)\, e^{\langle \ln(1+\theta),\gamma\rangle},$$

showing that B_μ is a modified Laplace transform.

If the Bogoliubov functional B_μ corresponding to a probability measure μ exists, then clearly the domain of B_μ depends on the underlying measure. Conversely, the domain of a Bogoliubov functional B_μ reflects special properties over the measure μ [14]. For instance, if μ has finite local exponential moments, i.e.,

$$\int_\Gamma d\mu(\gamma)\, e^{\alpha|\gamma_\Lambda|} < \infty \quad \text{for all } \alpha > 0 \text{ and all } \Lambda \in \mathcal{B}_c(\mathbb{R}^d),$$

then B_μ is well-defined for instance on all bounded functions θ with compact support. The converse is also true. In fact, for each $\alpha > 0$ and each $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ the latter integral is equal to $B_\mu((e^\alpha - 1)1\!\!1_\Lambda)$. In this situation, to a such measure μ one may associate the correlation measure ρ_μ, and equalities (2.3) and (2.4) then yield a description of the functional B_μ in terms of either the measure ρ_μ:

$$B_\mu(\theta) = \int_\Gamma d\mu(\gamma)\ (Ke_\lambda(\theta))\,(\gamma) = \int_{\Gamma_0} d\rho_\mu(\eta)\, e_\lambda(\theta,\eta),$$

or the correlation function k_μ, if ρ_μ is absolutely continuous with respect to the Lebesgue–Poisson measure λ:

$$B_\mu(\theta) = \int_{\Gamma_0} d\lambda(\eta)\, e_\lambda(\theta,\eta) k_\mu(\eta).$$

Within the framework of Subsection 2.2, this gives us a way to express the dynamics of an infinite particle system in terms of the Bogoliubov functionals

$$B_t(\theta) = \int_{\Gamma_0} d\lambda(\eta)\, e_\lambda(\theta,\eta) k_t(\eta)$$

corresponding to the states of the system at each time $t \geq 0$, provided the functionals exist. Informally,

$$\frac{\partial}{\partial t} B_t(\theta) = \int_{\Gamma_0} d\lambda(\eta)\, e_\lambda(\theta,\eta) \frac{\partial}{\partial t} k_t(\eta) = \int_{\Gamma_0} d\lambda(\eta)\, (\hat{L} e_\lambda(\theta))(\eta) k_t(\eta). \tag{2.7}$$

In other words, given the operator $\tilde{L}$ defined at

$$B(\theta) := \int_{\Gamma_0} d\lambda(\eta)\, e_\lambda(\theta,\eta)k(\eta) \quad (k : \Gamma_0 \to \mathbb{R}_0^+ := [0,+\infty))$$

by

$$(\tilde{L}B)(\theta) := \int_{\Gamma_0} d\lambda(\eta)\, (\hat{L}e_\lambda(\theta))(\eta)k(\eta),$$

heuristically (2.7) means that the Bogoliubov functionals B_t, $t \geq 0$, are a solution of the equation

$$\frac{\partial}{\partial t} B_t = \tilde{L}B_t. \tag{2.8}$$

Besides the problem of the existence of the Bogoliubov functionals B_t, $t \geq 0$, let us also observe that if a solution of equation (2.8) exists, a priori it does not have to be a Bogoliubov functional corresponding to some measure. The verification requests an additional analysis, see e.g. [14], [23].

In applications below, in order to derive explicit formulas for $\tilde{L}$, the next result turns out to be useful. Here and below, all $L^p_{\mathbb{C}}$-spaces, $p \geq 1$, consist of p-integrable complex-valued functions.

Proposition 2.2. *Given a measure $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ assume that the corresponding Bogoliubov functional B_μ is entire on $L^1_{\mathbb{C}}(\mathbb{R}^d, dx)$. Then each differential of n-th order of B_μ, $n \in \mathbb{N}$, at each $\theta_0 \in L^1_{\mathbb{C}}(\mathbb{R}^d, dx)$ is defined by a symmetric kernel in $L^\infty_{\mathbb{C}}((\mathbb{R}^d)^n, dx_1...dx_n)$ denoted by $\frac{\delta^n B_\mu(\theta_0)}{\delta\theta_0(x_1)...\delta\theta_0(x_n)}$ and called the variational derivative of n-th order of B_μ at θ_0. In other words,*

$$\frac{\partial^n}{\partial z_1 ... \partial z_n} B_\mu\left(\theta_0 + \sum_{i=1}^n z_i\theta_i\right)\Big|_{z_1=...=z_n=0}$$
$$= \int_{\mathbb{R}^d} dx_1\, \theta_1(x_1) \cdots \int_{\mathbb{R}^d} dx_n\, \theta_n(x_n) \frac{\delta^n B_\mu(\theta_0)}{\delta\theta_0(x_1)...\delta\theta_0(x_n)},$$

for all $\theta_1, ..., \theta_n \in L^1_{\mathbb{C}}(\mathbb{R}^d, dx)$. Furthermore, using the notation

$$\left(D^{|\eta|}B_\mu\right)(\theta_0;\eta) := \frac{\delta^n B_\mu(\theta_0)}{\delta\theta_0(x_1)...\delta\theta_0(x_n)} \quad \text{for } \eta = \{x_1, ..., x_n\} \in \Gamma^{(n)}, n \in \mathbb{N},$$

the Taylor expansion of B_μ at each $\theta_0 \in L^1_{\mathbb{C}}(\mathbb{R}^d, dx)$ may be written in the form

$$B_\mu(\theta_0 + \theta) = \int_{\Gamma_0} d\lambda(\eta)\, e_\lambda(\theta,\eta) \left(D^{|\eta|}B_\mu\right)(\theta_0;\eta), \quad \theta \in L^1_{\mathbb{C}}(\mathbb{R}^d, dx).$$

In terms of the measure μ, the holomorphy asssumption in Proposition 2.2 implies that μ is locally absolutely continuous with respect to the measure π and the correlation function k_μ is given for λ-a.a $\eta \in \Gamma_0$ by $k_\mu(\eta) = \left(D^{|\eta|}B_\mu\right)(0;\eta)$. Moreover, for all $\theta \in L^1_{\mathbb{C}}(\mathbb{R}^d, dx)$ the following relation holds

$$\left(D^{|\eta|}B_\mu\right)(\theta;\eta) = \int_{\Gamma_0} d\lambda(\xi)\, k_\mu(\eta \cup \xi) e_\lambda(\theta,\xi), \quad \lambda-\text{a.e.}, \tag{2.9}$$

showing that the Bogoliubov functional B_μ is the generating functional for the correlation functions $k_\mu\restriction_{\Gamma^{(n)}}$, $n \in \mathbb{N}_0$. For more details and proofs see e.g. [14].

2.4 Algebraic properties

As discussed before, the description of the dynamics of a particle system is closely related to the operators L, $\hat{L}$, and $\hat{L}^*$. To explicitly describe these operators in the examples below, the following algebraic properties turn out to be powerful tools for a simplification of calculations.

Given G_1 and G_2 two $\mathcal{B}(\Gamma_0)$-measurable functions, let us consider the $\star$-convolution between G_1 and G_2,

$$\begin{aligned}(G_1 \star G_2)(\eta) &:= \sum_{(\eta_1,\eta_2,\eta_3)\in\mathcal{P}_3(\eta)} G_1(\eta_1 \cup \eta_2) G_2(\eta_2 \cup \eta_3) \\ &= \sum_{\xi\subset\eta} G_1(\xi) \sum_{\zeta\subset\xi} G_2((\eta\setminus\xi)\cup\zeta), \quad \eta\in\Gamma_0,\end{aligned}$$

where $\mathcal{P}_3(\eta)$ denotes the set of all partitions of η in three parts which may be empty, [10]. It is straightforward to verify that the space of all $\mathcal{B}(\Gamma_0)$-measurable functions endowed with this product has the structure of a commutative algebra with unit element $e_\lambda(0)$. Furthermore, for every $G_1, G_2 \in B_{bs}(\Gamma_0)$ we have $G_1 \star G_2 \in B_{bs}(\Gamma_0)$, and

$$K(G_1 \star G_2) = (KG_1)\cdot(KG_2) \tag{2.10}$$

cf. [10]. Concerning the action of the $\star$-convolution on coherent states one finds

$$e_\lambda(f) \star e_\lambda(g) = e_\lambda(f + g + fg) \tag{2.11}$$

for all $\mathcal{B}(\mathbb{R}^d)$-measurable functions f and g. More generally, for all $\mathcal{B}(\Gamma_0)$-measurable functions G and all $\mathcal{B}(\mathbb{R}^d)$-measurable functions f we have

$$(G \star e_\lambda(f))(\eta) = \sum_{\xi\subset\eta} G(\xi)\, e_\lambda(f+1,\xi)\, e_\lambda(f,\eta\setminus\xi). \tag{2.12}$$

Technically the next result turns out to be very useful. We refer e.g. to [32] for its proof. In particular, for $n = 3$, it yields an integration result for the $\star$-convolution.

Lemma 2.3. *Let $n\in\mathbb{N}$, $n\geq 2$, be given. Then*

$$\begin{aligned}&\int_{\Gamma_0} d\lambda(\eta_1)\ldots\int_{\Gamma_0} d\lambda(\eta_n)\, G(\eta_1\cup\ldots\cup\eta_n) H(\eta_1,\ldots,\eta_n) \\ &= \int_{\Gamma_0} d\lambda(\eta)\, G(\eta) \sum_{(\eta_1,\ldots,\eta_n)\in\mathcal{P}_n(\eta)} H(\eta_1,\ldots,\eta_n)\end{aligned}$$

for all positive measurable functions $G:\Gamma_0\to\mathbb{R}$ and $H:\Gamma_0\times\ldots\times\Gamma_0\to\mathbb{R}$. Here $\mathcal{P}_n(\eta)$ denotes the set of all partitions of η in n parts, which may be empty.

Lemma 2.4. *For all positive measurable functions $H, G_1, G_2 : \Gamma_0\to\mathbb{R}$ one has*

$$\begin{aligned}&\int_{\Gamma_0} d\lambda(\eta)\, H(\eta)(G_1\star G_2)(\eta) \\ &= \int_{\Gamma_0} d\lambda(\eta_1)\int_{\Gamma_0} d\lambda(\eta_2)\int_{\Gamma_0} d\lambda(\eta_3) H(\eta_1\cup\eta_2\cup\eta_3) G_1(\eta_1\cup\eta_2) G_2(\eta_2\cup\eta_3).\end{aligned}$$

3 Markovian birth-and-death dynamics in configuration spaces

In a birth-and-death dynamics, at each random moment of time and at each site in $\mathbb{R}^d$, a particle randomly appears or disappears according to birth and death rates which depend on the configuration of the whole system at that time. Informally, in terms of Markov generators, this behaviour is described through the operators D_x^- and D_x^+ defined at each $F:\Gamma\to\mathbb{R}$ by[1]

$$(D_x^-F)(\gamma):=F(\gamma\setminus x)-F(\gamma),\quad (D_x^+F)(\gamma):=F(\gamma\cup x)-F(\gamma),$$

corresponding, respectively, to the annihilation and creation of a particle at a site x. More precisely,

$$(LF)(\gamma):=\sum_{x\in\gamma}d(x,\gamma\setminus x)(D_x^-F)(\gamma)+\int_{\mathbb{R}^d}dx\,b(x,\gamma)(D_x^+F)(\gamma),\tag{3.1}$$

where the coefficient $d(x,\gamma)\geq 0$ indicates the rate at which a particle located at x in a configuration γ dies or disappears, while $b(x,\gamma)\geq 0$ indicates the rate at which, given a configuration γ, a new particle is born or appears at a site x.

3.1 Markovian birth-and-death generators

In order to give a meaning to (3.1) let us consider the class of measures $\mu\in\mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ such that $d(x,\cdot),b(x,\cdot)\in L^1(\Gamma,\mu)$, $x\in\mathbb{R}^d$, and for all $n\in\mathbb{N}_0$ and all $\Lambda\in\mathcal{B}_c(\mathbb{R}^d)$ the following integrability condition is fulfilled:

$$\int_\Gamma d\mu(\gamma)\,|\gamma_\Lambda|^n\sum_{x\in\gamma_\Lambda}d(x,\gamma\setminus x)+\int_\Gamma d\mu(\gamma)\,|\gamma_\Lambda|^n\int_\Lambda dx\,b(x,\gamma)<\infty.\tag{3.2}$$

For $F\in\mathcal{FP}(\Gamma)=K(B_{bs}(\Gamma_0))$, this condition is sufficient to insure that LF is μ-a.e. well-defined on Γ. This follows from the fact that for each $G\in B_{bs}(\Gamma_0)$ there are $\Lambda\in\mathcal{B}_c(\mathbb{R}^d), N\in\mathbb{N}_0$ and a $C\geq 0$ such that G has support in $\cup_{n=0}^N\Gamma_\Lambda^{(n)}$ and $|G|\leq C$, which leads to a cylinder function $F=KG$ such that $|F(\gamma)|=|F(\gamma_\Lambda)|\leq C(1+|\gamma_\Lambda|)^N$ for all $\gamma\in\Gamma$ (cf. Subsection 2.1). Hence (3.1) and (3.2) imply that $LF\in L^1(\Gamma,\mu)$.

Given a family of functions $B_x,D_x:\Gamma_0\to\mathbb{R}$, $x\in\mathbb{R}^d$, such that $KB_x\geq 0$, $KD_x\geq 0$, in the following we wish to consider KB_x and KD_x as birth and death rates, i.e.,

$$b(x,\gamma)=(KB_x)(\gamma),\ d(x,\gamma)=(KD_x)(\gamma).\tag{3.3}$$

We shall then restrict the previous class of measures in $\mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ to the set of all measures $\mu\in\mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ such that $B_x,D_x\in L^1(\Gamma_0,\rho_\mu)$, $x\in\mathbb{R}^d$, and

$$\int_\Gamma d\mu(\gamma)\,|\gamma_\Lambda|^n\left\{\sum_{x\in\gamma_\Lambda}(K|D_x|)(\gamma\setminus x)+\int_\Lambda dx\ (K|B_x|)(\gamma)\right\}<\infty\tag{3.4}$$

[1]Here and below, for simplicity of notation, we have just written x instead of $\{x\}$.

for all $n \in \mathbb{N}_0$ and all $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$. Under these assumptions, the K-transform of each B_x and each D_x, $x \in \mathbb{R}^d$, is well-defined. Moreover, $KB_x, KD_x \in L^1(\Gamma, \mu)$, cf. Subsection 2.1. Of course, all previous considerations hold. In addition, we have the following result for the operator $\hat{L}$ on quasi-observables.

Proposition 3.1. *The action of $\hat{L}$ on functions $G \in B_{bs}(\Gamma_0)$ is given for ρ_μ-almost all $\eta \in \Gamma_0$ by*

$$(\hat{L}G)(\eta) = -\sum_{x\in\eta} (D_x \star G(\cdot \cup x))(\eta \setminus x) + \int_{\mathbb{R}^d} dx\ (B_x \star G(\cdot \cup x))(\eta). \tag{3.5}$$

Moreover, $\hat{L}(B_{bs}(\Gamma_0)) \subset L^1(\Gamma_0, \rho_\mu)$.

Proof. By the definition of the K-transform, for all $G \in B_{bs}(\Gamma_0)$ we find

$$(KG)(\gamma \setminus x) - (KG)(\gamma) = -(K(G(\cdot \cup x)))(\gamma \setminus x), \quad x \in \gamma,$$
$$(KG)(\gamma \cup x) - (KG)(\gamma) = (K(G(\cdot \cup x)))(\gamma), \quad x \notin \gamma.$$

Given a $F \in \mathcal{FP}(\Gamma)$ of the form $F = KG$, $G \in B_{bs}(\Gamma_0)$, these equalities combined with the algebraic action (2.10) of the K-transform yield

$$\begin{aligned}(LF)(\gamma) &= -\sum_{x\in\gamma} d(x, \gamma\setminus x)\,(K\,(G(\cdot\cup x)))\,(\gamma\setminus x)\\ &\quad+\int_{\{x:x\notin\gamma\}} dx\, b(x,\gamma)\,(K\,(G(\cdot\cup x)))\,(\gamma)\\ &= -\sum_{x\in\gamma} (K\,(D_x\star G(\cdot\cup x)))\,(\gamma\setminus x) + \int_{\mathbb{R}^d} dx\,(K\,(B_x\star G(\cdot\cup x)))\,(\gamma).\end{aligned}$$

Hence, for $\hat{L}G = K^{-1}(LF)$, we have

$$(\hat{L}G)(\eta) = -\sum_{\xi\subset\eta}(-1)^{|\eta\setminus\xi|}\sum_{x\in\xi}(K\,(D_x\star G(\cdot\cup x)))\,(\xi\setminus x) \tag{3.6}$$
$$+\int_{\mathbb{R}^d} dx\, K^{-1}\,(K\,(B_x\star G(\cdot\cup x)))\,(\eta). \tag{3.7}$$

A direct application of the definitions of the K-transform and K^{-1} yields for the sum in (3.6)

$$\begin{aligned}&\sum_{\xi\subset\eta}(-1)^{|\eta\setminus\xi|}\sum_{x\in\xi}(K\,(D_x\star G(\cdot\cup x)))\,(\xi\setminus x)\\ &=\sum_{x\in\eta}\sum_{\xi\subset\eta\setminus x}(-1)^{|\eta\setminus(\xi\cup x)|}\,(K\,(D_x\star G(\cdot\cup x)))\,((\xi\cup x)\setminus x)\\ &=\sum_{x\in\eta}\sum_{\xi\subset\eta\setminus x}(-1)^{|(\eta\setminus x)\setminus\xi|}\,(K\,(D_x\star G(\cdot\cup x)))\,(\xi)\\ &=\sum_{x\in\eta}K^{-1}\,(K\,(D_x\star G(\cdot\cup x)))\,(\eta\setminus x)\\ &=\sum_{x\in\eta}(D_x\star G(\cdot\cup x))\,(\eta\setminus x),\end{aligned}$$

and for the integral (3.7)

$$\int_{\mathbb{R}^d} dx\, K^{-1}\left(K\left(B_x \star G(\cdot \cup x)\right)\right)(\eta) = \int_{\mathbb{R}^d} dx\, \left(B_x \star G(\cdot \cup x)\right)(\eta).$$

In order to prove the integrability of $|\hat{L}G|$ for $G \in B_{bs}(\Gamma_0)$, first we note that each $G \in B_{bs}(\Gamma_0)$ can be majorized by $|G| \leq C 1\!\!1_{\bigsqcup_{n=0}^N \Gamma_\Lambda^{(n)}}$ for some $C \geq 0$ and for the indicator function $1\!\!1_{\bigsqcup_{n=0}^N \Gamma_\Lambda^{(n)}} \in B_{bs}(\Gamma_0)$ of some disjoint union $\bigsqcup_{n=0}^N \Gamma_\Lambda^{(n)}$, $N \in \mathbb{N}_0, \Lambda \in \mathcal{B}_c(\mathbb{R}^d)$. Hence the proof amounts to show the integrability of $|\hat{L} 1\!\!1_{\bigsqcup_{n=0}^N \Gamma_\Lambda^{(n)}}|$ for all $N \in \mathbb{N}$ and all $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$. This follows from

$$\begin{aligned}
&\int_{\Gamma_0} d\rho_\mu(\eta) \sum_{x\in\eta} \left(|D_x| \star 1\!\!1_{\bigsqcup_{n=0}^N \Gamma_\Lambda^{(n)}}(\cdot \cup x)\right)(\eta \setminus x) \\
&\quad + \int_{\Gamma_0} d\rho_\mu(\eta) \int_{\mathbb{R}^d} dx\, \left(|B_x| \star 1\!\!1_{\bigsqcup_{n=0}^N \Gamma_\Lambda^{(n)}}(\cdot \cup x)\right)(\eta) \\
&\leq \int_{\Gamma_0} d\rho_\mu(\eta) \sum_{x\in\eta} 1\!\!1_\Lambda(x) \left(|D_x| \star 1\!\!1_{\bigsqcup_{n=0}^{N-1} \Gamma_\Lambda^{(n)}}\right)(\eta \setminus x) \qquad (3.8)\\
&\quad + \int_\Lambda dx \int_{\Gamma_0} d\rho_\mu(\eta) \left(|B_x| \star 1\!\!1_{\bigsqcup_{n=0}^{N-1} \Gamma_\Lambda^{(n)}}\right)(\eta) \qquad (3.9)\\
&= \int_\Gamma d\mu(\gamma)\, K\left(\sum_{x\in\cdot} 1\!\!1_\Lambda(x) \left(|D_x| \star 1\!\!1_{\bigsqcup_{n=0}^{N-1} \Gamma_\Lambda^{(n)}}\right)(\cdot \setminus x)\right)(\gamma) \qquad (3.10)\\
&\quad + \int_\Lambda dx \int_\Gamma d\mu(\gamma)\, K\left(|B_x| \star 1\!\!1_{\bigsqcup_{n=0}^{N-1} \Gamma_\Lambda^{(n)}}\right)(\gamma), \qquad (3.11)
\end{aligned}$$

where a direct calculation using the definition of the K-transform gives for the integral (3.10)

$$\begin{aligned}
&\int_\Gamma d\mu(\gamma) \sum_{x\in\gamma} 1\!\!1_\Lambda(x) K\left(|D_x| \star 1\!\!1_{\bigsqcup_{n=0}^{N-1} \Gamma_\Lambda^{(n)}}\right)(\gamma \setminus x) \\
&= \int_\Gamma d\mu(\gamma) \sum_{x\in\gamma_\Lambda} (K|D_x|)(\gamma \setminus x) \left(K 1\!\!1_{\bigsqcup_{n=0}^{N-1} \Gamma_\Lambda^{(n)}}\right)(\gamma \setminus x),
\end{aligned}$$

cf. (2.10).

Taking into account that $1\!\!1_{\bigsqcup_{n=0}^{N-1} \Gamma_\Lambda^{(n)}} \in B_{bs}(\Gamma_0)$, and thus

$$\left(K 1\!\!1_{\bigsqcup_{n=0}^{N-1} \Gamma_\Lambda^{(n)}}\right)(\gamma) \leq (1 + |\gamma_\Lambda|)^{N-1},$$

one may then bound the sum of the integrals (3.10) and (3.11) by

$$\int_\Gamma d\mu(\gamma)\, |\gamma_\Lambda|^{N-1} \sum_{x\in\gamma_\Lambda} (K|D_x|)(\gamma \setminus x) + \int_\Gamma d\mu(\gamma)\, (1+|\gamma_\Lambda|)^{N-1} \int_\Lambda dx\, (K|B_x|)(\gamma),$$

which, by (3.4), shows the required integrability. □

Remark 3.2. *Integrability condition (3.4) is presented for general measures $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ and generic birth and death rates of the type (3.3). From the previous proof it is clear that (3.4) is the weakest possible integrability condition to state Proposition 3.1. In addition, its proof also shows that for each measure $\rho \in \mathcal{M}_{\mathrm{lf}}(\Gamma_0)$ such that $B_x, D_x \in L^1(\Gamma_0, \rho)$ and such that for all $n \in \mathbb{N}_0$ and all $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$*

$$\int_{\Gamma_0} d\rho(\eta) \left\{ \sum_{x\in\eta_\Lambda} \left(|D_x| \star 1\!\!1_{\Gamma_\Lambda^{(n)}} \right)(\eta \setminus x) + \int_\Lambda dx \left(|B_x| \star 1\!\!1_{\Gamma_\Lambda^{(n)}} \right)(\eta) \right\} < \infty,$$

one has $\hat{L}\left(B_{bs}(\Gamma_0)\right) \subset L^1(\Gamma_0, \rho)$. Moreover, this integrability condition on $\rho \in \mathcal{M}_{\mathrm{lf}}(\Gamma_0)$ is the weakest possible one to yield such an inclusion. This follows from (3.8), (3.9) and the fact that $1\!\!1_{\bigsqcup_{n=0}^N \Gamma_\Lambda^{(n)}} = \sum_{n=0}^N 1\!\!1_{\Gamma_\Lambda^{(n)}}$.

Remark 3.3. *Taking into account (2.12), we note that:*
(1) if each D_x is of the type $D_x = e_\lambda(d_x)$, then the sum in (3.5) is given by

$$\sum_{x\in\eta} \sum_{\xi\subset\eta\setminus x} G(\xi \cup x) e_\lambda(d_x + 1, \xi) e_\lambda(d_x, (\eta \setminus x) \setminus \xi);$$

(2) analogously, if $B_x = e_\lambda(b_x)$, then the integral in (3.5) is equal to

$$\sum_{\xi\subset\eta} \int_{\mathbb{R}^d} dx\, G(\xi \cup x) e_\lambda(b_x + 1, \xi) e_\lambda(b_x, \eta \setminus \xi).$$

Remark 3.4. *For birth and death rates such that $|B_x| \leq e_\lambda(b_x), |D_x| \leq e_\lambda(d_x)$, for some $0 \leq b_x, d_x \in L^1(\mathbb{R}^d, dx)$, and for measures $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ that are locally absolutely continuous with respect to π and the correlation function k_μ fulfills the so-called Ruelle bound, i.e., $k_\mu \leq e_\lambda(C)$ for some constant $C > 0$, one may replace (3.4) by the stronger integrability condition*

$$\int_\Lambda dx \left(\exp\left(2C\|b_x\|_{L^1(\mathbb{R}^d,dx)}\right) + \exp\left(2C\|d_x\|_{L^1(\mathbb{R}^d,dx)}\right) \right) < \infty \tag{3.12}$$

for any $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$.

Corollary 3.5. *Let $k : \Gamma_0 \to \mathbb{R}_0^+$ be such that*

$$\int_{\Gamma_\Lambda^{(n)}} d\lambda(\eta)\, k(\eta) < \infty \quad \text{for all } n \in \mathbb{N}_0 \text{ and all } \Lambda \in \mathcal{B}_c(\mathbb{R}^d). \tag{3.13}$$

If $B_x, D_x \in L^1(\Gamma_0, k\lambda)$ and for all $n \in \mathbb{N}_0$ and all $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ we have

$$\int_{\Gamma_0} d\lambda(\eta)\, k(\eta) \left\{ \sum_{x\in\eta_\Lambda} \left(|D_x| \star 1\!\!1_{\Gamma_\Lambda^{(n)}} \right)(\eta \setminus x) + \int_\Lambda dx \left(|B_x| \star 1\!\!1_{\Gamma_\Lambda^{(n)}} \right)(\eta) \right\} < \infty,$$

then

$$(\hat{L}^*k)(\eta) = -\int_{\Gamma_0} d\lambda(\zeta)\, k(\zeta\cup\eta) \sum_{x\in\eta}\sum_{\xi\subset\eta\setminus x} D_x(\zeta\cup\xi) \tag{3.14}$$

$$+\int_{\Gamma_0} d\lambda(\zeta) \sum_{x\in\eta} k(\zeta\cup(\eta\setminus x)) \sum_{\xi\subset\eta\setminus x} B_x(\zeta\cup\xi), \tag{3.15}$$

for λ-almost all $\eta\in\Gamma_0$.

Proof. According to the definition of the dual operator $\hat{L}^*$, for all $G\in B_{bs}(\Gamma_0)$ we have

$$\int_{\Gamma_0} d\lambda(\eta)\, (\hat{L}^*k)(\eta)G(\eta) = \int_{\Gamma_0} d\lambda(\eta)\, (\hat{L}G)(\eta)k(\eta). \tag{3.16}$$

Due to (3.13), we observe that the measure $k(\eta)\lambda(d\eta)$ on Γ_0 is in $\mathcal{M}_{\mathrm{lf}}(\Gamma_0)$. Therefore, according to Remark 3.2, under the fixed assumptions the integral on the right-hand side of (3.16) is always finite. The proof then follows by successive applications of Lemmata 2.3 and 2.4 to this integral. This procedure applied to the sum in (3.5) gives rise to

$$\begin{aligned}
&\int_{\Gamma_0} d\lambda(\eta)\, k(\eta) \sum_{x\in\eta} \left(D_x \star G(\cdot\cup x)\right)(\eta\setminus x)\\
&= \int_{\mathbb{R}^d} dx \int_{\Gamma_0} d\lambda(\eta)\, \left(D_x \star G(\cdot\cup x)\right)(\eta) k(\eta\cup x)\\
&= \int_{\mathbb{R}^d} dx \int_{\Gamma_0} d\lambda(\eta_1) \int_{\Gamma_0} d\lambda(\eta_2) D_x(\eta_1\cup\eta_2)\\
&\quad\times \int_{\Gamma_0} d\lambda(\eta_3)\, G(\eta_2\cup\eta_3\cup x) k(\eta_1\cup\eta_2\cup\eta_3\cup x)\\
&= \int_{\Gamma_0} d\lambda(\eta_1) \int_{\Gamma_0} d\lambda(\eta)\, G(\eta) k(\eta_1\cup\eta) \sum_{x\in\eta}\sum_{\xi\subset\eta\setminus x} D_x(\eta_1\cup\xi).
\end{aligned}$$

Similarly, for the integral expression which appears in (3.5) we find

$$\begin{aligned}
&\int_{\Gamma_0} d\lambda(\eta)\, k(\eta) \int_{\mathbb{R}^d} dx\, \left(B_x \star G(\cdot\cup x)\right)(\eta)\\
&= \int_{\mathbb{R}^d} dx \int_{\Gamma_0} d\lambda(\eta_1) \int_{\Gamma_0} d\lambda(\eta_2) \int_{\Gamma_0} d\lambda(\eta_3)\, B_x(\eta_1\cup\eta_2)\\
&\quad\times G(\eta_2\cup\eta_3\cup x) k(\eta_1\cup\eta_2\cup\eta_3)\\
&= \int_{\Gamma_0} d\lambda(\eta_1) \int_{\mathbb{R}^d} dx \int_{\Gamma_0} d\lambda(\eta_2) \int_{\Gamma_0} d\lambda(\eta_3)\, G(\eta_2\cup\eta_3\cup x)\\
&\quad\times B_x(\eta_1\cup\eta_2) k(\eta_1\cup\eta_2\cup\eta_3)\\
&= \int_{\Gamma_0} d\lambda(\eta_1) \int_{\Gamma_0} d\lambda(\eta)\, G(\eta) \sum_{x\in\eta}\sum_{\xi\subset\eta\setminus x} B_x(\eta_1\cup\xi) k(\eta_1\cup(\eta\setminus x)).
\end{aligned}$$

Taking into account the density of the space $B_{bs}(\Gamma_0)$ in $L^1(\Gamma_0, \lambda)$, the required explicit formula follows. □

Remark 3.6. *Concerning Corollary 3.5, observe that:*
(1) if each D_x is of the type $D_x = e_\lambda(d_x)$, then the integral in (3.14) is given by

$$\int_{\Gamma_0} d\lambda(\zeta) k(\eta \cup \zeta) \sum_{x\in\eta} e_\lambda(d_x + 1, \eta \setminus x) e_\lambda(d_x, \zeta);$$

(2) analogously, if $B_x = e_\lambda(b_x)$, then (3.15) is equal to

$$\int_{\Gamma_0} d\lambda(\zeta) \sum_{x\in\eta} k(\zeta \cup (\eta \setminus x)) e_\lambda(b_x + 1, \eta \setminus x) e_\lambda(b_x, \zeta).$$

Under quite general assumptions we have derived an explicit form for the operators $\hat{L}$, $\hat{L}^*$ related to the generator of a birth-and-death dynamics. Within Subsection 2.2 framework, this means that we may describe the underlying dynamics through the time evolution equations (KE), (QKE), and (QKE)*, respectively, for observables, quasi-observables, and correlation functions. The next result concerns a dynamical description through Bogoliubov functionals.

Proposition 3.7. *Let $k : \Gamma_0 \to \mathbb{R}_0^+$ be such that for all $\theta \in L^1_{\mathbb{C}}(\mathbb{R}^d, dx)$ one has $e_\lambda(\theta) \in L^1_{\mathbb{C}}(\Gamma_0, k\lambda)$, and the functional*

$$B(\theta) := \int_{\Gamma_0} d\lambda(\eta)\, e_\lambda(\theta, \eta) k(\eta)$$

is entire on the space $L^1_{\mathbb{C}}(\mathbb{R}^d, dx)$. Suppose also that $B_x, D_x \in L^1(\Gamma_0, k\lambda)$ and $\hat{L}e_\lambda(\theta) \in L^1_{\mathbb{C}}(\Gamma_0, k\lambda)$ for all $\theta \in L^1_{\mathbb{C}}(\mathbb{R}^d, dx)$. Then

$$\begin{aligned}(\tilde{L}B)(\theta) = &-\int_{\Gamma_0} d\lambda(\eta)\, e_\lambda(\theta + 1, \eta) \int_{\mathbb{R}^d} dx\, \theta(x) (D^{|\eta|+1}B)(\theta, \eta \cup x) D_x(\eta) \\ &+ \int_{\Gamma_0} d\lambda(\eta)\, (D^{|\eta|}B)(\theta, \eta) e_\lambda(\theta + 1, \eta) \int_{\mathbb{R}^d} dx\, \theta(x) B_x(\eta),\end{aligned}$$

for all $\theta \in L^1_{\mathbb{C}}(\mathbb{R}^d, dx)$.

Proof. In order to calculate

$$(\tilde{L}B)(\theta) = \int_{\Gamma_0} d\lambda(\eta)\, (\hat{L}e_\lambda(\theta))(\eta) k(\eta),$$

first we observe that the stated assumptions allow an extension of the operator $\hat{L}$ to coherent states $e_\lambda(\theta)$ with $\theta \in L^1_{\mathbb{C}}(\mathbb{R}^d, dx)$:

$$(\hat{L}e_\lambda(\theta)(\eta) = -\sum_{x\in\eta} \theta(x) \left(D_x \star e_\lambda(\theta)\right)(\eta \setminus x) + \int_{\mathbb{R}^d} dx\, \theta(x) \left(B_x \star e_\lambda(\theta)\right)(\eta).$$

Using the special simple form (2.12) for the $\star$-convolution, a direct application of Lemma 2.3 for $n = 2$ yields

$$\int_{\Gamma_0} d\lambda(\eta)\, k(\eta) \sum_{x\in\eta} \theta(x)\, (D_x \star e_\lambda(\theta))\, (\eta \setminus x)$$
$$= \int_{\mathbb{R}^d} dx\, \theta(x) \int_{\Gamma_0} d\lambda(\eta)\, D_x(\eta) e_\lambda(\theta + 1, \eta) \int_{\Gamma_0} d\lambda(\xi) k(\eta \cup \xi \cup x) e_\lambda(\theta, \xi).$$

Due to the holomorphicity of B on $L^1_{\mathbb{C}}(\mathbb{R}^d, dx)$, the latter integral is equal to $(D^{|\eta\cup x|}B)(\theta, \eta \cup x)$ cf. equality (2.9). Similarly,

$$\int_{\Gamma_0} d\lambda(\eta)\, k(\eta) \int_{\mathbb{R}^d} dx\, \theta(x)\, (B_x \star e_\lambda(\theta))\, (\eta)$$
$$= \int_{\Gamma_0} d\lambda(\eta)\, (D^{|\eta|}B)(\theta, \eta) e_\lambda(\theta + 1, \eta) \int_{\mathbb{R}^d} dx\, \theta(x) B_x(\eta). \qquad \square$$

Remark 3.8. *For functions $k : \Gamma_0 \to \mathbb{R}_0^+$ such that $k \leq e_\lambda(C)$ for some constant $C > 0$, the functionals B defined as in Proposition 3.7 are well-defined on the whole space $L^1_{\mathbb{C}}(\mathbb{R}^d, dx)$, cf. Example 2.1. Moreover, they are entire on $L^1_{\mathbb{C}}(\mathbb{R}^d, dx)$, see e.g. [12], [14]. For such functions k, one may then state Proposition 3.7 just under the assumptions $B_x, D_x \in L^1(\Gamma_0, k\lambda)$ and $\hat{L}e_\lambda(\theta) \in L^1_{\mathbb{C}}(\Gamma_0, k\lambda)$ for all $\theta \in L^1_{\mathbb{C}}(\mathbb{R}^d, dx)$.*

Remark 3.9. *Proposition 3.7 is stated for generic birth and death rates of the type (3.3). In applications, the concrete explicit form of such rates allows a reformulation of Proposition 3.7, generally under much weaker analytical assumptions. For instance, if B_x and D_x are of the type $B_x = e_\lambda(b_x), D_x = e_\lambda(d)$, where d is independent of x, then the expression for $\tilde{L}B$ given in Proposition 3.7 reduces to*

$$(\tilde{L}B)(\theta) = \int_{\mathbb{R}^d} dx\, \theta(x) \left(B(\theta(b_x + 1) + b_x) - \frac{\delta B(\theta(d+1) + d)}{\delta(\theta(d+1) + d)(x)} \right).$$

In contrast to the general formula, which depends of all variational derivatives of B at θ, this closed formula only depends on B and its first variational derivative on a shifted point. Further examples are presented in Subsection 3.2 below. Although in all these examples Proposition 3.7 may clearly be stated under much weaker analytical assumptions, the assumptions in Proposition 3.7 are sufficient to state a general result.

3.2 Particular models

Special birth-and-death type models will be presented and discussed within Subsection 3.1 framework. By analogy, all examples presented are a continuous version of models already known for lattices systems, see e.g. [27], [28].

3.2.1 Glauber dynamics

In this birth-and-death type model, particles appear and disappear according to a death rate identically equal to 1 and to a birth rate depending on the interaction between particles. More precisely, let $\phi : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ be a pair potential, that is, a Borel measurable function such that $\phi(-x) = \phi(x) \in \mathbb{R}$ for all $x \in \mathbb{R}^d \setminus \{0\}$, which we assume to be bounded from below, namely, $\phi \geq -2B_\phi$ on $\mathbb{R}^d$ for some $B_\phi \geq 0$, and which fulfills the standard integrability condition

$$\int_{\mathbb{R}^d} dx \left| e^{-\phi(x)} - 1 \right| < \infty. \tag{3.17}$$

Given a configuration γ, the birth rate of a new particle at a site $x \in \mathbb{R}^d \setminus \gamma$ is then given by $b(x, \gamma) = \exp(-E(x, \gamma))$, where $E(x, \gamma)$ is a relative energy of interaction between a particle located at x and the configuration γ defined by

$$E(x,\gamma) := \begin{cases} \sum_{y\in\gamma} \phi(x-y), & \text{if } \sum_{y\in\gamma} |\phi(x-y)| < \infty \\ \\ +\infty, & \text{otherwise} \end{cases}. \tag{3.18}$$

In this special example the required conditions (3.3) for the birth and death rates are clearly verified:

$$d \equiv 1 = Ke_\lambda(0), \quad b(x,\gamma) = e^{-E(x,\gamma)} = \left(Ke_\lambda(e^{-\phi(x-\cdot)} - 1)\right)(\gamma).$$

Comparing with the general case (Subsection 3.1), the conditions imposed to the potential ϕ lead to a simpler situation. In fact, the integrability condition (3.17) implies that for any $C > 0$ and any $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ the integral appearing in (3.12) is always finite. According to Remark 3.4, this implies that for each measure $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$, locally absolutely continuous with respect to π, for which the correlation function fulfills the Ruelle bound we have $L(\mathcal{FP}(\Gamma)) \subset L^1(\Gamma, \mu)$.

The especially simple form of the functions $B_x = e_\lambda(e^{-\phi(x-\cdot)} - 1)$ and $D_x = e_\lambda(0)$ also allows a simplification of the expressions obtained in Subsection 3.1. First, as D_x is the unit element of the $\star$-convolution, using (2.12) we obtain for (3.5)

$$\begin{aligned}(\hat{L}G)(\eta) &= -|\eta|G(\eta) + \int_{\mathbb{R}^d} dx \left(e_\lambda(e^{-\phi(x-\cdot)} - 1) \star G(\cdot \cup x)\right)(\eta) \\ &= -|\eta|G(\eta) + \sum_{\xi\subset\eta} \int_{\mathbb{R}^d} dx\, e^{-E(x,\xi)} G(\xi \cup x) e_\lambda(e^{-\phi(x-\cdot)} - 1, \eta \setminus \xi).\end{aligned} \tag{3.19}$$

Due to the semi-boundedness of ϕ, we note that this expression is well-defined on the whole space Γ_0. This follows from the fact that any $G \in B_{bs}(\Gamma_0)$ may be bounded by $|G| \leq Ce_\lambda(1\!\!1_\Lambda)$, for some $C \geq 0$ and some $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, and thus, by (2.11),

$$\int_{\mathbb{R}^d} dx \left| \left(e_\lambda(e^{-\phi(x-\cdot)} - 1) \star G(\cdot \cup x) \right)(\eta) \right|$$
$$\leq C \int_{\mathbb{R}^d} dx \, 1\!\!1_\Lambda(\eta) \left(e_\lambda(|e^{-\phi(x-\cdot)} - 1|) \star e_\lambda(1\!\!1_\Lambda) \right)(\eta) \leq C\,|\Lambda|\,(3 + 2e^{2B_\phi})^{|\eta|}.$$

Here $|\Lambda|$ denotes the volume of the set Λ. Second, by Remark 3.6, for λ-almost all $\eta \in \Gamma_0$ we find

$$\begin{aligned}
&(\hat{L}^* k)(\eta) \qquad (3.20)\\
&= -\int_{\Gamma_0} d\lambda(\zeta)\, k(\eta \cup \zeta) \sum_{x \in \eta} e_\lambda(1, \eta \setminus x) e_\lambda(0, \zeta)\\
&\quad + \int_{\Gamma_0} d\lambda(\zeta) \sum_{x \in \eta} k(\zeta \cup (\eta \setminus x)) e_\lambda(e^{-\phi(x-\cdot)}, \eta \setminus x) e_\lambda(e^{-\phi(x-\cdot)} - 1, \zeta)\\
&= -|\eta| k(\eta) + \sum_{x \in \eta} e^{-E(x, \eta \setminus x)} \int_{\Gamma_0} d\lambda(\zeta)\, e_\lambda(e^{-\phi(x-\cdot)} - 1, \zeta) k((\eta \setminus x) \cup \zeta).
\end{aligned}$$

According to Remark 3.9, we also have a simpler form for $\tilde{L}$,

$$(\tilde{L}B)(\theta) = -\int_{\mathbb{R}^d} dx\, \theta(x) \left(\frac{\delta B(\theta)}{\delta \theta(x)} - B((1+\theta)(e^{-\phi(x-\cdot)} - 1) + \theta) \right). \qquad (3.21)$$

The Glauber dynamics is the first example which emphasizes the technical efficacy of our approach to dynamical problems. As a matter of fact, for a quite general class of pair potentials one may apply standard Dirichlet forms techniques to L to construct an equilibrium Glauber dynamics, that is, a Markov process on Γ with initial distribution an equilibrium state. This scheme was used in [19] for pair potentials either positive or superstable. Recently, in [20], this construction was extended to a general case of equilibrium birth-and-death dynamics. However, starting with a non-equilibrium state, the Dirichlet forms techniques do not work. Such states can be so far from the equilibrium ones that one cannot even use the equilibrium Glauber dynamics (obtained through Dirichlet forms techniques) to construct the non-equilibrium ones. Within this context, in a recent work [18] the authors have used the $(\mathrm{QKE})^*$ equation to construct a non-equilibrium Glauber dynamics. That is, a Markov process on Γ starting with a distribution from a wide class of non-equilibrium initial states, also identified in [18]. The scheme used is the one described in Subsection 2.2.

3.2.2 Linear voter model

In contrast to the lattice case (see, e.g., [27], [28]) we may consider a voter type model in the continuum for the non-symmetric situation when there is a configuration of members of only one (political) organization. This configuration may obtain a new member somewhere in the society due to an influence of the existing ones. At the same time, the configuration may lose an existing member due to contradictions between its members. Mathematically, this means that, given a population γ of possible voters, an individual $x \in \gamma$ loses his willingness to vote according to a rate

$$d(x,\gamma) = \sum_{y\in\gamma} a_-(x,y) = (Ka_-(x,\cdot))(\gamma),$$

for some symmetric function $a_- : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_0^+$ such that

$$\sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} dy\, a_-(x,y) < \infty;$$

while an individual x wins a perception of the importance of joining the population γ according to a rate

$$b(x,\gamma) = \sum_{y\in\gamma} a_+(x,y) = (Ka_+(x,\cdot))(\gamma),$$

for some symmetric function $a_+ : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_0^+$ such that

$$\sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} dy\, a_+(x,y) < \infty.$$

Here $a_\mp(x,\cdot)$ are understood as functions on Γ_0, namely,

$$a_\mp(x,\eta) = 1\!\!1_{\left\{\eta\in\Gamma^{(1)},\eta=\{y\}\right\}} a_\mp(x,y).$$

Within Subsection 3.1 framework, one straightforwardly derives from the general case corresponding expressions for this special case:

$$\begin{aligned}(\hat{L}G)(\eta) = &-\sum_{x\in\eta}\sum_{y\in\eta\setminus x} a_-(x,y)\,(G(\eta\setminus y) + G(\eta)) \\ &+ \sum_{y\in\eta}\int_{\mathbb{R}^d} dx\, a_+(x,y)\,(G(\eta\cup x) + G((\eta\setminus y)\cup x)),\end{aligned} \tag{3.22}$$

and

$$\begin{aligned}(\hat{L}^*k)(\eta) = &-\int_{\mathbb{R}^d} dy\, k(\eta\cup y)\sum_{x\in\eta} a_-(x,y) - k(\eta)\sum_{x\in\eta}\sum_{y\in\eta\setminus x} a_-(x,y) \\ &+ \int_{\mathbb{R}^d} dy \sum_{x\in\eta} k((\eta\setminus x)\cup y)a_+(x,y) + \sum_{x\in\eta} k(\eta\setminus x)\sum_{y\in\eta\setminus x} a_+(x,y).\end{aligned} \tag{3.23}$$

In addition,

$$\begin{aligned}(\tilde{L}B)(\theta) = &\int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\, a_+(x,y)(1+\theta(y))\theta(x)\frac{\delta B(\theta)}{\delta\theta(y)} \\ &- \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\, a_-(x,y)(1+\theta(y))\theta(x)\frac{\delta^2 B(\theta)}{\delta\theta(x)\delta\theta(y)}.\end{aligned} \tag{3.24}$$

3.2.3 Polynomial voter model

More generally, one may consider rates of polynomial type, that is, the birth and the death rates are of the type

$$d(x,\gamma) = \sum_{\{x_1,...,x_q\}\subset\gamma} a_x^{(q)}(x_1,...,x_q), \qquad b(x,\gamma) = \sum_{\{x_1,...,x_p\}\subset\gamma} a_x^{(p)}(x_1,...,x_p),$$
$$= (K\tilde{a}_x^{(q)})(\gamma) \qquad\qquad = (K\tilde{a}_x^{(p)})(\gamma)$$

for some symmetric functions $0 \leq a_x^{(q)} \in L^1((\mathbb{R}^d)^q, dx_1...dx_q)$, $0 \leq a_x^{(p)} \in L^1((\mathbb{R}^d)^p, dx_1...dx_p)$, $x \in \mathbb{R}^d$, $p, q \in \mathbb{N}$, where

$$\tilde{a}_x^{(i)}(\eta) := \begin{cases} a_x^{(i)}(x_1,...,x_i), & \text{if } \eta = \{x_1,...,x_i\} \in \Gamma^{(i)} \\ 0, & \text{otherwise} \end{cases}, \quad i = p, q.$$

A straightforward application of the general results obtained in Subsection 3.1 yields for this case the expressions

$$\begin{aligned}(\hat{L}G)(\eta) &= -\sum_{x\in\eta}\left(\tilde{a}_x^{(q)} \star G(\cdot\cup x)\right)(\eta\setminus x) + \int_{\mathbb{R}^d} dx\, \left(\tilde{a}_x^{(p)} \star G(\cdot\cup x)\right)(\eta) \\ &= -\sum_{x\in\eta}\sum_{\substack{\xi\subset\eta\setminus x\\|\xi|=q}} \tilde{a}_x^{(q)}(\xi)\sum_{\zeta\subset\xi} G(\zeta\cup(\eta\setminus x)\setminus\xi) \\ &\quad + \sum_{\substack{\xi\subset\eta\\|\xi|=p}}\sum_{\zeta\subset\xi}\int_{\mathbb{R}^d} dx\, \tilde{a}_x^{(p)}(\xi) G(\zeta\cup(\eta\setminus\xi)\cup x) \end{aligned} \tag{3.25}$$

and

$$\begin{aligned}(\hat{L}^*k)(\eta) &= -\sum_{i=0}^{q}\frac{1}{i!}\int_{\Gamma^{(i)}} dm^{(i)}(\zeta)\, k(\zeta\cup\eta)\sum_{x\in\eta}\sum_{\substack{\xi\subset\eta\setminus x\\|\xi|=q-i}} \tilde{a}_x^{(q)}(\zeta\cup\xi) \\ &\quad + \sum_{i=0}^{p}\frac{1}{i!}\int_{\Gamma^{(i)}} dm^{(i)}(\zeta)\sum_{x\in\eta} k(\zeta\cup(\eta\setminus x))\sum_{\substack{\xi\subset\eta\setminus x\\|\xi|=p-i}} \tilde{a}_x^{(p)}(\zeta\cup\xi),\end{aligned} \tag{3.26}$$

where $m^{(i)}$ is the measure on $\Gamma^{(i)}$ defined in Example 2.1 (Subsection 2.1). Moreover,

$$\begin{aligned}&(\tilde{L}B)(\theta) \\ &= -\frac{1}{q!}\int_{\Gamma^{(q)}} dm^{(q)}(\eta)\, e_\lambda(\theta+1,\eta)\int_{\mathbb{R}^d} dx\,\theta(x)(D^{q+1}B)(\theta,\eta\cup x)\tilde{a}_x^{(q)}(\eta) \\ &\quad + \frac{1}{p!}\int_{\Gamma^{(p)}} dm^{(p)}(\eta)\,(D^pB)(\theta,\eta)e_\lambda(\theta+1,\eta)\int_{\mathbb{R}^d} dx\,\theta(x)\tilde{a}_x^{(p)}(\eta).\end{aligned} \tag{3.27}$$

3.2.4 Contact model

The dynamics of a contact model describes the spread of an infectious disease in a population. Given the set γ of infected individuals, an individual $x \in \gamma$ recovers at a constant rate $d(x,\gamma) = 1 = e_\lambda(0)$, while an healthy individual $x \in \mathbb{R}^d \setminus \gamma$ becomes infected according to an infection spreading rate which depends on the presence of infected neighbors,

$$b(x,\gamma) = \lambda \sum_{y\in\gamma} a(x-y) = \left(K(\lambda a(x-\cdot))\right)(\gamma)$$

for some function $0 \leq a \in L^1(\mathbb{R}^d, dx)$ and some coupling constant $\lambda \geq 0$. For this particular model, the application of the general results then yields the following expressions

$$(\hat{L}G)(\eta) = -|\eta|G(\eta) + \lambda \sum_{y\in\eta} \int_{\mathbb{R}^d} dx\, a(x-y) \left(G(\eta \cup x) + G((\eta \setminus y) \cup x)\right), \quad (3.28)$$

and

$$(\hat{L}^* k)(\eta) = -|\eta|k(\eta) + \lambda \int_{\mathbb{R}^d} dy \sum_{x\in\eta} k((\eta \setminus x) \cup y) a(x-y) + \lambda \sum_{x\in\eta} k(\eta \setminus x) \sum_{y\in\eta\setminus x} a(x-y). \quad (3.29)$$

In addition,

$$(\tilde{L}B)(\theta) = -\int_{\mathbb{R}^d} dx\, \theta(x) \frac{\delta B(\theta)}{\delta\theta(x)} + \lambda \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dx\, a(x-y)(1+\theta(y))\theta(x) \frac{\delta B(\theta)}{\delta\theta(y)}.$$

Concerning the corresponding time evolution equation (2.8), the contact model gives a meaning to the considerations done in Subsection 2.3. As a matter of fact, one can show that there is a solution of equation (2.8) only for each finite interval of time. Such a solution has a radius of analyticity which depends on t. For $\lambda \geq 1$ the radius of analyticity decreases when t increases [17]. Therefore, for $\lambda \geq 1$ equation (2.8) cannot have a global solution on time.

For finite range functions $0 \leq a \in L^1(\mathbb{R}^d, dx)$, $\|a\|_{L^1(\mathbb{R}^d,dx)} = 1$, being either $a \in L^\infty(\mathbb{R}^d, dx)$ or $a \in L^{1+\delta}(\mathbb{R}^d, dx)$ for some $\delta > 0$, the authors in [22] have proved the existence of a contact process, i.e., a Markov process on Γ, starting with an initial configuration of infected individuals from a wide set of possible initial configurations. Having in mind that the contact model under consideration is a continuous version of the well-known contact model for lattice systems [27], [28], the assumptions in [22] are natural. In particular the finite range assumption, meaning that the infection spreading process only depends on the influence of infected neighbors on healthy ones. Concerning the infection spreading rate itself, its additive character implies that each individual recovers, independently of the others, after a random exponentially distributed time [22].

Within Subsection 2.2 framework, in a recent work [17] the authors have used the (QKE)* equation to extend the previous existence result to Markov processes on Γ starting with an initial distribution. Besides the construction of the processes, the scheme used allows to identify all invariant measures for such contact processes.

4 Conservative dynamics

In contrast to the birth-and-death dynamics, in the following dynamics there is conservation on the number of particles involved.

4.1 Hopping particles: the general case

Dynamically, in a hopping particles system, at each random moment of time particles randomly hop from one site to another according to a rate depending on the configuration of the whole system at that time. In terms of generators this behaviour is informally described by

$$(LF)(\gamma) = \sum_{x\in\gamma} \int_{\mathbb{R}^d} dy\, c(x,y,\gamma) \left(F(\gamma \setminus x \cup y) - F(\gamma)\right), \tag{4.1}$$

where the coefficient $c(x,y,\gamma) \geq 0$ indicates the rate at which a particle located at x in a configuration γ hops to a site y.

To give a rigorous meaning to the right-hand side of (4.1), we shall consider measures $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ such that $c(x,y,\cdot) \in L^1(\Gamma,\mu)$, $x,y \in \mathbb{R}^d$ and which fulfil, for all $n \in \mathbb{N}_0$ and all $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, the integrability condition

$$\int_\Gamma d\mu(\gamma)\, |\gamma_\Lambda|^n \sum_{x\in\gamma} \int_{\mathbb{R}^d} dy\, c(x,y,\gamma) \left(1\!\!1_\Lambda(x) + 1\!\!1_\Lambda(y)\right) < \infty. \tag{4.2}$$

In this way, given a cylinder function $F \in \mathcal{FP}(\Gamma)$, $|F(\gamma)| = |F(\gamma_\Lambda)| \leq C(1 + |\gamma_\Lambda|)^N$ for some $\Lambda \in \mathcal{B}_c(\mathbb{R}^d), N \in \mathbb{N}_0, C \geq 0$, for all $\gamma \in \Gamma$ one finds

$$|F(\gamma \setminus x \cup y) - F(\gamma)| \leq 2C(2 + |\gamma_\Lambda|)^N (1\!\!1_\Lambda(x) + 1\!\!1_\Lambda(y)).$$

By (4.2), this implies that μ-a.e. the right-hand side of (4.1) is well-defined and finite and, moreover, it defines an element in $L^1(\Gamma,\mu)$.

Given a family of functions $C_{x,y} : \Gamma_0 \to \mathbb{R}$, $x,y \in \mathbb{R}^d$, such that $KC_{x,y} \geq 0$, in the following we wish to consider the case

$$c(x,y,\gamma) = (KC_{x,y})(\gamma \setminus x). \tag{4.3}$$

Therefore, we shall restrict the previous class of measures in $\mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ to all measures $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ such that $C_{x,y} \in L^1(\Gamma_0,\rho_\mu)$, $x,y \in \mathbb{R}^d$, and

$$\int_\Gamma d\mu(\gamma)\, |\gamma_\Lambda|^n \sum_{x\in\gamma} \int_{\mathbb{R}^d} dy\ (K|C_{x,y}|)\, (\gamma\setminus x)\, (1\!\!1_\Lambda(x) + 1\!\!1_\Lambda(y)) < \infty \tag{4.4}$$

for all $n \in \mathbb{N}_0$ and all $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$. In this way, the K-transform of each $C_{x,y}$, $x, y \in \mathbb{R}^d$, is well-defined, $KC_{x,y} \in L^1(\Gamma, \mu)$, and $L(\mathcal{FP}(\Gamma)) \subset L^1(\Gamma, \mu)$.

Proposition 4.1. *The action of the operator $\hat{L}$ on functions $G \in B_{bs}(\Gamma_0)$ is given by*

$$(\hat{L}G)(\eta) = \sum_{x\in\eta} \int_{\mathbb{R}^d} dy \, (C_{x,y} \star (G(\cdot \cup y) - G(\cdot \cup x))) \, (\eta \setminus x),$$

for ρ_μ-almost all $\eta \in \Gamma_0$. We have $\hat{L}\left(B_{bs}(\Gamma_0)\right) \subset L^1(\Gamma_0, \rho_\mu)$.

Proof. By the definition of the space $\mathcal{FP}(\Gamma)$, any element $F \in \mathcal{FP}(\Gamma)$ is of the form $F = KG$ for some $G \in B_{bs}(\Gamma_0)$. The properties of the K-transform, namely, its algebraic action (2.10), then allow to rewrite LF as

$$\begin{aligned}(LF)(\gamma) &= \sum_{x\in\gamma} \int_{\{y:y\notin\gamma\setminus x\}} dy \, c(x, y, \gamma) \left(K\left(G(\cdot \cup y) - G(\cdot \cup x)\right)\right)(\gamma\setminus x) \\ &= \sum_{x\in\gamma} \int_{\mathbb{R}^d} dy \, \left(K\left(C_{x,y} \star (G(\cdot \cup y) - G(\cdot \cup x))\right)\right)(\gamma\setminus x).\end{aligned}$$

Hence

$$\begin{aligned}(\hat{L}G)(\eta) &= K^{-1}\left(\sum_{x\in\cdot} \int_{\mathbb{R}^d} dy \, \left(K\left(C_{x,y} \star (G(\cdot \cup y) - G(\cdot \cup x))\right)\right)(\cdot \setminus x)\right)(\eta) \\ &= \sum_{\xi\subset\eta} (-1)^{|\eta\setminus\xi|} \sum_{x\in\xi} \int_{\mathbb{R}^d} dy \, \left(K\left(C_{x,y} \star (G(\cdot \cup y) - G(\cdot \cup x))\right)\right)(\xi\setminus x) \\ &= \int_{\mathbb{R}^d} dy \sum_{\xi\subset\eta} (-1)^{|\eta\setminus\xi|} \sum_{x\in\xi} \left(K\left(C_{x,y} \star (G(\cdot \cup y) - G(\cdot \cup x))\right)\right)(\xi\setminus x) \\ &= \int_{\mathbb{R}^d} dy \sum_{x\in\eta} \sum_{\xi\subset\eta\setminus x} (-1)^{|(\eta\setminus x)\setminus\xi|} \left(K\left(C_{x,y} \star (G(\cdot \cup y) - G(\cdot \cup x))\right)\right)(\xi) \\ &= \sum_{x\in\eta} \int_{\mathbb{R}^d} dy \, (C_{x,y} \star (G(\cdot \cup y) - G(\cdot \cup x)))(\eta\setminus x).\end{aligned}$$

As in the proof of Proposition 3.1, to check the required inclusion amounts to prove that for all $N \in \mathbb{N}$ and all $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ one has $\hat{L}1\!\!1_{\bigsqcup_{n=0}^N \Gamma_\Lambda^{(n)}} \in L^1(\Gamma_0, \rho_\mu)$. Similar arguments then yield

$$\begin{aligned}&\int_{\Gamma_0} d\rho_\mu(\eta) \left| \left(\hat{L}1\!\!1_{\bigsqcup_{n=0}^N \Gamma_\Lambda^{(n)}}\right)(\eta) \right| \\ &\leq \int_{\Gamma_0} d\rho_\mu(\eta) \sum_{x\in\eta} \int_{\mathbb{R}^d} dy \, \left(|C_{x,y}| \star 1\!\!1_{\bigsqcup_{n=0}^N \Gamma_\Lambda^{(n)}}(\cdot \cup y)\right)(\eta\setminus x) \\ &\quad + \int_{\Gamma_0} d\rho_\mu(\eta) \sum_{x\in\eta} \int_{\mathbb{R}^d} dy \, \left(|C_{x,y}| \star 1\!\!1_{\bigsqcup_{n=0}^N \Gamma_\Lambda^{(n)}}(\cdot \cup x)\right)(\eta\setminus x)\end{aligned}$$

$$\leq \int_\Lambda dy \int_\Gamma d\mu(\gamma)\,(1+|\gamma_\Lambda|)^{N-1} \sum_{x\in\gamma} (K|C_{x,y}|)\,(\gamma\backslash x)$$
$$+ \int_{\mathbb{R}^d} dy \int_\Gamma d\mu(\gamma)\,|\gamma_\Lambda|^{N-1} \sum_{x\in\gamma_\Lambda} (K|C_{x,y}|)\,(\gamma\backslash x),$$

which, by (4.4), complete the proof. □

Remark 4.2. *Similarly to the proof of Proposition 3.1, the proof of Proposition 4.1 shows that (4.4) is the weakest possible integrability condition to state Proposition 4.1 for generic measures* $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ *and generic rates* c *of the type (4.3). Its proof also shows that for each measure* $\rho \in \mathcal{M}_{\mathrm{lf}}(\Gamma_0)$ *such that* $C_{x,y} \in L^1(\Gamma_0,\rho)$ *and such that for all* $n \in \mathbb{N}_0$ *and all* $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$

$$\int_{\Gamma_0} d\rho(\eta) \sum_{x\in\eta} \int_{\mathbb{R}^d} dy\, \left(|C_{x,y}| \star 1\!\!1_{\Gamma^{(n)}_\Lambda}\right)(\eta\backslash x)\,(1\!\!1_\Lambda(x) + 1\!\!1_\Lambda(y)) < \infty,$$

we have $\hat{L}\,(B_{bs}(\Gamma_0)) \subset L^1(\Gamma_0,\rho)$*. This integrability condition for measures* $\rho \in \mathcal{M}_{\mathrm{lf}}(\Gamma_0)$ *is the weakest possible one to yield this inclusion.*

Remark 4.3. *Concerning Proposition 4.1 we note that if each* $C_{x,y}$ *is of the type* $C_{x,y} = e_\lambda(c_{x,y})$*, then*

$$(\hat{L}G)(\eta) = \sum_{x\in\eta} \sum_{\xi\subset\eta\backslash x} \int_{\mathbb{R}^d} dy\,(G(\xi\cup y) - G(\xi\cup x)) e_\lambda(c_{x,y}+1,\xi) e_\lambda(c_{x,y},(\eta\setminus x)\setminus\xi),$$

cf. equality (2.12).

Remark 4.4. *For rates* $C_{x,y}$ *such that* $|C_{x,y}| \leq e_\lambda(c_{x,y})$ *for some* $0 \leq c_{x,y} \in L^1(\mathbb{R}^d, dx)$*, and for measures* $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ *that are locally absolutely continuous with respect to* π *and the correlation function* k_μ *fulfills the Ruelle bound for some constant* $C > 0$*, one may replace (4.4) by the stronger integrability condition*

$$\int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\, \exp(2C\|c_{x,y}\|_{L^1(\mathbb{R}^d,dx)})\,(1\!\!1_\Lambda(x) + 1\!\!1_\Lambda(y)) < \infty, \quad \forall\, \Lambda \in \mathcal{B}_c(\mathbb{R}^d).$$

Similarly to the proof of Corollary 3.5, successive applications of Lemmata 2.3 and 2.4 lead to the next result.

Proposition 4.5. *Let* $k : \Gamma_0 \to \mathbb{R}^+_0$ *be such that*

$$\int_{\Gamma^{(n)}_\Lambda} d\lambda(\eta)\, k(\eta) < \infty \quad \textit{for all } n \in \mathbb{N}_0 \textit{ and all } \Lambda \in \mathcal{B}_c(\mathbb{R}^d).$$

If $C_{x,y} \in L^1(\Gamma_0, k\lambda)$ *and for all* $n \in \mathbb{N}_0$ *and all* $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ *we have*

$$\int_{\Gamma_0} d\lambda(\eta)\, k(\eta) \sum_{x\in\eta} \int_{\mathbb{R}^d} dy\, \left(|C_{x,y}| \star 1\!\!1_{\Gamma^{(n)}_\Lambda}\right)(\eta\backslash x)\,(1\!\!1_\Lambda(x) + 1\!\!1_\Lambda(y)) < \infty,$$

then the action of the operator $\hat{L}^*$ on k is given by

$$(\hat{L}^*k)(\eta) = \sum_{y\in\eta}\int_{\mathbb{R}^d} dx \int_{\Gamma_0} d\lambda(\xi)\, k(\xi\cup(\eta\backslash y)\cup x) \sum_{\zeta\subset\eta\backslash y} C_{x,y}(\xi\cup\zeta)$$
$$- \int_{\Gamma_0} d\lambda(\xi)\, k(\xi\cup\eta)\sum_{x\in\eta}\sum_{\zeta\subset\eta\backslash x}\int_{\mathbb{R}^d} dy\, C_{x,y}(\xi\cup\zeta),$$

for λ-almost all $\eta\in\Gamma_0$.

Remark 4.6. *Under the conditions of Proposition 4.5, if each $C_{x,y}$ is of the type $C_{x,y} = e_\lambda(c_{x,y})$, then*

$$(\hat{L}^*k)(\eta) = \sum_{y\in\eta}\int_{\mathbb{R}^d} dx\, e_\lambda(c_{x,y}+1,\eta\backslash y) \int_{\Gamma_0} d\lambda(\xi)\, k(\xi\cup(\eta\backslash y)\cup x) e_\lambda(c_{x,y},\xi)$$
$$- \int_{\Gamma_0} d\lambda(\xi)\, k(\xi\cup\eta)\sum_{x\in\eta}\int_{\mathbb{R}^d} dy\, e_\lambda(c_{x,y}+1,\eta\backslash x) e_\lambda(c_{x,y},\xi).$$

Proposition 4.7. *Let $k:\Gamma_0\to\mathbb{R}_0^+$ be such that $e_\lambda(\theta)\in L^1_{\mathbb{C}}(\Gamma_0,k\lambda)$ for all $\theta\in L^1_{\mathbb{C}}(\mathbb{R}^d,dx)$, and the functional*

$$B(\theta) := \int_{\Gamma_0} d\lambda(\eta)\, e_\lambda(\theta,\eta)k(\eta)$$

is entire on the space $L^1_{\mathbb{C}}(\mathbb{R}^d,dx)$. If $C_{x,y}\in L^1(\Gamma_0,k\lambda)$ and $\hat{L}e_\lambda(\theta)\in L^1_{\mathbb{C}}(\Gamma_0,k\lambda)$ for all $\theta\in L^1_{\mathbb{C}}(\mathbb{R}^d,dx)$, then for all $\theta\in L^1_{\mathbb{C}}(\mathbb{R}^d,dx)$ we have

$$(\tilde{L}B)(\theta) = \int_{\Gamma_0} d\lambda(\eta) e_\lambda(\theta+1,\eta)\int_{\mathbb{R}^d} dx\, (D^{|\eta|+1}B)(\theta,\eta\cup x)$$
$$\times \int_{\mathbb{R}^d} dy\, (\theta(y)-\theta(x)) C_{x,y}(\eta).$$

Proof. This proof follows similarly to the proof of Proposition 3.7. In this case we obtain

$$\begin{aligned}
&(\hat{L}e_\lambda(\theta))(\eta)\\
&= \sum_{x\in\eta}\int_{\mathbb{R}^d} dy\, (\theta(y)-\theta(x))(C_{x,y}\star e_\lambda(\theta))(\eta\backslash x)\\
&= \sum_{x\in\eta}\int_{\mathbb{R}^d} dy\, (\theta(y)-\theta(x)) \sum_{\xi\subset\eta\backslash x} C_{x,y}(\xi) e_\lambda(\theta+1,\xi)\, e_\lambda(\theta,(\eta\setminus x)\setminus\xi),
\end{aligned}$$

where we have used the expression (2.12) concerning the $\star$-convolution. Arguments similar to those used in the proof of Proposition 3.7 lead then to

$$\begin{aligned}
&\int_{\Gamma_0} d\lambda(\eta)\, k(\eta)\, (\hat{L}e_\lambda(\theta))(\eta)\\
&= \int_{\mathbb{R}^d} dx \int_{\Gamma_0} d\lambda(\eta)\, (D^{|\eta\cup x|}B)(\theta,\eta\cup x) e_\lambda(\theta+1,\eta)\int_{\mathbb{R}^d} dy\, (\theta(y)-\theta(x)) C_{x,y}(\eta). \quad\square
\end{aligned}$$

Remark 4.8. *According to Remark 3.8, for functions $k : \Gamma_0 \to \mathbb{R}_0^+$ such that $k \leq e_\lambda(C)$ for some constant $C > 0$, one may state Proposition 4.7 just under the assumptions $C_{x,y} \in L^1(\Gamma_0, k\lambda)$ and $\hat{L}e_\lambda(\theta) \in L^1_{\mathbb{C}}(\Gamma_0, k\lambda)$ for all $\theta \in L^1_{\mathbb{C}}(\mathbb{R}^d, dx)$.*

Remark 4.9. *As before, in applications, the concrete explicit form of the rate $C_{x,y}$ allows a reformulation of Proposition 4.7, in general under much weaker analytical assumptions. For instance, if $C_{x,y} = e_\lambda(c_y)$ for some function c_y which is independent of x, then the expression for $\tilde{L}B$ given in Proposition 4.7 reduces to*

$$(\tilde{L}B)(\theta) = \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\, (\theta(y) - \theta(x)) \frac{\delta B(\theta(c_y + 1) + c_y)}{\delta(\theta(c_y + 1) + c_y)(x)}.$$

In contrast to the general formula, which depends of all variational derivatives of B at θ, this closed formula only depends on the first variational derivative of B on a shifted point. Further examples are presented in Subsection 4.2. Although in all such examples Proposition 4.7 may clearly be stated under much weaker analytical assumptions, the assumptions in Proposition 4.7 are sufficient to state a general result.

4.2 Particular models

Special hopping particles models will be presented and discussed within Subsection 4.1 framework. By analogy, such examples are a continuous version of models already known for lattice systems.

4.2.1 Kawasaki dynamics

In such a dynamics particles hop over the space $\mathbb{R}^d$ according to a rate which depends on the interaction between particles. This means that given a pair potential $\phi : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$, the rate c is of the form

$$\begin{aligned} c(x, y, \gamma) &= c_s(x, y, \gamma) = a(x - y)e^{sE(x,\gamma\setminus x)-(1-s)E(y,\gamma)} \\ &= K\Big(a(x - y)e^{(s-1)\phi(x-y)}e_\lambda(e^{s\phi(x-\cdot)-(1-s)\phi(y-\cdot)} - 1)\Big)(\gamma\setminus x) \end{aligned} \quad (4.5)$$

for some $s \in [0, 1]$. Here $a : \mathbb{R}^d \to \mathbb{R}_0^+$ and E is a relative energy defined as in (3.18).

For $a \in L^1(\mathbb{R}^d, dx)$ and for ϕ bounded from below and fulfilling the integrability condition (3.17), the condition (4.4) is always fulfilled, for instance, by any Gibbs measure $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ corresponding to ϕ for which the correlation function fulfills the Ruelle bound. We recall that a probability measure μ on $(\Gamma, \mathcal{B}(\Gamma))$ is called a Gibbs or an equilibrium measure if it fulfills the integral equation

$$\int_\Gamma d\mu(\gamma) \sum_{x\in\gamma} H(x, \gamma\setminus x) = \int_\Gamma d\mu(\gamma) \int_{\mathbb{R}^d} dx\, H(x, \gamma)e^{-E(x,\gamma)} \quad (4.6)$$

for all positive measurable functions $H : \mathbb{R}^d \times \Gamma \to \mathbb{R}$ ([31, Theorem 2], see also [11, Theorem 3.12], [23, Appendix A.1]). Correlation measures corresponding to such a class of measures are always absolutely continuous with respect to the Lebesgue–Poisson measure λ. For Gibbs measures described as before, the integrability condition (4.4) follows as a consequence of (4.6), applying the assumptions on ϕ and the Ruelle boundedness. For such Gibbs measures μ and for a being, in addition, an even function, it is shown in [20] the existence of an equilibrium Kawasaki dynamics, i.e., a Markov process on Γ which generator is given by (4.1) for c defined as in (4.5). Such a process has μ as an invariant measure.

The general results obtained in Subsection 4.1 yield for the Kawasaki dynamics the expressions

$$\begin{aligned}(\hat{L}G)(\eta) = \sum_{x\in\eta}\sum_{\xi\subset\eta\setminus x} e^{sE(x,\xi)}\int_{\mathbb{R}^d} dy\, a(x-y)e^{(s-1)E(y,\xi\cup x)} \\ \times e_\lambda(e^{s\phi(x-\cdot)-(1-s)\phi(y-\cdot)}-1,(\eta\setminus x)\setminus\xi)(G(\xi\cup y)-G(\xi\cup x)),\end{aligned} \tag{4.7}$$

and

$$\begin{aligned}(\hat{L}^*k)(\eta) = \sum_{y\in\eta}\int_{\mathbb{R}^d} dx\, a(x-y)e^{sE(x,\eta\setminus y)-(1-s)E(y,\eta\setminus y\cup x)} \\ \times \int_{\Gamma_0} d\lambda(\xi)\, k(\xi\cup(\eta\setminus y)\cup x)e_\lambda(e^{s\phi(x-\cdot)-(1-s)\phi(y-\cdot)}-1,\xi) \\ -\int_{\Gamma_0} d\lambda(\xi)\, k(\xi\cup\eta)\sum_{x\in\eta}\int_{\mathbb{R}^d} dy\, a(x-y) \\ \times e^{sE(x,\eta\setminus x)-(1-s)E(y,\eta)}e_\lambda(e^{s\phi(x-\cdot)-(1-s)\phi(y-\cdot)}-1,\xi),\end{aligned} \tag{4.8}$$

where we have taken into account Remark 4.6. In terms of Bogoliubov functionals, Proposition 4.7 leads to

$$\begin{aligned}(\tilde{L}B)(\theta) = \int_{\Gamma_0} d\lambda(\eta)e_\lambda(\theta+1,\eta)\int_{\mathbb{R}^d} dx(D^{|\eta|+1}B)(\theta,\eta\cup x)\int_{\mathbb{R}^d} dya(x-y) \\ \times e^{(s-1)\phi(x-y)}(\theta(y)-\theta(x))e_\lambda(e^{s\phi(x-\cdot)-(1-s)\phi(y-\cdot)}-1,\eta).\end{aligned} \tag{4.9}$$

In particular, for $s = 0$, one obtains

$$\begin{aligned}(\tilde{L}B)(\theta) = \int_{\mathbb{R}^d} dx\int_{\mathbb{R}^d} dy\, a(x-y)e^{-\phi(x-y)}(\theta(y)-\theta(x)) \\ \times \frac{\delta B((1+\theta)(e^{-\phi(y-\cdot)}-1)+\theta)}{\delta((1+\theta)(e^{-\phi(y-\cdot)}-1)+\theta)(x)},\end{aligned} \tag{4.10}$$

cf. Remark 4.9.

Remark 4.10. *In the case $s = 0$, in a recent work [5] the authors have shown that in the high-temperature-low activity regime the scaling limit (of a Kac type)* *of an equilibrium Kawasaki dynamics yields in the limit an equilibrium Glauber dynamics. More precisely, given an even function $0 \leq a \in L^1(\mathbb{R}^d, dx)$ and a stable pair potential ϕ, i.e.,*

$$\exists B_\phi \geq 0 : \sum_{\{x,y\}\subset\eta} \phi(x-y) \geq -B_\phi|\eta|, \ \forall\, \eta \in \Gamma_0,$$

such that

$$\int_{\mathbb{R}^d} dx \left|e^{-\phi(x)} - 1\right| < \left(2e^{1+2B_\phi}\right)^{-1}$$

(high temperature-high temperature regime), the authors have considered an equilibrium Kawasaki dynamics which generator L_ε is given by (4.1) for c defined as in (4.5) for $s = 0$ and a replaced by the function $\varepsilon^d a(\varepsilon\cdot)$. We observe that such a dynamics exists due to [20]. Then it has been shown that the generators L_ε converge to

$$-\alpha \sum_{x\in\gamma} (F(\gamma \setminus x) - F(\gamma)) - \alpha \int_{\mathbb{R}^d} dx\, e^{-E(x,\gamma)} \left(F(\gamma \cup x) - F(\gamma)\right),$$

which is the generator of an equilibrium Glauber dynamics. Here the constant α is defined by $\alpha := k_\mu^{(1)} \int_{\mathbb{R}^d} dx\, a(x)$ for $k_\mu^{(1)} := k_\mu\!\upharpoonright_{\Gamma^{(1)}}$ being the first correlation function of the initial distribution μ.

4.2.2 Free hopping particles

In the free Kawasaki dynamics case one has $\phi \equiv 0$, meaning that particles hop freely over the space $\mathbb{R}^d$. Therefore, all previous considerations hold for this special case. In particular, for every even function $0 \leq a \in L^1(\mathbb{R}^d, dx)$ the construction done in [20] yields the existence of an equilibrium free Kawasaki dynamics. Actually, in this case the generator L is a second quantization operator which leads to a simpler situation. The existence result extends to the non-equilibrium case [21] for a wide class of initial configurations also identified in [21]. This allows the study done in [15] of the large time asymptotic behaviours and hydrodynamical limits.

4.2.3 Polynomial rates

In applications one may also consider rates of polynomial type, i.e.,

$$c(x,y,\gamma) = \sum_{\{x_1,...,x_p\}\subset\gamma\setminus x} c_{x,y}^{(p)}(x_1,...,x_p) = (K\tilde{c}_{x,y}^{(p)})(\gamma \setminus x)$$

for some symmetric function $0 \leq c_{x,y}^{(p)} \in L^1((\mathbb{R}^d)^p, dx_1...dx_p)$, $x \in \mathbb{R}^d$, $p \in \mathbb{N}$, where

$$(\tilde{c}_{x,y}^{(p)})(\eta) := \begin{cases} c_{x,y}^{(p)}(x_1,...,x_p), & \text{if } \eta = \{x_1,...,x_p\} \in \Gamma^{(p)} \\ 0, & \text{otherwise} \end{cases}.$$

A straightforward application of the general results obtained in Subsection 4.1 yields for this case the expressions

$$(\hat{L}G)(\eta) = \sum_{x\in\eta} \int_{\mathbb{R}^d} dy \, \left(\tilde{c}^{(p)}_{x,y} \star (G(\cdot \cup y) - G(\cdot \cup x)) \right) (\eta \setminus x), \tag{4.11}$$

and

$$\begin{aligned}
&(\hat{L}^* k)(\eta) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (4.12)\\
&= \sum_{y\in\eta} \sum_{i=0}^{p} \frac{1}{i!} \int_{\Gamma^{(i)}} dm^{(i)}(\xi) \int_{\mathbb{R}^d} dx \, k(\xi \cup (\eta\backslash y) \cup x) \sum_{\substack{\zeta\subset\eta\backslash y\\ |\zeta|=p-i}} \tilde{c}^{(p)}_{x,y}(\xi \cup \zeta)\\
&\quad - \sum_{i=0}^{p} \frac{1}{i!} \int_{\Gamma^{(i)}} dm^{(i)}(\xi) \, k(\xi \cup \eta) \sum_{x\in\eta} \sum_{\substack{\zeta\subset\eta\backslash x\\ |\zeta|=p-i}} \int_{\mathbb{R}^d} dy \, \tilde{c}^{(p)}_{x,y}(\xi \cup \zeta), \qquad (4.13)
\end{aligned}$$

where $m^{(i)}$ is the measure on $\Gamma^{(i)}$ defined in Example 2.1 (Subsection 2.1). In terms of Bogoliubov functionals, the statement of Proposition 4.7 leads now to

$$\begin{aligned}
(\tilde{L}B)(\theta) = \frac{1}{p!} \int_{\Gamma^{(p)}} dm^{(p)}(\eta) \, e_\lambda(\theta + 1, \eta) \int_{\mathbb{R}^d} dx \, (D^{p+1}B)(\theta, \eta \cup x)\\
\times \int_{\mathbb{R}^d} dy \, \tilde{c}^{(p)}_{x,y}(\eta)(\theta(y) - \theta(x)). \qquad (4.14)
\end{aligned}$$

As a particular realization, one may consider

$$c(x, y, \gamma) = b(x, y) + \sum_{x_1\in\gamma\backslash x} c^{(1)}_{x,y}(x_1) = K \left(b(x, y) e_\lambda(0) + \tilde{c}^{(1)}_{x,y} \right) (\gamma \setminus x),$$

where b is a function independent of γ. From the previous considerations we obtain

$$\begin{aligned}
(\hat{L}G)(\eta) = &\sum_{x\in\eta} \int_{\mathbb{R}^d} dy \, b(x, y)(G((\eta \setminus x) \cup y) - G(\eta)) \qquad (4.15)\\
&+ \sum_{x\in\eta} \sum_{x_1\in\eta\backslash x} \int_{\mathbb{R}^d} dy \, c^{(1)}_{x,y}(x_1)(G((\eta \setminus \{x, x_1\}) \cup y) - G(\eta \setminus x_1))\\
&+ \sum_{x\in\eta} \int_{\mathbb{R}^d} dy \, (G((\eta \setminus x) \cup y) - G(\eta)) \sum_{x_1\in\eta\backslash x} c^{(1)}_{x,y}(x_1),
\end{aligned}$$

and

$$\begin{aligned}
(\hat{L}^* k)(\eta) = &\sum_{y\in\eta} \int_{\mathbb{R}^d} dx_1 \int_{\mathbb{R}^d} dx \, k(x_1 \cup (\eta \setminus y) \cup x) c^{(1)}_{x,y}(x_1) \qquad (4.16)\\
&- \int_{\mathbb{R}^d} dx_1 \, k(\eta \cup x_1) \sum_{x\in\eta} \int_{\mathbb{R}^d} dy \, c^{(1)}_{x,y}(x_1)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{y\in\eta}\int_{\mathbb{R}^d} dx\, k((\eta\setminus y)\cup x)\Big(b(x,y)+\sum_{x_1\in\eta\setminus y} c^{(1)}_{x,y}(x_1)\Big)\\
&-k(\eta)\sum_{x\in\eta}\int_{\mathbb{R}^d} dy\,\Big(b(x,y)+\sum_{x_1\in\eta\setminus x} c^{(1)}_{x,y}(x_1)\Big).
\end{aligned}
$$

In addition,

$$
\begin{aligned}
(\tilde{L}B)(\theta) = &\int_{\mathbb{R}^d} dx\,\frac{\delta B(\theta)}{\delta\theta(x)}\int_{\mathbb{R}^d} dy\, b(x,y)(\theta(y)-\theta(x)) \\
&+\int_{\mathbb{R}^d} dx_1\,(\theta(x_1)+1)\int_{\mathbb{R}^d} dx\,\frac{\delta^2 B(\theta)}{\delta\theta(x_1)\delta\theta(x)}\\
&\times\int_{\mathbb{R}^d} dy\, c^{(1)}_{x,y}(x_1)(\theta(y)-\theta(x)).
\end{aligned}
\tag{4.17}
$$

4.3 Other conservative jump processes

Before we have analyzed individual hops of particles. We may also analyze hops of groups of $n\geq 2$ particles. Dynamically this means that at each random moment of time a group of n particles randomly hops over the space $\mathbb{R}^d$ according to a rate which depends on the configuration of the whole system at that time. In terms of generators this behaviour is described by

$$
\begin{aligned}
(LF)(\gamma) = &\sum_{\{x_1,\ldots,x_n\}\subset\gamma}\int_{\mathbb{R}^d} dy_1\ldots\int_{\mathbb{R}^d} dy_n\, c(\{x_1,\ldots,x_n\},\{y_1,\ldots,y_n\},\gamma)\\
&\times\left(F(\gamma\setminus\{x_1,\ldots,x_n\}\cup\{y_1,\ldots,y_n\})-F(\gamma)\right),
\end{aligned}
\tag{4.18}
$$

where $c(\{x_1,\ldots,x_n\},\{y_1,\ldots,y_n\},\gamma)\geq 0$ indicates the rate at which a group of n particles located at $x_1,\ldots,x_n$ ($x_i\neq x_j$, $i\neq j$) in a configuration γ hops to the sites $y_1,\ldots,y_n$ ($y_i\neq y_j$, $i\neq j$). As before, we consider the case

$$
c(\{x_1,...,x_n\},\{y_1,...,y_n\},\gamma) = (KC_{\{x_i\},\{y_i\}})(\gamma\setminus\{x_1,...,x_n\})\geq 0,
$$

where $C_{\{x_i\},\{y_i\}} := C_{\{x_1,\ldots,x_n\},\{y_1,\ldots,y_n\}}$. Similar calculations lead then to the expressions

$$
\begin{aligned}
(\hat{L}G)(\eta) = &\,1\!\!1_{\bigsqcup_{k=n}^{\infty}\Gamma^{(k)}}(\eta)\sum_{\{x_1,\ldots,x_n\}\subset\eta}\int_{\mathbb{R}^d} dy_1...\int_{\mathbb{R}^d} dy_n\\
&\times\sum_{\xi\subset\{y_1,\ldots,y_n\}}\left(C_{\{x_i\},\{y_i\}}\star G(\cdot\cup\xi)\right)(\eta\setminus\{x_1,...,x_n\})\\
&-1\!\!1_{\bigsqcup_{k=n}^{\infty}\Gamma^{(k)}}(\eta)\sum_{\{x_1,\ldots,x_n\}\subset\eta}\int_{\mathbb{R}^d} dy_1...\int_{\mathbb{R}^d} dy_n\\
&\times\sum_{\xi\subset\{x_1,\ldots,x_n\}}\left(C_{\{x_i\},\{y_i\}}\star G(\cdot\cup\xi)\right)(\eta\setminus\{x_1,...,x_n\}),
\end{aligned}
\tag{4.19}
$$

and

$$(\hat{L}^* k)(\eta) = \int_{\Gamma_0} d\lambda(\zeta) \int_{\Gamma^{(n)}} dm^{(n)}(\xi) \sum_{\eta_1 \subset \eta} k(\zeta \cup (\eta \setminus \eta_1) \cup \xi) \tag{4.20}$$

$$\times \int_{\Gamma_0} d\lambda(\tau) \mathbb{1}_{\Gamma^{(n)}}(\eta_1 \cup \tau) \sum_{\eta_2 \subset \eta \setminus \eta_1} C_{\xi, \eta_1 \cup \tau}(\zeta \cup \eta_2)$$

$$- \int_{\Gamma_0} d\lambda(\zeta) \int_{\Gamma_0} d\lambda(\xi)\, k(\zeta \cup \eta \cup \xi) \sum_{\eta_1 \subset \eta} \mathbb{1}_{\Gamma^{(n)}}(\eta_1 \cup \xi)$$

$$\times \int_{\Gamma^{(n)}} dm^{(n)}(\tau) \sum_{\eta_2 \subset \eta \setminus \eta_1} C_{\eta_1 \cup \xi, \tau}(\zeta \cup \eta_2).$$

Moreover

$$(\tilde{L}B)(\theta) = \frac{1}{n!} \int_{\Gamma_0} d\lambda(\eta)\, e_\lambda(\theta + 1, \eta) \int_{\Gamma^{(n)}} dm^{(n)}(\xi)\, (D^{|\eta|+n} B)(\theta, \eta \cup \xi)$$

$$\times \int_{\Gamma^{(n)}} dm^{(n)}(\zeta)\, C_{\xi,\zeta}(\eta) \big(e_\lambda(\theta + 1, \zeta) - e_\lambda(\theta + 1, \xi)\big). \tag{4.21}$$

Remark 4.11. *If the rate c does not depend on the configuration,*

$$c(\{x_1, \dots, x_n\}, \{y_1, \dots, y_n\}, \gamma) = c(\{x_1, \dots, x_n\}, \{y_1, \dots, y_n\}),$$

one can show that each Poisson measure π_z, $z > 0$, is invariant. If, in addition, the rate $c(\{x_1, \dots, x_n\}, \{y_1, \dots, y_n\})$ is symmetric in $x_1, \dots, x_n, y_1, \dots, y_n$, then these Poisson measures are symmetrizing.

In particular, the conditions of the previous remark hold for $n = 2$ and

$$C_{\{x_1,x_2\},\{y_1,y_2\}} = p(x_1 - y_1)p(x_1 - y_2)p(x_2 - y_1)p(x_2 - y_2)e_\lambda(0),$$

where $p : \mathbb{R}^d \to \mathbb{R}_0^+$ is either an even or an odd function. In this case, denoting by $c(x_1, x_2, y_1, y_2) = p(x_1 - y_1)p(x_1 - y_2)p(x_2 - y_1)p(x_2 - y_2)$, one obtains the following explicit formulas

$$(\hat{L}G)(\eta) \tag{4.22}$$

$$= \mathbb{1}_{|\eta| \ge 2} \sum_{\{x,y\} \subset \eta} \int_{\mathbb{R}^d} dx' \int_{\mathbb{R}^d} dy'\, c(x, y, x', y') \left[G(\eta \cup \{x', y'\} \setminus \{x, y\}) - G(\eta)\right]$$

$$+ 2 \cdot \mathbb{1}_{|\eta| \ge 2} \sum_{\{x,y\} \subset \eta} \int_{\mathbb{R}^d} dx'\, G(\eta \cup x' \setminus \{x, y\}) \int_{\mathbb{R}^d} dy'\, c(x, y, x', y')$$

$$- \mathbb{1}_{|\eta| \ge 2} \sum_{\{x,y\} \subset \eta} (G(\eta \setminus x) + G(\eta \setminus y)) \int_{\mathbb{R}^d} dx' \int_{\mathbb{R}^d} dy'\, c(x, y, x', y'),$$

and

$$(\hat{L}^* k)(\eta) \tag{4.23}$$
$$= 1\!\!1_{|\eta|\geq 2} \sum_{\{x,y\}\subset\eta} \int_{\mathbb{R}^d} dx' \int_{\mathbb{R}^d} dy' c\,(x,y,x',y')\,[k\,(\eta\cup\{x',y'\}\setminus\{x,y\}) - k\,(\eta)]$$
$$+ \sum_{x\in\eta} \int_{\mathbb{R}^d} dx' \int_{\mathbb{R}^d} dy' \int_{\mathbb{R}^d} dy\, c\,(x,y,x',y')\,[k(\eta\cup\{x',y'\}\setminus x) - k(\eta\cup y)]\,.$$

Additionally,

$$(\tilde{L}B)(\theta) = \frac{1}{2} \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\, \frac{\delta^2 B(\theta)}{\delta\theta(x)\delta\theta(y)} \int_{\mathbb{R}^d} dx' \int_{\mathbb{R}^d} dy'\, c(x,y,x',y') \times [(\theta(x')+1)(\theta(y')+1) - (\theta(x)+1)(\theta(y)+1)]\,. \tag{4.24}$$

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