EUROPEAN CONFERENCE ON ITERATION THEORY

Yalta, Crimea, Ukraine

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European Conference on Iteration Theory (ECIT 2008) is organized by the *Institute of Mathematics, National Academy of Sciences of Ukraine* and *National Taurida University.*

The conference follows the sequence of European Conferences on Iteration Theory, which started in Toulouse (France 1973) and continued in Graz (Austria 1977), Marburg (Germany 1980), Toulouse (France 1982), Lochau (Austria 1984), Caldes de Malavella (Spain 1987), Batschuns (Austria 1989), Lisbon (Portugal 1991), Batschuns (Austria 1992), Opava (Czech Republic 1994), Urbino (Italy 1996), Muszyna (Poland 1998), La Manga del Mar Menor (Spain 2000), Evora (Portugal 2002), Batschuns (Austria 2004), Gargnano (Italy 2006).

ECIT 2008 takes place in *Yalta*, which is a city in Crimea, southern Ukraine, situated on the Black Sea coast.

Traditionally, *Dynamical Systems* and *Functional Equations* are the main topics of the conference.

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BIFURCATION CURVE STRUCTURE OF A FAMILY OF LINEAR DISCONTINUOUS MAPS

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We consider a two-sector growth model describing the role of distribution of factor incomes when the capital is owned by heterogeneous consumers.

The research on this topic has developed during the Fifties and the Sixties, engaging many important economic scientists ([1–3], to cite a few). In these pioneering papers the economic systems are studied only on the steady state and no attention is paid to the dynamic behavior of the models. After these studies, the problem has been neglected for many years and only recently has been reconsidered in [4], where even a local stability analysis of the steady states is proposed.

In particular, we embed the problem in a overlapping generations model (OLG), consider a Leontiev technology and assume a constant capital pay out ratio of profit to capital owners. So doing, the model results in a two-dimensional linear discontinuous map T, admitting an invariant submanifold $\mathcal{M}$.

As a first step in the study of such a model, in this paper we shall analyze the one-dimensional map f, restriction of T on $\mathcal{M}$. Having show that f is topologically conjugated to a linear discontinuous map depending on 3 parameters, we shall describe the different bifurcation curves of the parameter space.

Particular attention will be devoted to the border collision bifurcations, showing the corresponding box-in-file structure, and to the parameter space regions where two attractors coexist.

[1] N. Kaldor, *Alternative theories of distribution.* Review of Economic Studies **23** (1955 - 1956), 83-100.

[2] L.L. Pasinetti, *Rate of profit and income distribution in relation to the rate of economic growth.* Review of Economic Studies **29** (1962), 267-279.

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SOME PROBLEMS ON TOPOLOGICAL AND COMBINATORIAL DYNAMICS ON DENDRITES

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The interest in considering dynamical systems on dendrites is double. By one hand, they appear frequently on Julia sets associated to transforms in complex dynamics. On the other hand, dendrites are examples of Peano continua with complicate topological structure.

In our talk we will present new results in two senses: 1) a complete characterization of minimal sets on dendrites, dendroids and local dendrites, 2) in what extent is true the periodic-recurrent property, that is, if for the family of dynamical systems on dendrites is $\overline{P(f)} = \overline{R(f)}$ where $P(f)$ and $R(f)$ are respectively the set of periodic and recurrent points.

Finally some remarks will be given on the periodic structure (of Sharkovsky type) associated to those dynamical systems on dendrites.).

TOPOLOGICAL INVARIANTS FOR THE LOZI MAPS

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In [1], R.Lozi studied the interesting dynamic behavior associated with the two parameters family of piecewise affine homeomorphisms of the plane. This family called Lozi mappings $L_{a,b} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is defined by

$$L_{a,b}\left(x, y\right) = \left(1 - a\left|x\right| + y, bx\right), \ a, b \in \mathbb{R}, \ b \neq 0.$$

R.Lozi found a strange attractor for the parameters values $a = 1.7$ and $b = 0.5$. A few years later, M.Misiurewicz [2] characterized the set of parameters for which Lozi mappings do have a strange attractor and proved the existence of a non-empty compact invariant set $\mathcal{F}$ for which (i) there exists a neighborhood G of $\mathcal{F}$ such that the distance between $\mathcal{F}$ and $L_{a,b}(x, y)$ tends to zero when n goes to $+\infty$ for every $(x, y) \in G$, (ii) the unstable manifold of the hyperbolic fixed point in the first quadrant is dense in $\mathcal{F}$, and (iii) $L_{a,b}$ on $\mathcal{F}$ is topologically mixing. From the kneading theory for Lozi mappings, presented by Y.Ishii [3], we study its topological invariants.

[1] R. Lozi, *Un attracteur étrange du type attracteur de Hénon.* J. Physique (Paris) **39** (Coll. C5) (1978), 9-10.

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ERGODIC PROPERTIES OF ERDÖS MEASURE FOR THE GOLDEN RATIO

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Let Y be the compact space $\{0,1\}^N$ with the usual topology, and σ is the shift on Y. The Fibonacci compact space X is the closed σ-invariant set

$$X = \{x \in Y : x_j x_{j+1} = 0, j = 1, 2, \ldots\}.$$

Define a map Φ from Y to X by

$$(\Phi(y))_j = x_j, \quad \sum_{j=1}^{\infty} y_j \varrho^{j+1} = \sum_{j=1}^{\infty} x_j \varrho^j,$$

$0 < \varrho < 1$, where ϱ^{-1} is the golden ratio.

Let m be the product measure on Y with the multipliers $(p, 1-p)$, where $0 < p < 1$. The Erdös measure μ_m on X is defined by

$$\mu_m(A) = m(\Phi^{-1}A).$$

The invariant Erdös measure μ is the unique σ-invariant measure in the equivalence class of μ_m. We obtain a convenient "regenerative representation" for μ. The entropy of μ is obtained with the help of this representation. We determine the Hausdorff dimension of Erdös measure with less than 10^{-15} accuracy (i.e., with machine precision) for $p = 0.05, 0.1, 0.2, 0.3, 0.4, 0.5$.

SCHRÖDER EQUATION AND COMMUTING FUNCTIONS

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The main purpose of this talk is to show that if $\mathcal{F}$ is a non–trivial continuous and disjoint iteration group or semigroup on the unit circle $\mathbf{S}^1$ and a continuous at least at one point function $G : \mathbf{S}^1 \longrightarrow \mathbf{S}^1$ commutes with a suitable element of $\mathcal{F}$, then $G \in \mathcal{F}$. In order to do this we use a result concerning continuous at a point of the limit set of F solutions of the Schröder equation

$$\Phi(F(z)) = e^{2\pi i\alpha(F)}\Phi(z), \qquad z \in \mathbf{S}^1,$$

where $F : \mathbf{S}^1 \longrightarrow \mathbf{S}^1$ is a homeomorphism with irrational rotation number $\alpha(F)$.

SYMBOLIC DYNAMICS FOR ITERATED SMOOTH FUNCTIONS

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We consider the discrete dynamical system (A, T), where A is a space of smooth real functions defined on some interval and $T : A \to A$ is an operator $T\phi := f \circ \phi$, where f is a polynomial function on the real line. Iteration of smooth maps appears naturally in the study of continuous difference equations and boundary value problems such as those in [1-5]. In our work we develop some techniques of symbolic dynamics for the discrete dynamical system (A, T). We analyze some concrete examples when f is one-parameter family of quadratic maps.

[1] A. N. Sharkovsky, Yu. L. Maistrenko and E. Yu. Romanenko, *Difference equations and their applications.* Kluwer Academic Publishers (1993).

[2] E. Yu. Romanenko and A. N. Sharkovsky, *From boundary value problems to difference equations: a method of investigation of chaotic vibrations.* Int. J. Bif. Chaos **9** (1999), 1285-1306.

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[5] S. Vinagre, R. Severino and J. Sousa Ramos, *Topological invariants in nonlinear boundary value problems.* Chaos, Solitons & Fractals **25** (2005), 65-78.

NONLINEARLY PERTURBED HEAT EQUATION

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We consider the linear heat equation with appropriate boundary conditions in order to model the temperature on a wire with adiabatic endpoints. We assume there is some physical perturbation which provokes a sudden change in the temperature of the wire. This perturbation is modelled by an iterated nonlinear map of the interval f. When the perturbation occurs periodically we have two clearly different behaviours: either the temperature stabilizes or evolves oscillating with increasing frequencies. We study the parameters in order to characterize these behaviours and its dependence on the topological invariants of f, with similar methods and ideas as those in [1-6].

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[4] A. N. Sharkovsky, *Ideal Turbulence.* Nonlinear Dynamics **44** (2006), 15-27.

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[6] S. Vinagre, R. Severino and J. Sousa Ramos, *Topological invariants in nonlinear boundary value problems.* Chaos, Solitons & Fractals **25** (2005), 65-78.

ASYMPTOTICS OF THE POINCARE FUNCTIONS

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In 1890 H. Poincare has studied the equation

$$f(\lambda z) = R(f(z)), \quad z \in \mathbf{C}, \tag{1}$$

where $R(z)$ is a rational function and $\lambda \in \mathbf{C}$. He proved that, if $R(0) = 0$, $R'(0) = \lambda$ and $|\lambda| > 1$, then there exists a meromorphic or entire solution of (1). After Poincare, solutions of (1) are called *the Poincare functions, satisfying the multiplication theorem.* Later on, Valiron elaborated the case, where $R(z) = P(z)$ is a polynomial, i.e.

$$f(\lambda z) = P(f(z)), \quad z \in \mathbf{C}, \tag{2}$$

and obtained conditions for the existence of an entire solution $f(z)$. Furthermore, he derived for $M(r) = \max_{|z|\le r} |f(z)|$ the following asymptotic formula:

$$\ln M(r) \sim r^{\rho} Q\left(\frac{\ln r}{\ln|\lambda|}\right), \quad r \to \infty. \tag{3}$$

Here $Q(z)$ is a 1-periodic function bounded between two positive constants, $\rho = \frac{\ln m}{\ln|\lambda|}$ and $m = \deg P(z)$.

An interesting example of the Poincare equations is the equation $f(5z) = 4f^2(z) - 3f(z)$, which appears in the theory of branching processes and in the description of Brownian motion on Sierpinski's gasket.

We develop the above results further. Namely, in addition to (3) we obtain asymptotics of entire solutions $f(z)$ on various rays $re^{i\vartheta}$ of the complex plane. It turns out that this heavily depends on the arithmetic nature of λ.

Further refinements are possible when $\lambda > 1$ is real and $P(z)$ is a polynomial with real positive coefficients.

This is joint work with Peter Grabner (Technical University of Graz) and Fritz Vogl (Technical University of Vienna).

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[2] G. Derfel, P. Grabner and F. Vogl, *Complex asymptotics of Poincare functions and properties of Julia sets* (to appear in Math. Proc. Camb. Phil. Soc.).

DYNAMICS OF INTERVALS IN ONE-DIMENSIONAL SYSTEMS

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Let $I = [0, 1]$ and $f \in C^0(I, I)$, i.e. f be a continuous map of the interval I into itself. The sequence of sets $f^n(J)$, $n = 0, 1, 2...$, is called the trajectory of the closed interval $J \subset I$ of the dynamical system generated by the map f. Each set $f^n(J)$, $n = 0, 1, 2...$, is an closed interval or a point, which we denote by $[a_n, b_n]$. The trajectory of the interval J is called convergent, if the sequences a_n and b_n are convergent under $n \to \infty$. The trajectory of J is called asymptotically periodic if there exits a natural number p such that the trajectory of the interval J under the map f^p is convergent. The smallest of such number p is called the period of this asymptotically periodic trajectory.

Theorem 1 [1-4]. The trajectory of the interval J is an asymptotically periodic if and only if J contains an asymptotically periodic point. If the interval J contains an asymptotically periodic point of the period p, then the period of the trajectory of the interval J is a divisor of number $2p$.

Theorem 2 $[3, 4]$. If 1) $f \in C^0(I, I)$ is a piecewise linear map and it has no intervals on which f is a constant, or 2) $f \in C^2(I, I)$ is non-flat at each critical point, then the asymptotically periodic points of the map f are everywhere dense in I, and, consequently, the trajectory of any interval is an asymptotically periodic.

[1] V. V. Fedorenko, *Topological limit of trajectories of intervals of the simplest one-dimensional dynamical systems.* Ukrainian Math. J. **54** (2002), 527-532.

[2] V. V. Fedorenko, *Topological limit of trajectories of intervals of one-dimensional dynamical systems.* Grazer Math. Ber. **346** (2004), 107-111.

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PILGERSCHRITT TRANSFORM IN THE GROUP $Aff(1,\mathbb{C})$

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The method of Pilgerschritt transform can be used to find one-parameter-subgroups of topological groups connecting the unit of the group with a given element. It may be described as the iteration of transforming a path joining the unit with the given element – and this process should converge.
However, convergence could be proved only for paths to elements close to the unit element.

In the group $Aff(1,\mathbb{C})$ elements can be written as $\begin{pmatrix} 1 & y \\ 0 & \mathrm{e}^a \end{pmatrix}$ with $a, y \in \mathbb{C}$. It was known that the method of Pilgerschritt transform converges for elements with a close to 0 (cf. Schloss Hofen, ECIT, 1984), and if a is a positive real number.

Now it could be shown that the region of convergence (with respect to a) contains a <u>cone</u> with nonempty interior, vertex at 0, and containing the positive real half line.

This is a joint work with Christian Winklmair (University of Innsbruck, Austria).

BIFURCATIONS ON THE POINCARÉ EQUATOR IN THE 2D PIECEWISE LINEAR CANONICAL FORM

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The object of the present work is to describe some bifurcations occurring in the 2D piecewise smooth continuous map in canonical form:

$$X' = \begin{cases} A_l X + B & if \quad x < 0 \\ A_r X + B & if \quad x > 0 \end{cases}$$

with

$$X = \begin{bmatrix} x \\ y \end{bmatrix} \quad , \quad A_l = \begin{bmatrix} \tau_l & 1 \\ -\delta_l & 0 \end{bmatrix} \quad , \quad A_r = \begin{bmatrix} \tau_r & 1 \\ -\delta_r & 0 \end{bmatrix} \quad , \quad B = \begin{bmatrix} \mu \\ 0 \end{bmatrix}$$

involving the Poincaré Equator (P.E.). This map has been investigated in several recent papers (see [1–4]). Some particular behaviors related with divergent trajectories are the so-called *dangerous Border Collision Bifurcation* (see [2]- [4]). We shall see that this is not a 'rare' phenomenon, several bifurcation curves in the parameter space involve cycles belonging to the P.E. and the related dynamic behaviors (that is, the dynamics occurring crossing such bifurcation curves) are very rich.

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[4] I. Sushko, and L. Gardini *Center Bifurcation for a Two-Dimensional Border-Collision Normal Form*. Int. J. Bif. Chaos, **18** (2008), 1029-1050.

ON ANALOGUE OF THE CENTER BIFURCATION AT INFINITY

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We consider a family of 2Dim piecewise smooth discontinuous maps F representing the so-called relative dynamics of a Hicksian business cycle model [1]. The map F is given by two maps F_1 and F_2 which are defined in the regions R_1 and R_2, respectively:

$$F:(x,y)\mapsto\begin{cases} F_1(x,y), & (x,y)\in R_1, \\ F_2(x,y), & (x,y)\in R_2, \end{cases}$$

where

$$F_1:\begin{pmatrix} x \\ y \end{pmatrix}\mapsto\begin{pmatrix} x/y+a(1-1/y) \\ c+a(1-1/y) \end{pmatrix},\quad R_1 = \{(x,y):\ x(a(y-1)+rx)\geq 0\};$$

$$F_2:\begin{pmatrix} x \\ y \end{pmatrix}\mapsto\begin{pmatrix} x(1-r)/y \\ c-rx/y \end{pmatrix},\quad R_2 = \{(x,y):\ x(a(y-1)+rx)<0\}.$$

Here a, c and r are real parameters such that $a>0$, $0<c<1$, $0<r<1$.

We explain the origin of the periodicity regions in the parameter space, namely, we show that at $a=1$ an analogue of the center bifurcation [2] occurs for a point at infinity. Depending on other parameters such a bifurcation can give birth to a couple of cycles of any period, an attracting and saddle, located at a hyperbola-like attracting invariant set.

The bifurcations associated with the periodicity regions are studied making use of the first-return map on a suitable segment of the phase plane. The bifurcation curves which are the boundaries the periodicity region are related to saddle-node and border-collision bifurcations of the related attracting cycle of the first return map.

[1] I. Sushko, L. Gardini, T. Puu, *Tongues of periodicity in a family of two-dimensional discontinuous maps of real Möbius type.* Chaos, Solitons & Fractals **21** (2004), 403-412.

[2] I. Sushko and L. Gardini, *Center Bifurcation for Two-Dimensional Border-Collision Normal Form.* Int. J. Bif. Chaos **18** (2008), 1029-1050.

SOME APPLICATIONS OF ORIENTED GRAPHS TO DYNAMICAL SYSTEMS AND TO MATRIX THEORY

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To every finite or countable infinite oriented graph G there corresponds a shift transformation T in the space $X = X(G)$ of its infinite (one-sided or two sided) paths. The dynamical system (X, T), called a topological Markov shift, is of interest in its own right and is finding increasing use both within dynamical systems theory and beyond.

The first application of this kind was apparently performed in [1] for constructing a homeomorphism of a compact metric space that has no measure with maximal entropy. The most resent application [2] is the proof of the fact that the Teichmüller geodesic flow on the moduli space of Abelian differentials introduced by H. Masur and W. Veech (see, for example, [3]) has a unique measure with maximal entropy.

Moreover, using the same method, one can give a new proof of the classical Perron-Frobenius theorem and its extension to infinite non-negative matrices. In this proof, unlike traditional ones, the relevant eigenvectors can be indicated in some sense explicitly.

[1] B. M. Gurevich, *Topological entropy of a denumerable Markov chain.* Soviet Math. Doklady **187** (1969), 715-718.

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JOINT SIGNAL-SYSTEM APPROACH FOR CHAOTIC GENERATORS SELECTION

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Spread-spectrum signal processing techniques are encountering a growing interest thanks to the increasing number of applications where they pair or outclass the performances of more conventional approaches. As for telecommunications, several chaotic cryptographic schemes have been proposed, where globally the plain text is mixed up with a chaotic signal to form the ciphertext which is to be transmitted though an insecure channel. The richness of non-linear dynamics of these systems is exploited, since it is generated by deterministic models, but also exhibits a wide spectrum similar to pseudo-random generators, which are conventionally used for encryption. Thus the purpose of our work is to evaluate the spectral properties of the chaotic generators using jointly nonlinear dynamics tools, such as the Lyapunov exponents, and signal processing tools, such as the Fourier transform.

Two main families of chaotic generators exist: continuous-time generators defined by ordinary differential equations, and discrete time ones governed by recurrence equations. One of the main drawbacks of the continuous-time systems are the implementation problems due to uncertainties in the real parameter values. Moreover, the intrinsically discrete nature of the telecommunication systems makes them being naturally modelled by discrete maps. Thus, we deal with 2D discrete maps defined by: $x_{k+1} = F(x_k), \quad x \in \mathbb{R}^2$.

Most of the papers devoted to chaotic encryption use only the statistical properties of the signal. Unlike these works, we investigate in addition the Lyapunov exponents of the chaotic generators, and their impact on the spectrum. Indeed, it is well known that chaotic systems are characterised by at least one positive Lyapunov exponent and the bigger the exponents are, the more complex the chaotic dynamics is. On this basis, different hyperchaotic systems have been analysed in this work; using the Fourier transform, it is demonstrated that the positive Lyapunov exponents don't always give rise to satisfactory spectral properties. Finally, the obtained results allow to refine the appropriate criteria for the selection of the most suitable chaotic generator.

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ON RANDOM ITERATION

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Discrete random dynamical systems may be defined as solutions of random difference equations. Namely

$$X_{n+1}(\omega) = f(\theta^n \omega, X_n(\omega)), \qquad X_0 = x.$$

As in the deterministic case, fixed points, omega-limit sets, attractors, etc. may be defined, but as random sets. Moreover, new specific notions arise in the random case (conditional expectations, increments, etc.). In this talk, we present also some classical and recent models.

ON LIMITS OF ITERATES OF RV-FUNCTIONS

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Fix a probability space $(\Omega, \mathcal{A}, P)$, a separable Banach space X and a product measurable function $f : X \times \Omega \to X$. Assuming that the sequence of iterates (in the sense of K. Baron and M. Kuczma [1]) of f converges to $\xi : X \times \Omega^{\infty} \to X$, we investigate the problem of continuity of the limit with respect to the first variable.

[1] K. Baron, M. Kuczma, *Iteration of random-valued functions on the unit interval.* Colloq. Math. **37** (1977), 263-269.

SINGULAR HYPERBOLIC ATTRACTOR AS INVERSE SPECTRUM OF SEMIFLOW ON BRANCHED MANIFOLD

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Report is devoted to singular hyperbolic attractors [1],[2].

Consider singular hyperbolic local flow Φ_t on open bounded set $U \subset R^3$. Suppose that for all $t > 0$ $\Phi_t(x)$ is defined for every point $x \in U$ and $clos\Phi_t(U) \subset U$. The set $\Lambda = \cap_{t \geq 0} \Phi_t U$ is singular hyperbolic attractor.

In paper [3] it is proved that every expanding compact hyperbolic attractor of diffeomorphism of a smooth manifold can be represented in the form of generalized solenoid (inverse limit of branched manifold).

We present analogous theorem for singular hyperbolic attractors.

Theorem. Every compact topologically transitive singular hyperbolic attractor $\Lambda = \cap_{t \geq 0} \Phi_t U$ in R^3, containing at least one fixed point can be represented as inverse limit of semiflow on branched manifold

$$\widehat{L} = \lim_{\leftarrow}(K, \phi_t, t \geq 0)$$

i.e., there exists a branched manifold K with semiflow ϕ_t and homeomorphism $H : \Lambda \to \widehat{L}$, conjugating flows Φ_t and $\widehat{\phi}_t$.

It is constructed a cardinal valued topological invariant *L-manuscript* which can distinguish uncountable set of non-homeomorphic attractors [4].

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ON THE SPACE OF ω-LIMIT SETS OF A CONTINUOUS MAP ON A DENDRITE

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If $(\omega_k)_{k=1}^{\infty}$ is a sequence of ω-limit sets of a continuous interval map f such that $\omega_k \subset \omega_{k+1}$, for every k, then the closure of their union is also an ω-limit set of f. We construct an example of a map showing that it does not hold for dendrites. Consequently, the space of all ω-limit sets of a continuous self-map of a dendrite endowed with the Hausdorff metric need not to be compact.

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GEOMETRICAL CONTINUED FRACTIONS AS INVARIANTS IN THE PROBLEM ON TOPOLOGICAL CLASSIFICATION OF ANOSOV DIFFEOMORPHISMS OF n-TORUS

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Recently it was found deep connection between geometrical continued fractions in the sense of Klein and the problem on topological classification of Anosov diffeomorphisms of n-torus.

The problem on topological classification of Anosov diffeomorphisms of n-torus appeared in sixties. The main progress in this problem was obtained by J. Franks, 1969 and C. Manning, 1973. They proved that every Anosov diffeomorfism of torus (of every dimension bigger than one) is topological conjugate to a linear hyperbolic automorphism. So initial problem was reduced to linear classification of hyperbolic (Anosov, linear) automorphisms of n-torus.

In the case $n = 2$ a solution of the last problem goes back to Gauss and Lagrange. A full invariant is the couple - the trace of the linear hyperbolic operator and the period of decomposition of the slope of a eigenvector of the operator into continued fraction. A geometrical interpretation of this invariant is the geometrical version of continued fraction constructed by Klein. So it is interesting to find "good" generalization of continued fraction for multidimensional case.

The main result of our work is that two hyperbolic automorphisms of n-torus are linear conjugated if and only if two corresponding geometrical continued fractions are linear equivalent.

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ON ITERATION GROUPS CONTAINING GENERALIZED CONVEX AND CONCAVE FUNCTIONS

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Let $I \subset \mathbb{R}$ be an open interval and let $M, N : I^2 \to I$ be continuous functions. A function $f : I \to I$ is said to be (M, N)-convex ((M, N)-concave) if

$$f(M(x, y)) \leq (\geq) N(f(x), f(y)), \qquad x, y \in I.$$

A function $f : I \to I$ simultaneously (M, N)-convex and (M, N)-concave is called (M, N)-affine [1].

We prove that if in a continuous iteration group $\{f^t : t \in \mathbb{R}\}$, functions f^t for every $t > 0$ are (M, N)-convex or (M, N)-concave, then either all f^t for $t > 0$ are (M, N)-convex or all f^t for $t > 0$ are (M, N)-concave. Moreover, if in such a group there are two (M, N)-affine elements, then $M = N$ and every element of the group is (M, M)-affine.

We consider also the case where $M = N$ and we show that if in a continuous iteration semigroup $\{f^t : t \geq 0\}$, there exist $f^r < id$ and $f^s < id$ such that $\frac{r}{s} \notin \mathbb{Q}$ and f^r is (M, M)-convex and f^s is (M, M)-concave, then every element of the semigroup is (M, M)-affine.

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FUZZY DYNAMICAL SYSTEMS

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A (crisp) *discrete dynamical system* on a (locally) compact metric space X is defined as a pair (X, φ) where $\varphi : X \to X$ is a continuous map. A *fuzzy set* A on the space X is a map $A : X \to [0, 1]$. By $F(X)$ we denote a space of nonempty compact fuzzy sets on X. It is shown in [1] that there is a natural way (called a *usual fuzzification*) allowing us to define a continuous map $\Phi : F(X) \to F(X)$ and, consequently, to define a *fuzzy discrete dynamical system* on the same space X.

It is natural that the behavior of such a fuzzy dynamical system on the space X can somehow depend on the initial crisp dynamical system on the same space and, conversely, some properties of the fuzzified dynamical system $(F(X), \Phi)$ can be "inherited" to the initial dynamical system (X, φ). This talk is a contribution to the theory studying possible connections between certain crisp and fuzzy discrete dynamical systems.

During the talk we introduce basic notions and three metrics on the space of fuzzy sets. We discuss some basic topological properties ((local) compactness, completeness and separability) on newly obtained topological spaces. We show that the situation in these topological spaces has not been clarified sufficiently up to now. Moreover, the fixed point property of the space of fuzzy sets is discussed.

In the second part of the talk we generalize the notion of the usual fuzzification and we define more general fuzzifications. Several basic properties (mainly the continuity) of the considered fuzzifications are presented afterwards and some relations between the crisp discrete dynamical system and the fuzzified one are shown at the end of the talk.

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SPECIFICATION PROPERTY VERSUS OMEGA-CHAOS

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The first notion of chaos was introduced by Li and Yorke in [1] and later there emerged many others. Namely, Devaney chaos appeared in [2], distributional chaos in [3] and omega chaos in [4]. It was also proved that Devaney, omega chaos and specification property (due to [5]) will not occur on minimal systems. The main aim of this talk is to describe the relations between those notions of chaos that can not appear on minimal systems.

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SYNCHRONIZATION OF ONE-DIMENSIONAL CHAOTIC QUADRATIC MAPS BY A NON-SYMMETRIC COUPLING

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We present a systematic study of a bidirectional coupling without symmetry of non-identical one-dimensional chaotic quadratic maps. This particular coupling was suggested by complex analytic quadratic maps, where we proceed to the decomposition of real and imaginary parts.

For the coupled maps, we identify the range of coupling strengths for which is allowed practical synchronization in the Kapitaniak sense and stable asymptotic synchronization. When discrete dynamical systems are non-symmetrically coupled, in order to achieve synchronization it is necessary to employ techniques from linear algebra, stability theory and control. We use two methods to study the stability of synchronous state: the linear stability of the fixed points and the Lyapunov functional analysis of the transversal system.

ON FRACTIONAL ITERATES OF A FREE MAPPING

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We present a method for finding iterative roots of a free mapping which is embeddable in a flow. To obtain the roots we use a countable family of maximal parallelizable regions of the flow which is a cover of the plane and the relations between parallelizing homeomorphisms defined on the regions of the family.

GLOBAL ORBIT PATTERNS FOR DISCRETE MAPS

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Simple dynamical systems often involve periodic motion. Quasi-periodic or chaotic motion is frequently present in more complicated dynamical systems. However, mathematical results concerning periodic orbits are often obtained using sets of real numbers.

On the other hand O.E. Lanford III [1] reports the results of some computer experiments on the orbit structure of the discrete maps on a finite set which arise when an expanding map of the circle is iterated "naively" on the computer.

Due to the discrete nature of floating points used by computers, there is a huge gap between these results and the theoretical results obtained when this map is considered on a real interval.

In order to understand precisely which periodic orbit can be observed numerically, we study the orbits generated by the iterations of a one-dimensional system on a finite set X_N with a cardinal N. On finite set, only periodic orbits can exist.

Let F_N the set of the maps from X_N to X_N. This set is finite with N^N elements. There is a natural order on it. At every map f belonging to F_N it is associated a unique global orbit pattern (g.o.p.) which is the ordered set of all the periods of the different orbits obtained when all the possible initial points belonging to X_N are used in increasing (or decreasing) order.

We have obtained several results on the g.o.p.:

- there are exactly $2^N - 1$ different g.o.p.,

- the set of all the g.o.p. of X_N is totally ordered,

- a threshold function is associated to every g.o.p., it is the first one for which this g.o.p. appears. Given a g.o.p. this function is explicitly determined.

The main result is that for a given g.o.p. we know explicitly, using a closed formula, the number of functions of F_N having such a pattern. For N greater than 100, it is impossible to compute numerically such a result even using a farm of computers. Instead this formula give very simply the result.

Others results depending on the local properties of the functions f are also explored numerically, but not yet proven such as all the g.o.p. obtained for a function for which the difference between the image of two neighbour elements is bounded.

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ON SOME SET-VALUED ITERATION SEMIGROUPS

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Let X be an arbitrary set, $A\colon X \to 2^{\mathbb{R}}$ be a set-valued function with non-empty values and $q := \sup A(X)$. We present the necessary and sufficient conditions for A under which a multifunction $F\colon (0,\infty) \times X \to 2^X$ given by the formula

$$F(t,x) = A^-\big(A(x) + \min\big\{t, q - \inf A(x)\big\}\big),$$

where A^- denotes the lower preimage under A, is an iteration semigroup.

GENERAL SOLUTIONS AND STABILITY OF SOME FUNCTIONAL EQUATIONS INVOLVING BABBAGE EQUATION

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Some theorems characterizing general solutions of the equation $f(x) = f(\varphi(x)) + g(x)$, where φ is a solution of Babbage equation $\varphi^n(x) = x$, are presented. Moreover, the stability of this equation, non-stability of Babbage equation and, by remark of Z. Moszner, non-stability of the translation equation in a class of functions are proved.

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A 3-DIMENSIONAL PIECEWISE AFFINE MAP USED AS A CHAOTIC GENERATOR

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Piecewise affine maps exhibiting chaotic behavior are of importance in various electrical engineering applications. When such maps are continuous with the standard topology on a Euclidean subspace, their circuit realization makes them more amenable for practical use. Further, in applications such as chaos based secure communication, higher dimensional maps exhibiting chaos are very useful when they are nonlinearly coupled. Unfortunately, most of the well known examples of coupled chaotic piecewise affine maps are on subsets of $\mathbb{R}$, or on the n-dimensional torus (the class of hyperbolic toral automorphisms). These higher dimensional maps on the torus are not continuous on the standard topology of the Euclidean space since they have jump discontinuities. The realization of jump discontinuities in a circuit implementation is not reliable.

In this paper we analyze a piecewise affine map defined on a compact space of $\mathbb{R}^3$ which is a generalization of the classical tent map to three dimensions. We show that this map exhibits sensitive dependence on initial conditions and has positive topological entropy.

ITERATIONS OF THE MEAN-TYPE MAPPINGS

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Let an interval $I \subset \mathbb{R}$ and $p \in \mathbb{N}, p \geq 2$, be fixed. Assuming that the continuous means $M_i : I^p \to I$, $i = 1, ..., p$, are such that

$$\begin{aligned} &\min(x_1, ...x_p) + \max(M_1(x_1, ...x_p), ..., M_p(x_1, ...x_p)) \\ &< \min(M_1(x_1, ...x_p), ..., M_p(x_1, ...x_p)) + \max(x_1, ...x_p) \end{aligned}$$

if $\min(x_1, ...x_p) < \max(x_1, ...x_p)$, we prove that the sequence of iterates of the mean-type mapping $(M_1, ..., M_p) : I^p \to I^p$ converges to a mean-type mapping $(K, ..., K)$, where $K : I^p \to I$ is a unique continuous and $(M_1, ..., M_p)$-invariant mean. This improves an earlier result from [1] where it is assumed that at most one of the means $M_1, ..., M_p$ is not strict. As an application, for some families of mean-type mappings $(M_1, ..., M_p)$, the effective form of real continuous solutions F of the functional equation $F \circ (M_1, ..., M_p) = F$ is given.

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ON TRIANGULAR EXTENSIONS OF TRANSITIVE MAPS

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Let f be invertible selfmap of a compact metric space X. We consider the class $\mathcal{F}_f$ of all its triangular extensions $F = (f, g_x)$ on the space $X \times [0,1]$ which have the property: every fibre map g_x is increasing homeomorphism of $[0,1]$. It is known [1] that every map in $\mathcal{F}_f$ has the same topological entropy like f.

For the case of minimal f we describe the closure of all transitive maps in $\mathcal{F}_f$.

Using ideas similar to those used in the proof of the previous fact we generalize the result proved in [2]. Namely we prove that every transitive selfmap of a compact metric space X can be extended to a triangular transitive map on $X \times [0,1]$ without increasing the entropy.

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COMMON PERIODIC TRAJECTORIES OF INTERVAL MAPS

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For a pair of maps $f, \widetilde{f} \in C^0(I, I)$ we define $J(f, \widetilde{f}) = \left\{x \in I | f^i(x) = \widetilde{f}^i(x), i = 0, 1, 2, ..\right\}$ – the set of their common trajectories. Let $Per(f)$ be the set of periodic trajectories of f, and $Per(f, \widetilde{f}) = J(f, \widetilde{f}) \cap Per(f)$ be the set of their common periodic trajectories. We also denote by $p(f)$ and $p(f, \widetilde{f})$ the sets of periods of trajectories from $Per(f)$ and $Per(f, \widetilde{f})$, respectively.

For given a map $f \in C^r(I, I), r \geq 0$, a set $M \subseteq p(f)$ and a positive ε we study the following question: *Is there $f_\varepsilon \in C^r(I, I)$ satisfying conditions $p(f, f_\varepsilon) = M$ and $\max_{x \in I} |f(x) - f_\varepsilon(x)| < \varepsilon$?*

We prove that the answer is affirmative if f satifies at least one of the conditions:

a) f is not a 2^∞-map,
b) f is a C^1-map,
c) f is a piecewise monotone map.

Whatever $m \geq 0$, one can find a continuous 2^∞-map f such that for any infinite set $\mathbb{M} \subseteq \{1, 2, 2^2, 2^3, ...\} \setminus \{2^m\}$, there does not exist a continuous map $\widetilde{f}$ with $p\,(f, \widetilde{f}) = \mathbb{M}$.

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THE CHAOTIC MOTION OF PROFITS IN A STRUCTURE OF A FIRM: MEASURE AND CONTROL

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The last two decades have witnessed strong revival of interest in nonlinear endogenous business chaotic models. In the present work we study a specific economic model, proposed by S. Bouali ([1]), which is a system of three ordinary differential equations gathering the variables of profits, reinvestments and financial flow of borrowings in the structure of a firm. Firstly, using results of symbolic dynamics, we characterize the topological entropy and the parameter space ordering of kneading sequences, associated with one-dimensional maps that reproduce significant aspects of the model dynamics. Finally, we apply the pole placement technique ([2]) in order to control complicated behavior, arising from the chaotic firm model, without changing its original properties, showing that the dynamics can be turned into a desired attracting time periodic motion (a stable steady state or a regular cycle).

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LEARNING TO PLAY NASH IN DETERMINISTIC UNCOUPLED DYNAMICS

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This paper is concerned with the following problem. In a bounded rational game where players cannot be as super-rational as in Kalai and Leher (1993), are there simple adaptive heuristics or rules that can be used in order to secure convergence to Nash equilibria, or convergence only to a larger set designated by correlated equilibria? Are there games with uncoupled deterministic dynamics in discrete time that converge to Nash equilibrium or not? Young (2008) argues that if an adaptive learning rule follows three conditions — (i) it is uncoupled, (ii) each player's choice of action depends solely on the frequency distribution of past play, and (iii) each player's choice of action, conditional on the state, is deterministic — no such rule leads the players' behavior to converge to Nash equilibrium. In this paper we present a counterexample, showing that there are simple adaptive rules that secure convergence, in fact fast convergence, in a fully deterministic and uncoupled game. We used the Cournot model with nonlinear costs and incomplete information for this purpose and also illustrate that this convergence can be achieved with or without any coordination of the players actions.

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LARGE CHAOTIC SETS FOR MAPS OF THE UNIT CUBE

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In this talk we will present how the knowledge about residual subsets of the hyperspace of all compact nonempty subsets of I^k can be used to obtain large chaotic sets for maps of I^k. Various definitions of chaotic sets will be considered.

Presented approach is motivated by results of Iwanik [1], Kato [2], Akin [3] and others. Presented approach simplifies previously developed techniques (e.g. [4, 5]) and provides a tool to obtain further stronger results.

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APPROXIMATION OF INVARIANT MESURES

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Let f be a homeomorphism of a compact manifold M. The Krylov-Bogoloubov Theorem guarantees the existence of a measure invariant to f. It is well known Ulam's method to approximate an invariant measure. In particular, the "natural" (Sinai-Bowen-Ruelle) measure was constructed in a variety of settings. The set of all invariant measures $\mathcal{M}(f)$ forms a convex compact in the weak topology. The goal is a construction of the set $\mathcal{M}(f)$.

To obtain an approximation of $\mathcal{M}(f)$ we used the symbolic image construction with respect to a partition $C = \{M(1),\ M(2),\ \dots,\ M(n)\}$ of M. The symbolic image G is a directed graph such that a vertex i corresponds the cell $M(i)$ and an edge $i \to j$ exists if $f(M(i)) \cap M(j) \neq \emptyset$. A flow on the symbol image G is a probability distribution $m = \{m_{ij}\}$ on the edges satisfying the invariant property in each vertex i: the incoming flow equals outgoing one

$$\sum_k m_{ki} = \sum_j m_{ij}.$$

The measure of the cell $M(i)$ or the vertex i is defined as $m_i = \sum_j m_{ij}$. The described distribution $\{m_i\}$ is an approximation to some invariant measure. The set of flows $\{m = (m_{ij})\}$ on the symbolic image G forms a convex polygon $\mathcal{M}(G)$ which is an approximation to the set of invariant measures $\mathcal{M}(f)$. By considering a sequence of subdivisions of the partitions C_k one gets sequence of symbolic images G_k and the approximations $\mathcal{M}(G_k)$ which tends to $\mathcal{M}(f)$ as diameter of cells goes to zero. If the flows m^k on each G_k are chosen in a special manner then the sequence $\{m^k\}$ converges to some invariant measure μ. A substantiation of the method and numerical examples are given.

SYNCHRONIZATION AND STABILITY IN A NON-AUTONOMOUS ITERATIVE SYSTEM

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Let $\Phi : (\mathbb{R}^2)^n \mapsto (\mathbb{R}^2)^n$ be a $2n$-dimensional map which generates the following iterative system:

$$\begin{aligned} q_i(t+1) &= \begin{cases} F(Q_i(t)), & t - mi \bmod T \neq 0, \\ G(Q_i(t)), & t - mi \bmod T = 0, \end{cases} \\ k_i(t+1) &= \begin{cases} k_i(t), & t - mi \bmod T \neq 0, \\ 2G(Q_i(t)), & t - mi \bmod T = 0, \end{cases} \end{aligned} \qquad i = \overline{1, n}, \qquad (*)$$

where

$$F(Q_i(t)) = \begin{cases} k_i(t) \dfrac{\sqrt{Q_i(t)/c} - Q_i(t)}{k_i(t) + \sqrt{Q_i(t)/c}}, & Q_i(t) \le 1/c, \\ 0, & Q_i(t) > 1/c, \end{cases}$$

$$G(Q_i(t)) = \begin{cases} 0.5\sqrt{Q_i(t)/c} - Q_i(t), & Q_i(t) \le 1/4c, \\ 0, & Q_i(t) > 1/4c, \end{cases}$$

and $Q_i(t) = \sum\limits_{j=1}^{j=n} q_j(t) - q_i(t)$.

The system (*) models a market with n different firms, where q_i is the output of the i-th firm and k_i corresponds to its capital stock. There are three parameters in the model, namely, the initial marginal cost $c \in \mathbb{R}^+$, the interval between entry of new firms $m \in \mathbb{Z}^+$, and the durability of the capital $T \in \mathbb{N}$.

The map Φ has a unique fixed point $(q_1^*, k_1^*, q_2^*, k_2^*, \ldots, q_n^*, k_n^*)$ (Cournot equlibrium) with the coordinates

$$q_i^* = \frac{1}{n} \frac{n-1}{n} \frac{1}{4c}, \quad k_i^* = 2q_i, \quad i = \overline{1, n},$$

which corresponds to the state of complete synchronization, when all firms behave equally.

Our goal is to study the stability of this fixed point and other different synchronization regimes which can appear in the model.

CONSTRUCTION OF A-ENDOMORPHISMS WHICH SATISFY NO CYCLE CONDITION AND STRUCTURE OF CENTRALIZERS OF ALGEBRAIC ANOSOV ENDOMORPHISMS ON THE TORUS

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Surgery of Smale is applied to algebraic Anosov endomorphism on the torus. Nonwandering set consists of unique attractor and unique zero-dimensional repeller (one-dimensional repeller and zero-dimensional attractor). In accordance with theorem F.Przytycki this endomorphism is topologically stable. The problem of topological distinguishing of these attractors (repellers) leads to the theorem on structure of centralizers of the algebraic Anosov endomorphism.

THE STABILITY OF THE TRANSLATION EQUATION

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We present the results concerning the stability of the translation equation obtained by A. Mach, Z. Moszner, J. Chudziak. Our contribution to this problem is, in particularly, the theorem which reads as follows. For every $\varepsilon > 0$ there is a $\delta > 0$ such that for every function $H\colon (0,\infty) \times I \to I$ satisfying the inequality $|H(s, H(t,x)) - H(s+t,x)| \le \delta$ as well as some additional conditions, we can find a solution of the translation equation F for which the inequality $|F(t,x) - H(t,x)| \le \varepsilon$ holds. Here I is a real interval.

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OLIGOPOLY AND STABILITY

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In this paper the so called Theocharis-Cournot problem is reconsidered. It concerns the relation between oligopoly and perfect competition, in particular the destabilization of Cournot equilibrium when the number of competitors increases. Using a CES production function where one input, capital, is fixed during periods of investment, a mixed short/long run market dynamics is set up. In the short run, with capital fixed, there is a capacity limit for production possibilities, whereas, at moments of capital renewal there are constant returns to scale. In this setting the local stability of Cournot equilibrium is reconsidered. It is demonstrated that if no more than two firms reinvest in the same time period, and the wage rate is not too high, then the Cournot equilibrium is stable.

THE SPACE OF OMEGA-LIMIT SETS OF A PIECEWISE CONTINUOUS INTERVAL MAP

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Assume that $T : [0,1] \to [0,1]$ is a piecewise continuous map. A map is called piecewise continuous, if there are $0 < c_1 < c_2 < \cdots < c_k < 1$, such that T is continuous on $[0,1] \setminus \{c_1, c_2, \ldots, c_k\}$. Moreover, it is assumed that for each $j \in \{1, 2, \ldots, k\}$ both one-sided limits $\lim_{x \to c_j^-} Tx$ and $\lim_{x \to c_j^+} Tx$ exist, and Tc_j coincides with one of these one-sided limits. Furthermore suppose that for all nonnegative integers n and all $j \in \{1, 2, \ldots, k\}$ the set $T^{-n}\{c_j\}$ is finite. Denote the space of all ω-limit sets of T by $H(T)$, where this set is endowed with the Hausdorff metric. In a joint work with Franz Hofbauer and Jaroslav Smítal some properties of the space $H(T)$ are investigated.

For continuous maps T it has been proved by A. Blokh, A. Bruckner, P. Humke and J. Smítal that $H(T)$ is compact. Furthermore, a closed subset of the interval is an ω-limit set if and only if it is locally expanding.

In the case of a piecewise continuous map T the space $H(\widehat{T})$ is compact, where $\widehat{T}$ is obtained from T using a standard doubling points construction. However, in general the space $H(T)$ is not compact. A condition implying the compactness of $H(T)$ is presented. Then the limit of a sequence in $H(T)$ is investigated. One obtains also in the case of piecewise continuous maps that every ω-limit set is locally expanding.

Unfortunately the converse is not true in general. Nonetheless a condition is given, which implies that every locally expanding set satisfying this condition is an ω-limit set.

TOPOLOGICAL MARKOV CHAINS AND KNEADING THEORY

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It is widely recognized that the kneading theory, introduced by Milnor and Thurston in the 1970s, [1], is a powerful technique in the study of the iteration of one-dimensional maps. Over the last two decades, Sousa Ramos and its collaborators, [2] and [3], studied the dynamics of Markov maps of the interval via a transition matrix obtained from the periodic orbits of its critical points and proved some relationships between this approach and the kneading formalism. The aim of our work is to see which relationships are still valid in case the orbits of the critical points are not periodic.

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GENERATION OF CELLULAR AUTOMATA WITH GENETIC ALGORITHMS

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Cellular automata are discrete dynamical systems suitable to describe and model many different complex systems. Recently, genetic algorithms have been used successfully in discrete optimization and many other fields. Genetic algorithms applied to cellular automata modelling and to complex systems can be seen in [1], [2] and [3]. In our work, we generate cellular automata with a certain dynamical behaviour and consequently with a particular visual appearance. If we generate randomly in the parameter space it is very difficult to achieve the desired behaviour, since generally the obtained cellular automata will present either a very simple behaviour or an almost random behaviour. Then we use a genetic algorithm so that given some cellular automata, as initial conditions, we generate by recombination many other cellular automata, satisfying the desired characteristics. We discuss several variations of this algorithm.

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PIECEWISE LINEAR BOUNDARY VALUE PROBLEMS

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A significant field of application of Iteration Theory is boundary value problems (BVPs) for partial differential equations (PDEs). In a number of our works (see for instance [1-3]), there was set and studied an ample class of BVPs that are reducible to continuous time difference equations and show the chaotic behavior. This class consists of BVPs given by linear PDEs and nonlinear boundary conditions; just the nonlinearity of boundary conditions accounts for "chaos" in these BVPs. The possibility for reduction to difference equations is provided the fact that these BVPs have the property $\mathcal{GS}$: the general solution of their associated PDEs can be represented as an analytic formulae.

The next step along this line is to extend the above-mentioned studies to BVPs with nonlinear PDEs. Whereas in the general case this task requires the development of a perturbation theory, when investigating BVPs with piecewise-linear PDEs, the method of reduction to difference equations can be used directly since BVPs with piecewise-linear PDEs have the property $\mathcal{GS}$. In the report we consider the question:

"Which are the simplest boundary conditions that guarantee the chaotic behavior of BVPs with piecewise-linear PDEs? Is the simplest conditions piecewise-linear but irreversible, or is it possible to obtain "chaos" in piecewise-linear PDEs by the addition of linear boundary conditions?"

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DYNAMICS IN FAMILIES OF MAPS WITH A FLAT SEGMENT AT A PERIODIC POINT

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It is known that for the one-parameter family of quadratic maps $\lambda x(1-x)$, $x \in [0,1]$, $\lambda \in [0,4]$, its spectral decomposition (which depends on λ) can consist of finite or infinite "indecomposable" components, and the last one of these components attracts almost all trajectories of the dynamical system. For example, for the map $4x(1-x)$ in this family, the interval [0,1] is transitive and almost all trajectories of the map are dense. If we replace the graph of the map by a flat segment in a neighborhood of zero fixed point $x = 0$ (or some other periodic point), then the fixed point will attract almost all trajectories of the map. Varying the length of the segment (i.e., the neighborhood of attraction), we can see that the length of the spectral decomposition of the modified map varies as well. For example, if the neighborhood is large enough, then all trajectories eventually end at the fixed point. If the neighborhood is small enough, then there exist trajectories (e.g. periodic ones) that do not approach the fixed point. We study the spectral decomposition of this system and other similar systems of modified quadratic maps. The problem arises in a memory model suggested by A. N. Sharkovsky.

ON ITERATIVE SQUARE ROOTS OF LINEAR SET-VALUED FUNCTIONS

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Let K be a closed convex cone in a real Banach space and let $cc(K)$ denote the family of all non-empty compact convex subsets of K. The problem of the existence of continuous linear solutions $\Phi : K \to cc(K)$ of functional equation $\Phi^2(x) = F(x)$ be considered.

NUMERICAL STUDY OF QUADRATIC AND CUBIC RECURRENCE WITH DOUBLE INDICES

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We present in this talk the analytical and numerical results obtained in case of recurrences with double indices, more particularly in the non-linear case of quadratic and cubic recurrences.

The numerical results (a bifurcation diagram in parameter plane and a tree of bifurcations) were first obtained analytically, and then verified with a software that we developed.

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THE QUASIPERIODICALLY DRIVEN INTERVAL SHIFT IN DIGITAL PHASE-LOCKED LOOPS

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Interval shift maps appear in a number of mathematical models of electronic devices with discretisation, particularly in sigma-delta modulators and digital phase-locked loops (DPLL), see [1]. We consider the following model of the first-order DPLL with FM input: $F(\theta, \phi) = (\theta + \omega, \Phi_{\theta+\omega}(\phi))$, where $\theta \in \mathbf{S}_{2\pi}$ (the circle of length 2π), $\phi \in \mathbf{R}$, and

$$\Phi_\theta(\phi) = \phi + 2\pi\nu + f(\theta) - 2\pi Q_b(K_1 \sin\phi). \tag{4}$$

Here ϕ stands for the phase error, $Q_b(x) = \lfloor 2^b x \rfloor / 2^b$ is the b-bit quantizer function, ν is the carrier frequency of the FM input signal, $f(\theta)$ corresponds to its modulation signal of zero mean with the amplitude $A = \max_{\theta\in\mathbf{S}_{2\pi}} |f(\theta)|$, and K_1 is a positive gain. We assume that $\omega/2\pi$ is irrational, neither $2^b\nu$ nor $2^b K_1$ are integers, $K_1 < (2\pi)^{-1}$, and $\nu \in (Q_b(-K_1), Q_b(K_1))$.

$\Phi_\theta(\phi) - \phi$ is a 2π-periodic piecewise-constant function with discontinuity points $\sigma_k^+ = \arcsin(k/(2^b K_1))$ and $\sigma_k^- = \pi - \sigma_k^+$ for $|k| \le k_{\max} = \lfloor 2^b K_1 \rfloor$. Therefore, F is indeed a quasiperiodically driven interval shift map. The investigation of the dynamics produced by this map is interesting from the pure-mathematical point of view, because the unquantized and undriven version of the map F, i.e. $F_0(\phi) = \phi + 2\pi\nu - 2\pi K_1 \sin\phi$, is the famous Arnold map, which is the most studied nonlinear circle map.

We give a restriction on the amplitude A and show that inside the convergence region $D^+ = \mathbf{S}_{2\pi} \times (-2\pi + \sigma^-_{-k_{\max}}, \sigma^-_{k_{\max}}]$ there exists a unique invariant belt $B = [L, U)$ with continuous boundaries $\phi = L(\theta)$, $\phi = U(\theta)$, and constant thickness $U - L = 2\pi/2^b$. There also exists a continuous contour $\phi = R(\theta)$ in the divergence region $D^- = \mathbf{S}_{2\pi} \times [\sigma^+_{-k_{\max}}, 2\pi + \sigma^+_{k_{\max}})$ such that all trajectories starting between R and its shifted copy $R - 2\pi$ get absorbed by the belt B uniformly w.r.t. their initial values. The dynamics inside B (with its boundaries U and L glued together) is equivalent to a certain skew translation on a two-dimensional torus.

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BIFURCATION CURVES IN DISCONTINUOUS MAPS

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Recently several models (mainly in Engineering) are described by piecewise linear functions, in one- and two- dimensional spaces. Here we are interested in the dynamics of a one-dimensional map (arising from a duopoly model) which is made up of *three linear pieces (i.e. with two discontinuity points).*

The bifurcation structure associated with one-dimensional maps, with one discontinuity point ad an increasing or decreasing jump, has been the object of recent papers (see [1–3]), and several results where already presented by Leonov [4, 5] (also called boxes in files bifurcations in [6]).

In our case, the map has three linear pieces, and we shall investigate the border collision bifurcations involving the periodic orbits in such a new situation.

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ITERATED FUNCTION SYSTEMS IN BACKWARD MODELS

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In the last years several models in the applied context (specially in economics) appeared formulated in the so called 'backward dynamics', that is, as discrete models in the form:

$$x_t = F(x_{t+1})$$

and the interest is in the dynamic behavior of the *forward states* of the state variable $(x_t, x_{t+1}, x_{t+2}...)$. There are no problems when the function $F(.)$ is invertible, while difficulties arise in the cases in which the function $F(.)$ has not a unique inverse (also in the interpretation of the model). Mathematically this kind of models have been investigated considering the space of all possible sequences, which is a space of infinite dimension, and is known as Inverse Limit Theory. However, the inverse limit approach is rather abstract, and we prefer to follow a different approach, which is based on the theory of Iterated Function Systems (IFS, as described in [1] and [2]). We shall see how the *homoclinic orbits* existing in the forward dynamics of the map F (that is, in the standard dynamical system $x_{t+1} = F(x_t)$) give us a kind of 'bridge' between the theory of Dynamical Systems and the theory of IFS, to describe *fractal attractors in the backward models.*

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DISTRIBUTIONAL CHAOS IN DIFFERENTIAL EQUATIONS

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There are some papers devoted to the distributional chaos in differential equations cf. [1, 2]. In those papers the equation

$$\dot{x} = f(t, x) \tag{1}$$

is considered, where $f \in \mathcal{C}(\mathbf{R}^{n+1}, \mathbf{R}^n)$ is T-periodic in the time variable t. The periodicity of f allows to generate some discrete dynamical system on some compact subset of the Poincaré section which exhibits distributional chaos. The technique involves some semiconjugacy of the Poincaré map with chaotic system (e.g. shift space). This approach does not allow to deal with nonperiodic f. We try to omit this difficulty and introduce the notion of distributional chaos in the general case of the equation (1). We also present some examples of nonperiodic chaotic ODEs.

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ON COMMUTING HOMEOMORPHISMS AND GENERALIZED ITERATION GROUPS

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Let I be an open interval and $f, g : I \to I$ be commuting homeomorphisms and $f < id$, $g < id$. Let f and g be iteratively incommensurable that is for every $x \in I$ and every $n, m \in N$ $f^n(x) \neq g^m(x)$. Put

$$s := inf\{\frac{n}{m} : n, m \in N, f^n < g^m\}$$

and define

$$a^t(x) := sup\{f^n \circ g^{-m}(x), n, m \in N, n - sm > t\},$$
$$b^t(x) := inf\{f^n \circ g^{-m}(x), n, m \in N, n - sm < t\}.$$

The families of functions $\{a^t, t \in R\}$ and $\{b^t, t \in R\}$ are iteration groups, i.e. $a^u \circ a^v = a^{u+v}$ and $b^u \circ b^v = b^{u+v}$, $u, v \in R$. The functions a^t, b^t are nondecreasing and $t \to a^t(x)$, $t \to b^t(x)$ for $x \in R$ are strictly decreasing.

The family of set-valued functions

$$F^t(x) := [a^t(x), b^t(x)], \quad t \in R,$$

is an iteration group that is

$$F^u(F^v(x)) := \bigcup_{y \in F^v(x)} F^u(y) \ = \ F^{u+v}(x), \quad x \in I, \ u, v \in R,$$

such that $f(x) \in F^1(x)$ and $g(x) \in F^s(x)$.

Homeomorphisms f and g are embeddable in a continuous iteration group if and only if for every $x \in I$ and $t \in R$ $F^t(x)$ is a singleton.

In the talk we present many other properties of set valued iteration group $\{F^t : t \in R\}$.

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SOME RESULTS ON MULTIDIMENSIONAL PERTURBATIONS OF 1-DIM MAPS

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We consider continuous maps $F : \mathbb{R} \times \mathbb{R}^k \to \mathbb{R} \times \mathbb{R}^k$ which are close to $F_0(x, y) = (f(x), 0)$, where $f : \mathbb{R} \to \mathbb{R}$ is continuous.

We address the following question: Will the map F inherit some interesting dynamical properties of the map f if $|F(z) - F_0(z)|$ is small for z from a suitable compact set? In this context, we consider certain properties of the set of periods and the topological entropy. We present some positive answers in this direction.

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ON VARIOUS METHODS OF CONSTRUCTING ITERATION GROUPS IN RINGS OF FORMAL POWER SERIES

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One of the basic problems in iteration theory (for formal power series) is the explicit construction of one-parameter groups $(F_t)_{t\in\mathbb{C}}$ in the group Γ of invertible formal series in $\mathbb{C}[\![x]\!]$, i.e. of homomorphisms $\theta\colon G \to \Gamma$ from an abelian group $(G,+)$ into the group $(\Gamma,\circ)$, $\circ$ denoting the substitution. This problem is again equivalent to an infinite recursive system $(\mathcal{C})$ of inhomogeneous Cauchy type equations for the coefficient functions c_ν occuring in $F_t(x) = \sum_{\nu=1}^{\infty} c_\nu(t)x^\nu$. Here we consider only the case $G = (\mathbb{C},+)$. Two different approaches are possible:

1) (W. Jabłoński and L. R.) We construct a set of solutions of $(\mathcal{C})$, depending on an additive function $a \neq 0$ or a generalized exponential function $e \neq 1$ and a sequence of parameters, and show, using the already known regular solutions of $(\mathcal{C})$ and some basic facts about additive and generalized exponential functions, that the obtained set of solutions is the general solution of $(\mathcal{C})$.

2) (H. Fripertinger and L. R.) We observe, just from the beginning, that $(\mathcal{C})$ is equivalent to an infinite system of purely algebraic relations in the ring $\mathbb{C}[y][\![x]\!]$ and solve this system by using the formal differential equations of Aczél-Jabotinsky type valid for so called formal one-parameter groups.

Both, for 1) and 2) the two types of iteration groups require somewhat different methods. A modification of 2) arises from the possibility to arrange a formal iteration group $\sum_{\nu=1}^{\infty} P_\nu(y)x^\nu \in \mathbb{C}[y][\![x]\!]$ as $\sum_{\nu=0}^{\infty} Q_\nu(x)y^\nu$ with $Q_\nu(x) \in \mathbb{C}[\![x]\!]$ and to find the coefficients $Q_\nu(x)$.