GENERAL PROPERTIES OF SURFACE SINGULARITIES

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Contents

We fix $S = \operatorname{Spec} \mathbf{A}$, where $\mathbf{A}$ is a local normal ring of Krull dimension 2 (a "*normal surface singularity*"). Moreover, for the sake of simplicity we suppose that $\mathbf{A}$ is an algebra over an algebraically closed field $\mathbf{k}$ and $\mathbf{A}/\mathfrak{m} \simeq \mathbf{k}$, where $\mathfrak{m}$ denotes the maximal ideal of $\mathbf{A}$. Sometimes it is important that $\mathbf{A}$ be complete (or henselian), but we shall try to specify such places properly. We denote by p the unique closed point of $\mathbf{A}$ (corresponding to the maximal ideal $\mathfrak{m}$).

A *resolution* of S is, by definition, a projective morphism $X \to S$, where X is smooth, which induces an isomorphism $X \setminus \pi^{-1}(p) \to S \setminus \{p\}$. In particular, π is birational. It is known, due to Zariski and Abhyankar (cf. [**Lip**, § 2]), that every surface singularity has a resolution; moreover, it can be obtained by a sequence of *monoidal transformations* (blowing up closed singular points) and *normalizations*. Some examples of calculations are presented in Section 5. For such a resolution, we denote by E the *reduced* pre-image $\pi^{-1}(p)_{\mathrm{red}}$, which is a projective curve over $\mathbf{k}$ (it might be singular and reducible). We call E the *exceptional curve* of the resolution π and denote by E_i $(i = 1, \dots, s)$ its irreducible components. Remind that E is always *connected* (it follows from Zariski's Main Theorem [**Ha**, Corollary III.11.4]). We say that the resolution is *transversal* if E_i are smooth, pairwise transversal, and neither three of them have a common point. Especially, all singular points of E (if exist) are in this case ordinary double points (nodes).

An *exceptional cycle* (or simply *cycle*, or an *exceptional divisor*) of such a resolution is a divisor C on X with $\operatorname{supp} C \subseteq E$. It means that $C = \sum_{i=1}^{s} c_i E_i$. If $c_i \geqslant 0$ and $C \neq 0$, call C an *effective cycle*.

Then C is identified with the closed sub-scheme of X defined by the ideal sheaf $\mathcal{O}_X(-C) \subset \mathcal{O}_X$. We also denote by ω_X the *canonical* (or *dualizing*) line bundle over X and by K_X a *canonical divisor* of X; thus $\omega_X \simeq \mathcal{O}_X(K_X)$. Then for any effective cycle C there is a canonical (dualizing) line bundle

$$\omega_C = \mathcal{E}xt^1(\mathcal{O}_C, \omega_X) \simeq \mathcal{O}_C \otimes \omega_X(C)$$

(we always write $\otimes$ for $\otimes_{\mathcal{O}_X}$, if it not very ambiguous). It establishes the *Serre's duality* for any coherent sheaf $\mathcal{F}$ on C:

$$\text{(SD)} \qquad \mathrm{Ext}^i(\mathcal{F}, \omega_C) \simeq \mathrm{DH}^{1-i}(C, \mathcal{F}) \qquad (i = 0, 1)$$

[**Ha**, Theorem III.7.6], or, if $\mathcal{F}$ is a vector bundle (locally free sheaf),

$$\mathrm{DH}^i(C, \mathcal{F}) \simeq \mathrm{H}^{1-i}(C, \mathcal{F}^\vee \otimes \omega_C).$$

Here $\mathrm{D}V$ denotes the dual vector space $\mathrm{Hom}_{\mathbf{k}}(V, \mathbf{k})$ and $\mathcal{F}^\vee$ denotes the dual vector bundle $\mathcal{H}om_{\mathcal{O}_C}(\mathcal{F}, \mathcal{O}_C)$.

1. Intersection theory

Let C be a projective curve (possibly non-reduced); for instance, it may be an effective cycle of a resolution. For any locally free sheaf $\mathcal{F}$ of rank n on C define its *degree* $\deg \mathcal{F}$ (or $\deg_C \mathcal{F}$) as

$$\deg \mathcal{F} = \chi(\mathcal{F}) - n\chi(\mathcal{O}_C),$$

where χ is the Euler–Poincaré characteristic: $\chi(\mathcal{F}) = \mathrm{h}^0(\mathcal{F}) - \mathrm{h}^1(\mathcal{F})$. If C is an irreducible curve and $\mathcal{F} = \mathcal{O}_C(D)$, where D is a divisor supported on the set of regular points of C, the Riemann–Roch theorem gives $\deg \mathcal{F} = \deg D$, the usual degree of a divisor (cf. [**Ser**] or [**Ha**, Theorem IV.1.3 and Exercise IV.1.9]). This definition enjoys most properties of "usual" degree, which we collect in the following proposition. We write $\mathcal{F} \overset{g}{\sim} \mathcal{E}$ and say that the sheaves $\mathcal{F}$ and $\mathcal{E}$ are *generically isomorphic*, if $\mathcal{F}|U \simeq \mathcal{E}|U$ for an open dense subset $U \subseteq C$.

Proposition 1.1. (1) *If $\mathcal{F}_1 \overset{g}{\sim} \mathcal{F}_2$ and $\mathcal{E}$ is locally free of rank m, then*

$$\chi(\mathcal{E} \otimes \mathcal{F}_1) - \chi(\mathcal{F}' \otimes \mathcal{F}_2) = m(\chi(\mathcal{F}_1) - \chi(\mathcal{F}_2)).$$

Especially if $\mathcal{F}_1, \mathcal{F}_2$ are also locally free, the same holds for their degrees.

(2) *If $\mathcal{E}, \mathcal{F}$ are locally free of ranks, respectively, m, n, then*

$$\deg(\mathcal{E} \otimes \mathcal{F}) = n \deg \mathcal{E} + m \deg \mathcal{F}.$$

(3) *If $f : D \to C$ is a proper morphism of curves such that $f_*\mathcal{O}_D \overset{g}{\sim} m\mathcal{O}_C$ and $\mathcal{F}$ is a locally free sheaf on C, $\deg f^*\mathcal{F} = m \deg \mathcal{F}$.*

Proof. (1) Let $\mathcal{F}_1|U \simeq \mathcal{F}_2|U$, where U is open dense, $i : U \to X$ be the embedding. Then $\mathcal{F} = i^* i_* \mathcal{F}_1 \simeq i^* i_* \mathcal{F}_2$ and there is an exact sequence

$$0 \longrightarrow \mathcal{S}_{i1} \longrightarrow \mathcal{F}_i \longrightarrow \mathcal{F} \longrightarrow \mathcal{S}_{i2} \longrightarrow 0 \qquad (i = 1, 2),$$

where $\operatorname{supp} \mathcal{S}_{ij} \subseteq X \setminus U$, so it is 0-dimensional. Therefore $\chi(\mathcal{S}_{ij}) = \mathrm{h}^0(\mathcal{S}_{ij})$ and $\chi(\mathcal{E} \otimes \mathcal{S}_{ij}) = m\chi(\mathcal{S}_{ij})$. As $\chi(\mathcal{F}_i) = \chi(\mathcal{F}) + \chi(\mathcal{S}_{i1}) - \chi(\mathcal{S}_{i2})$, it implies the necessary formula.

(2) Here $\mathcal{E} \overset{g}{\sim} m\mathcal{O}_C$, $\mathcal{F} \overset{g}{\sim} n\mathcal{O}_C$, so using (1) we get

$$\begin{aligned} \chi(\mathcal{E} \otimes \mathcal{F}) - mn\chi(\mathcal{O}_C) &= \chi(\mathcal{E} \otimes \mathcal{F}) - \chi(m\mathcal{O}_C \otimes \mathcal{F}) + \\ &+ \chi(m\mathcal{O}_C \otimes \mathcal{F}) - \chi(m\mathcal{O}_C \otimes n\mathcal{O}_C) = n(\chi(\mathcal{E}) - m\chi(\mathcal{O}_C)) + \\ &+ m(\chi(\mathcal{F}) - n\chi(\mathcal{O}_C)) = n \deg \mathcal{E} + m \deg \mathcal{F}. \end{aligned}$$

(3) By definition, $\Gamma(C, f_*\mathcal{M}) = \Gamma(D, \mathcal{M})$ for any sheaf $\mathcal{M}$ on D. It gives a spectral sequence

$$\mathrm{H}^i(C, R^j f_* \mathcal{M}) \Longrightarrow \mathrm{H}^p(D, \mathcal{M}).$$

For $p = 1$ it gives an exact sequence

$$0 \to \mathrm{H}^1(C, f_*\mathcal{M}) \longrightarrow \mathrm{H}^1(D, \mathcal{M}) \longrightarrow \mathrm{H}^0(C, R^1 f_* \mathcal{M}) \to 0.$$

If $\mathcal{M} = f^*\mathcal{F}$ and $\mathcal{F}$ is locally free of rank n, $f_* f^* \mathcal{F} \simeq f_*\mathcal{O}_D \otimes \mathcal{F}$ and $R^1 f_*(f^*\mathcal{F}) \simeq R^1 f_* \mathcal{O}_D \otimes \mathcal{F}$ [**Ha**, Exercise III.8.3]. As $R^1 f_* \mathcal{O}_D$ has 0-dimensional support, it implies that $\mathrm{h}^0(R^1 f_*(f^*\mathcal{F})) = n\mathrm{h}^0(R^1 f_* \mathcal{O}_D)$ and

$$\begin{aligned} \deg(f^*\mathcal{F}) &= \chi(f^*\mathcal{F}) - n\chi(\mathcal{O}_D) = \\ &= \chi(f_*(f^*\mathcal{F})) + n\mathrm{h}^0(r^1 f_* \mathcal{O}_D) - n\chi(\mathcal{O}_D) = \\ &= \chi(f_*\mathcal{O}_D \otimes \mathcal{F}) - n\chi(f_*\mathcal{O}_D) = \\ &= \chi(f_*\mathcal{O}_D \otimes \mathcal{F}) - \chi(m\mathcal{O}_C \otimes \mathcal{F}) + \\ &\quad + \chi(m\mathcal{O}_C \otimes \mathcal{F}) - n\chi(f_*\mathcal{O}_D) = \\ (*) \qquad &= n\chi(f_*\mathcal{O}_D) - mn\chi(\mathcal{O}_C) + \\ &\quad + m\chi(\mathcal{F}) - n\chi(f_*\mathcal{O}_D) = \\ &= m \deg(\mathcal{F}) \end{aligned}$$

(equality (*) holds since $f_*\mathcal{O}_D \overset{g}{\sim} m\mathcal{O}_C$). □

Let now X be a smooth surface (not necessary projective!) and C be an effective divisor on X whose support is a projective curve. For instance, C may be an effective cycle on a resolution of a normal surface singularity. For every divisor D on X define the *intersection number* of D with C as $(D.C) = \deg_C(\mathcal{O}_C(D))$, where, as usually, we set $\mathcal{F}(D) = \mathcal{F} \otimes \mathcal{O}_X(D)$ for any coherent sheaf $\mathcal{F}$ on X. Again we gather the properties of these numbers in the following proposition. We denote $\chi(C) = \chi(\mathcal{O}_C)$.

Proposition 1.2. (1) $((D+D').C)=(D.C)+(D'.C)$.

(2) $(D.(C+C'))=(D.C)+(D.C')$.

(3) *If* D *is effective and* $\operatorname{supp} D$ *contains neither component* E_i, *then* $(D.C)\geqslant 0$*; moreover,* $(D.C)=0$ *if and only if* $\operatorname{supp} D\cap \operatorname{supp} C=\emptyset$.

(4) *If both* C *and* C' *are effective divisors with projective supports,*

$$(C'.C)=\chi(C')+\chi(C)-\chi(C+C'), \tag{1.1}$$

in particular $(C'.C)=(C.C')$.

(5) $\chi(C)=-(K+C.C)/2$, *where* K *is a canonical divisor of* X ("adjunction formula," cf. [**Ha**, Proposition V.1.5]).

Proof. (1) is obvious since $\mathcal{O}_X(D+D')=\mathcal{O}_X(D)\otimes\mathcal{O}_X(D')$.

(2) and (4) will be proved simultaneously. Tensoring the exact sequence $0\to\mathcal{O}_X(-C)\to\mathcal{O}_X\to\mathcal{O}_C\to 0$ with $\mathcal{O}_X(-C')$ we get

$$0\to\mathcal{O}_X(-C-C')\longrightarrow\mathcal{O}_X(-C')\longrightarrow\mathcal{O}_C(-C')\to 0.$$

Thus there is an exact sequence

$$0\to\mathcal{O}_C(-C')\longrightarrow\mathcal{O}_{C+C'}\longrightarrow\mathcal{O}_{C'}\to 0 \tag{1.2}$$

and all these sheaves are actually coherent sheaves on $C+C'$. So if $\mathcal{L}$ is an invertible sheaf on $C+C'$, we get, using Proposition 1.1(1) and denoting $\mathcal{L}_C=\mathcal{L}\otimes\mathcal{O}_C$,

$$\begin{aligned}(C'.C)&=\deg_C(\mathcal{O}_C(C'))=\chi(\mathcal{O}_C(C'))-\chi(\mathcal{O}_C)=\\ &=\chi(\mathcal{L}_C)-\chi(\mathcal{L}_C(-C'))=\chi(\mathcal{L}_C)+\chi(\mathcal{L}_{C'})-\chi(\mathcal{L})\end{aligned} \tag{1.3}$$

(to get the last equality, just tensor (1.2) by $\mathcal{L}$). If $\mathcal{L}=\mathcal{O}_{C+C'}$, it gives (1.1). Subtracting (1.1) from (1.3) gives $\deg_{C+C'}(\mathcal{L})=\deg_C(\mathcal{L}_C)+\deg_{C'}(\mathcal{L}_{C'})$. Taking $\mathcal{L}=\mathcal{O}_{C+C'}(D)$ we get the assertion (2).

(3) If D is effective, tensoring the exact sequence $0\to\mathcal{O}_X(-D)\to\mathcal{O}_X\to\mathcal{O}_D\to 0$ with $\mathcal{O}_C$ gives

$$0\to\mathcal{T}or_1(\mathcal{O}_C,\mathcal{O}_D)\to\mathcal{O}_C(-D)\to\mathcal{O}_C\to\mathcal{O}_C\otimes\mathcal{O}_D\to 0.$$

As $\operatorname{supp}\mathcal{T}or_1(\mathcal{O}_C,\mathcal{O}_D)\subseteq C\cap D$, it is a sky-scraper sheaf, so cannot be embedded into $\mathcal{O}_C(-D)$, which is locally free on C. Hence $\mathcal{T}or_1(\mathcal{O}_C,\mathcal{O}_D)=0$ and

$$(D.C)=\chi(\mathcal{O}_C)-\chi(\mathcal{O}_C(-D))=\chi(\mathcal{O}_C\otimes\mathcal{O}_D).$$

The latter sheaf is also skyscraper, so $(D.C)=\mathrm{h}^0(\mathcal{O}_C\otimes\mathcal{O}_D)\geqslant 0$. Moreover, if $\operatorname{supp} C\cap\operatorname{supp} D=\emptyset$, also $\mathcal{O}_C\otimes\mathcal{O}_D=0$. On the other hand, if $x\in\operatorname{supp} C\cap\operatorname{supp} D$, the residue field $\mathbf{k}(x)$ is a factor of both $\mathcal{O}_C$ and $\mathcal{O}_D$, hence of their tensor product, so $\mathcal{O}_C\otimes\mathcal{O}_D\neq 0$ and $(D.C)\neq 0$.

(5) Remind that $\omega_C=\omega_X(C)\otimes\mathcal{O}_C\simeq\mathcal{O}_X(K+C)\otimes\mathcal{O}_C$ and, by Serre's duality, $\chi(C)=-\chi(\omega_C)$, so $\deg_C(\omega_C)=-2\chi(C)$. But $\deg_C(\omega_C)=\deg_C(\mathcal{O}_X(K+C)\otimes\mathcal{O}_C)=(K+C.C)$. □

The main result of this intersection theory is

Theorem 1.3. *For every non-zero exceptional cycle* C *,* $(C.C)<0$ *.*

First prove the following

Lemma 1.4. *Let* $(\_.\_)$ *be a symmetric bilinear form on* $\mathbb{Z}^n$ *. Suppose that there is a basis* $e_1,e_2,\dots,e_n$ *such that*

(1) $(e_i.e_j)\geqslant 0$ *for* $i\neq j$ *,*

(2) *there is a vector* $z=\sum_{k=1}^n z_ke_k$ *with all* $z_k>0$ *such that* $(z.e_i)\leqslant 0$ *for all* i *,*

(3) *for each* i *there is* $j\neq i$ *such that* $(e_i.e_j)\neq 0$ *.*

Then $(\_.\_)$ *is negative semi-definite. If, moreover,* $(z.z)<0$ *, it is negative definite.*

Proof. Use induction by n to show that $(v.v)\leqslant 0$ for each v. If $n=1$, it is trivial. Suppose that $(v.v)>0$. It follows from (1) that replacing all coordinates of v by their absolute value cannot diminish $(v.v)$, so we may suppose that $v=\sum c_ie_i$ with $c_i\geqslant 0$. Set $r=\min\{a_i/z_i\}$. Then $v-rz$ has all coordinates non-negative and one of them zero. On the other hand, $(v-rz.v-rz)=(v.v)-(z.2v-rz)\geqslant(v.v)>0$ due to the condition (2). In particular, $v\neq rz$. Thus we may suppose that $c_i>0$ for $1\leqslant i\leqslant l$ and $c_i=0$ for $i>l$, where $l<n$. Consider the vector $z'=\sum_{k=1}^l z_ke_k$. If $i\leqslant l$, $(z'.e_i)\leqslant(z.e_i)\leqslant 0$, since $(e_j.e_i)\geqslant 0$ if $j>l$. As z' and v belong to a subspace generated by $\{e_1,e_2,\dots,e_l\}$, $(v.v)\leqslant 0$ by induction.

Suppose now that $(z.z)<0$ and $(v.v)=0$ for some v as above. Again we can choose v with at least one coordinate $c_j=0$ (note that $v=rz$ is impossible since $(z.z)<0$). Moreover, the condition (3) implies that we can choose j such that $(v.e_j)\neq 0$, hence $(v.e_j)>0$. Then $(av+e_j.av+e_j)=2a(v.e_j)+(e_j.e_j)>0$ for big enough a. As we have already seen, it is impossible. □

Proof of Theorem 1.3. We shall construct an effective cycle Z such that $(Z.E_i)\leqslant 0$ for all i and $(Z.Z)<0$. Since E is connected, we can apply lemma 1.4 afterwards, taking into account proposition 1.2(3). Consider a non-zero element $a\in\mathfrak{m}$ and its divisor (a) on X. Note that a has no poles, so (a) is effective. Let $(a)=\sum_{i=1}^s z_iE_i+D$, where $E_i\not\subseteq\operatorname{supp}D$. Certainly $z_i>0$ since $E_i\subseteq\pi^{-1}(p)$ and $a(p)=0$. Set $Z=\sum_{i=1}^s z_iE_i$. Then $Z\sim(-D)$ as divisor on X, so $(Z.E_i)=-(D.E_i)\leqslant 0$. On the other hand, since a is non-invertible element of $\mathbf{A}$, there is an irreducible curve C on S such that $a|C=0$ and $p\in C$. Hence $\operatorname{supp}D$ has a component that intersects E, so $(D.Z)>0$ by proposition 1.2(3). Thus $(Z.Z)=-(D.Z)<0$. □

It is known (cf. [**Gr**, **La1**]) that the converse holds in *analytic case*: if X is a smooth analytic surface and E is a projective curve on X such that the intersection form is negative definite on cycles with support in E, there is an analytic surface S, a point $p \in S$ and a proper birational mapping $\pi : X \to S$ such that $E = \pi^{-1}(p)_{\mathrm{red}}$ and the restriction of π on $X \setminus E$ is an isomorphism. I do not know whether it is true in *algebraic situation*. Some results can be found in [**Art**].

2. Minimal resolutions

Definition 2.1. A resolution $\pi : X \to S$ is said to be *minimal* if for any other resolution $\phi : Y \to S$ there is a morphism $\psi : Y \to X$ such that $\phi = \pi \circ \psi$.

Note that ψ is uniquely determined since π is dominant, so usual considerations show that a minimal resolution, whenever it exists, is unique up to a canonical isomorphism. To show existence we need some facts about birational transformations, especially about *monoidal transformations*, i.e. *blowing up* closed points [**Ha**, Sections II.7, V.3]. The main properties of monoidal transformations are collected in the following

Proposition 2.2. *Let X be a smooth 2-dimensional variety, $\tau : X' \to X$ be the blowing up of a closed point x (the* monoidal transformation *at the point x), and $L = \tau^{-1}(x)$. For any divisor D on X denote by $\tau^* D$ its pre-image and by $\tau' D$ its* strict transform *(for an effective D it is defined as the closure of $\tau^{-1}(D \setminus \{ x \})$). Let also m_D be the multiplicity of D at x, defined for an effective D as $\max \{ m \mid f \in \mathfrak{m}_x^m \}$, where f is a local equation of D in a neighbourhood of x (especially $m_D = 0$ if $x \notin \operatorname{supp} D$).*

1. $\operatorname{Pic} X' \simeq \operatorname{Pic} X \oplus \mathbb{Z}$, *where the latter summand is generated by the class of L.*
2. $L \simeq \mathbb{P}_1$ *and* $(L.L) = -1$.
3. $\tau^* D = \tau' D + m_D L$.
4. $(\tau^* D.\tau^* C) = (D.C)$ *and* $(\tau^* D.L) = 0$ *for every D.*
5. $(\tau' D.\tau' C) = (D.C) - m_D m_C$.
6. $K_{X'} = \tau^* K_X + L$.
7. $\chi(\tau' C) = \chi(C) + m_C(m_C - 1)/2$.

In these formulas C denotes a projective curve on X and intersection numbers are defined in the preceding section.

For the proofs, see [**Ha**, Section V.3]. Though it is supposed there that X is a projective surface, all these proofs are in fact local, so they remain valid in our situation. The last formula for $\chi(\tau' C)$ follows immediately from the preceding ones and the adjunction formula $\chi(C) = -(K + C.C)/2$ from Proposition 1.2(5).

We call a curve C on a smooth surface X a *contractible line* if $C \simeq \mathbb{P}^1$ and $(C.C) = -1$. The sense of this notion is clarified by the classical Castelnuovo theorem [**Ha**, Theorem III.5.7]. We formulate it in a bit more general form, though the proof essentially remains the same.

Theorem 2.3 (Castelnuovo)**.** *Let A be an affine variety, $\phi : X \to A$ be a projective morphism, where X is a smooth surface, and C be a contractible line on X. There is a projective morphism $\psi : Y \to A$, where Y is also a smooth surface, a monoidal transformation $\tau : Y' \to Y$ at a point y, and an isomorphism $\eta : X \to Y'$ such that $\phi = \psi \circ \tau \circ \eta$ and $\eta(C) = \psi^{-1}(y)$.*

We always use the isomorphism η from this theorem to identify X with Y' and C with $\tau^{-1}(y)$, and say that Y is obtained from X by *contracting* C.

The next important fact on birational transformations of surfaces is

Theorem 2.4. *Let X and Y be smooth surfaces, projective over some affine variety A, $\phi : Y \to X$ be a birational morphism (over A). Then ϕ decomposes into a product of monoidal transformations, i.e. there is a morphism $\psi : Y' \to X$ that is a product of monoidal transformations and an isomorphism $\eta : Y \to Y'$ such that $\phi = \psi \circ \eta$. Moreover, the number of monoidal factors in ϕ equals the number of irreducible curves C on Y such that $\phi(C)$ is a closed point.*

Again the proof from [**Ha**, Section V.5] can be applied with no changes in this situation, and we shall always identify Y with Y' and ϕ with ψ.

Now we are able to show that a minimal resolution always exists.

Theorem 2.5. *For any surface singularity S there is a minimal resolution. Namely, any resolution $\pi : X \to S$ such that $\pi^{-1}(p)$ contains no contractible lines are minimal.*

Proof. Consider any resolution $\psi : Z \to S$ and its exceptional curve E. If E has a component E_i that is a contractible line, we can decompose $\psi = \tau \circ \psi'$, where $\tau : Z \to Z'$ is a monoidal transformation and ψ' is again a resolution. Moreover, since $\tau(E_i)$ is a point, the exceptional curve of ψ' has less irreducible components. Therefore we can find a resolution $\pi : X \to S$ such that its exceptional curve contains no contractible lines. We shall prove that this resolution is minimal.

Indeed, consider any other resolution $\psi : Y \to S$. Let $P = X \times_S Y$. It is again a surface, though not necessarily smooth. Nevertheless, we can construct a resolution $Z \to P$, thus obtaining a commutative diagram of birational morphisms

$$\begin{array}{ccc} Z & \xrightarrow{\alpha} & X \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\phi} \\ Y & \xrightarrow[\psi]{} & S \end{array}$$

Moreover, we can choose Z minimal in the sense that there is no birational morphism $\theta : Z \to Z'$, which is not an isomorphism, but $\alpha = \alpha' \circ \theta$ and $\beta = \beta' \circ \theta$ for some $\alpha' : Z' \to X$ and $\beta' : Z' \to Y$. Suppose that β is not isomorphism. Then it decomposes into a product of monoidal transformations. In particular, there is a monoidal transformation $\tau : Z \to Y'$ at some point $y \in Y'$ such that $\beta = \beta' \circ \tau$. Let $L = \tau^{-1}(y)$. It is a contractible line. Set $C = \alpha(L)$. It is the total transform of y under the birational transformation $\alpha \circ \tau^{-1} : Y' \to X$, which is defined everywhere except maybe y. If it is also defined at y, then α factors through Y', in contradiction with the minimality of Z. Hence $\dim C = 1$ [**Ha**, Theorem V.5.2], so C is an irreducible curve and L is the strict transform of C under α. From Proposition 2.2(7) we know that $(C.C) + \chi(C) \geqslant (L.L) + \chi(L) = 0$. As $(C.C) \leqslant -1$ and $\chi(C) \leqslant 1$, necessarily $(C.C) = -1$ and $\chi(C) = 0$, so C is a contractible line, in contradiction with the choice of X. □

Theorem 2.6. *For any surface singularity S there is a* minimal transversal *resolution, i.e. a transversal resolution $\widetilde{\pi} : \widetilde{X} \to S$ such that any other transversal resolution factors through $\widetilde{\pi}$.*

Proof. Consider a minimal resolution $\pi : X \to S$ and construct morphisms $\phi_k : X_k \to X$ and $\pi_k = \pi \circ \phi_k : X_k \to S$ recursively. Namely, set $X_0 = X$ and $\phi_0 = \mathrm{Id}$. If $\phi_k : X_k \to X$ and $\pi_k : X_k \to S$ have been constructed, let $E^{(k)} = \pi_k^{-1}(p)$ and $E_1, E_2, \dots, E_s$ be the irreducible components of $E^{(k)}$. Define the set Γ_k of closed points of $E^{(k)}$ such that $x \in \Gamma_k$ if and only if one of the following conditions hold:

(i) x is a singular point of some E_i;
(ii) $x \in E_i \cap E_j$ $(i \neq j)$ and E_i, E_j are not transversal at x;
(iii) $x \in E_i \cap E_j \cap E_l$ with $i \neq j \neq l \neq i$.

Obviously Γ_k is finite. Define $\phi_k : X_{k+1} \to X_k$ as the result of monoidal transformations performed at all points of Γ_k and $\pi_{k+1} = \pi_k \circ \phi_k$. It is well-known [**Ha**, Theorem V.3.9] that finally we get l such that π_l is a transversal resolution. We show that it is even a minimal transversal resolution. Let $\pi' : X' \to S$ be any transversal resolution. As π is minimal, π' factors through π. We shall use induction to show that ψ can be factored through each π_k. We already know it for $k = 0$. Suppose that $\pi' = \pi_k \circ \psi$ for $k < l$, where $\psi : X' \to X_k$. The morphism ψ is a composition of monoidal transformations. Let $x \in \Gamma_k$. If $\tau : Y' \to X_k$ is a monoidal transformation at some point $y \neq x$, some neighbourhoods of x and $\tau^{-1}(x)$ are isomorphic. Hence $\tau^{-1}(x)$ also has one of the above properties (i–iii). On the other hand, monoidal transformations at y and at x commute. Therefore, one may suppose that all monoidal transformations at the points from Γ_k are among those that constitute ψ, i.e. ψ factors through ϕ_k and π'

factors through π_{k+1}. As a result, π' factors through π_l, hence the latter is indeed a minimal transversal resolution. □

If $\pi : X \to S$ is a minimal transversal resolution, define its *dual graph* as a *weighed graph* $\Gamma = \Gamma(S)$ such that:

- the vertices of Γ are the irreducible components of E, the exceptional curve of this resolution (or further their indices $i = 1, \dots, s$);
- the edges of Γ are singular points of E; if $x \in E_i \cap E_j$, the corresponding edge joins the vertices i and j;
- each vertex i has weight (g, d), where g is the genus of E_i and $d = -(E_i.E_i)$; if $g = 0$, i.e. $E_i \simeq \mathbb{P}^1$, we omit g in this pair writing d instead of $(0, d)$.

Note that there can be *multiple edges* between two vertices i, j in Γ: it just means that E_i and E_j have several intersection points.

3. Fundamental cycle

Consider a resolution $\pi : X \to S$ of a normal surface singularity. Let $E_1, E_2, \dots, E_s$ be irreducible components of the exceptional curve E. As we have already seen, there is an effective cycle $Z = \sum_{i=1}^{s} z_i E_i$ such that $(Z.E_i) \leqslant 0$ for all i. If $Z' = \sum_{i=1}^{s} z'_i E_i$ is another such cycle, one can easily see that $\min\{ Z, Z' \} = \sum_{i=1}^{s} \min\{ z_i, z'_i \} E_i$ also has this property. Hence there is the smallest effective cycle Z such that $(Z.E_i) \leqslant 0$ for all i. It is called the *fundamental cycle* of this resolution. Of course, if the exceptional curve E is irreducible, $Z = E$, but it is not the case in general situation (cf. Example 5.3).

There is a recursive procedure to calculate the fundamental cycle due to Laufer [**La2**]. It also gives information about the cohomologies of this cycle.

Proposition 3.1. *Define the cycles Z_k recursively:*

- $Z_0 = 0$,
- $Z_1 = E_{i_0}$ *for some (arbitrary)* i_0,
- $Z_{k+1} = Z_k + E_{i_k}$ *for some (arbitrary)* i_k *such that* $(Z_k.E_{i_k}) > 0$ *(if it exists).*

Then there is l such that $Z_l = Z$ is a fundamental cycle. Moreover, for each $k = 1, \dots, l$

(i) $$\mathrm{h}^0(\mathcal{O}_{Z_k}) = 1,$$

(ii) $$p(Z_k) = \sum_{j=0}^{k-1} \mathrm{h}^1(\mathcal{O}_{E_{i_j}}(-Z_j)),$$

where $p(C) = \mathrm{h}^1(\mathcal{O}_C)$ is the *arithmetic genus* of a curve C.

Proof. For the first assertion it is enough to verify that $Z_k \leqslant Z$ for all k such that Z_k can be constructed. It is so for $k = 1$. Let

$Z = \sum_{i=1}^s z_iE_i,\ Z_k = \sum_{i=1}^s c_iE_i$ with $c_i \leqslant z_i$, and Z_{k+1} can be constructed. If $c_i = z_i$, then $(Z_k.E_i) \leqslant (Z.C_i)$, because $(E_j.E_i) \geqslant 0$ for $j \neq i$. Hence $c_{i_k} < z_{i_k}$, so $Z_{k+1} \leqslant Z$.

Now the exact sequence (1.2) for $C' = Z_k,\ C = E_{i_k}$ (thus $C + C' = Z_{k+1}$) gives

$$0 \to \mathcal{O}_{E_{i_k}}(-Z_k) \longrightarrow \mathcal{O}_{Z_{k+1}} \longrightarrow \mathcal{O}_{Z_k} \to 0,$$

and $\mathrm{h}^0(\mathcal{O}_{E_{i_k}}(-Z_k)) = 0$ since $(Z_k.E_{i_k}) > 0$. So the exact sequence of cohomologies is

$$\begin{aligned}(3.1)\quad 0 \to \mathrm{H}^0(\mathcal{O}_{Z_{k+1}}) \longrightarrow \mathrm{H}^0(\mathcal{O}_{Z_k}) \longrightarrow& \\ \longrightarrow \mathrm{H}^1(\mathcal{O}_{E_{i_k}}(-C_k)) \longrightarrow \mathrm{H}^1(\mathcal{O}_{Z_{k+1}}) &\longrightarrow \mathrm{H}^1(\mathcal{O}_{Z_k}) \to 0.\end{aligned}$$

As Z_1 is an irreducible reduced curve, $\mathrm{h}^0(Z_1) = 1$, hence $\mathrm{h}^0(Z_k) = 0$ for all k and the first mapping in (3.1) is an isomorphism. Thus $\mathrm{h}^1(\mathcal{O}_{Z_{k+1}}) = \mathrm{h}^1(\mathcal{O}_{Z_k}) + \mathrm{h}^1(\mathcal{O}_{E_{i_k}}(-C_k))$, wherefrom (ii) follows. $\square$

Remark 3.2. Note that $\mathcal{O}_C(-C') \simeq \mathcal{O}_X(-C)/\mathcal{O}_X(-C-C')$, so the formula (ii) above can be rewritten as

$$p(Z_k) = \sum_{j=0}^{k-1} \mathrm{h}^1(\mathcal{O}_X(-Z_j)/\mathcal{O}_X(-Z_{j+1})).$$

Moreover,

$$\begin{aligned}\mathrm{h}^1(\mathcal{O}_{E_{i_j}}(-Z_j)) &= -\chi(\mathcal{O}_{E_{i_j}}(-Z_j)) = \\ &= -\deg_{E_j} \mathcal{O}_{E_{i_j}}(-Z_j) - \chi(E_{i_j}) = \\ &= (Z_j.E_{i_j}) - 1 + p(E_{i_j})\end{aligned}$$

for $j > 0$. Thus

$$(3.2)\qquad p(Z_k) = \sum_{j=0}^{k-1}(p(E_{i_j}) + (Z_j.E_{i_j})) - k + 1.$$

In particular, this rule shows that $p(Z)$ only depends on genera $p(E_i)$ and intersection numbers $(E_i.E_j)$, and if $Z = \sum_{i=1}^s z_iE_i$, then $p(Z) \geqslant \sum_{i=1}^s z_ip(E_i)$.

Proposition 3.3. *Let $\pi : X \to S$ be a resolution with fundamental cycle Z, $\phi : Y \to X$ be a birational projective morphism. Then $Z^* = \phi^*Z$ is the fundamental cycle of the resolution $\pi \circ \phi : Y \to S$.*

Proof. We only have to consider the case when ϕ is a monoidal transformation at a point x. We use the notations and assertions of Proposition 2.2. Let E_i be the components of the exceptional curve on X. The components of the exceptional curve on Y are E_i' (strict transforms of E_i) and $L = \phi^{-1}(x)$. Let m_i be the multiplicity of x on E_i, n be its multiplicity on Z. Then $(Z^*.E_i') = (Z^*.E_i' + m_iL) = (Z^*.E_i^*) = (Z.E_i) \leqslant 0$. On the contrary, we can write any effective cycle D on Y as a sum $C' + lL$, where C' is the strict transform of an effective cycle C on X. Then $(D.L) = (C^* + (l-m)L.L) = m - l$, where m is the multiplicity of x on C, so $(D.L) \leqslant 0$ implies $l \geqslant m$. Now $(D.E_i') = (C^* + (l-m)L.E_i') = (C^*.E_i') + (l-m)m_i = (C^*.E_i^*) + (l-m)m_i \geqslant (C.E_i)$. Hence $(D.E_i') \leqslant 0$ implies that $D \geqslant C^*$ and $(C.E_i) \leqslant 0$, i.e. $C \geqslant Z$ and $D \geqslant Z^*$. So Z^* is indeed the fundamental cycle on Y. □

4. Cohomological cycle

We study cohomological properties of the resolution $\pi : X \to S$, especially $R^1\pi_*\mathcal{O}_X$. As S is affine, we may (and shall) identify any coherent sheaf $\mathcal{F}$ on S with $\mathbf{A}$-module $\Gamma(S,\mathcal{F})$. In particular, we identify $R^1\pi_*\mathcal{O}_X$ with $\Gamma(S, R^1\pi_*\mathcal{O}_X)$. But this module is isomorphic to $\mathrm{H}^1(X,\mathcal{O}_X)$, since $\Gamma(S,\pi_*\mathcal{F}) \simeq \Gamma(X,\mathcal{F})$ for every $\mathcal{F}$ and the functor $\Gamma(S, \_)$ is exact. It so happens that $\mathrm{H}^1(X,\mathcal{O}_X)$ can be calculated from some effective cycle.

Theorem 4.1. *There is an effective cycle Z_h such that:*

1. $\mathrm{h}^1(\mathcal{O}_{Z_h}) \geqslant \mathrm{h}^1(\mathcal{O}_C)$ *for every effective cycle C.*
2. Z_h *is the smallest effective cycle with this property.*
3. $\mathrm{H}^1(X,\mathcal{O}_X) \simeq \mathrm{H}^1(\mathcal{O}_{Z_h})$.

The cycle Z_h is called the *cohomological cycle* of the resolution $\pi : X \to S$.

Proof. We start from the

Lemma 4.2. *Suppose that a symmetric bilinear form satisfies conditions of Lemma 1.4. Given any integers c_i, there is a vector v such that $(v.e_i) \leqslant c_i$ for all i.*

Proof. Use induction. For $s = 1$ the claim is obvious, and we have seen in the proof of lemma 1.4 that the conditions remain valid for the restriction of the form onto the subgroup generated by a part of basic elements. Find i such that $(z.e_i) < 0$, let it be $i = s$. We may suppose that there is $u \in \langle e_1, e_2, \dots, e_{s-1} \rangle$ such that $(u.e_i) \leqslant c_i$ for $i < s$. Then $(u + kz.e_i) \leqslant (u.e_i) \leqslant c_i$ for $i < s$, and $(u + kz.e_s) \leqslant c_s$ for big enough k. □

Find now an effective cycle D such that $(D.E_i) \leqslant -(K_X.E_i)$, so $(K_X + D.E_i) \leqslant 0$. For any positive cycle C the exact sequence

$$0 \to \mathcal{O}_C(-D) \longrightarrow \mathcal{O}_{D+C} \longrightarrow \mathcal{O}_D \to 0$$

induces the exact sequence

$$\mathrm{H}^1(\mathcal{O}_C(-D)) \longrightarrow \mathrm{H}^1(\mathcal{O}_{D+C}) \longrightarrow \mathcal{O}_D \to 0.$$

Moreover, by Serre's duality, $\mathrm{H}^1(\mathcal{O}_C(-D)) \simeq \mathrm{D}\mathrm{H}^0(\mathcal{O}_C(K + C + D))$, since $\omega_C \simeq \mathcal{O}_C \otimes \omega_X(C) \simeq \mathcal{O}_C(K + D)$. But $(K + C + D.C) \leqslant$

$(C.C) < 0$, so $\mathrm{H}^0(\mathcal{O}_C(K+C+D)) = 0$ and $\mathrm{H}^1(\mathcal{O}_{D+C}) \simeq \mathrm{H}^1(\mathcal{O}_D)$. Thus $\mathrm{h}^1(\mathcal{O}_D)$ is the maximal possible.

Let now C also have this property, $M = \min\{C, D\}$, $C = M + A$, $D = M + B$, where A, B are effective cycles without common components. Set $N = A + B + M$. Then we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_A(-D) & \longrightarrow & \mathcal{O}_N & \longrightarrow & \mathcal{O}_D & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & \mathcal{O}_C(-M) & \longrightarrow & \mathcal{O}_C & \longrightarrow & \mathcal{O}_M & \longrightarrow & 0 \\ & & & & \big\downarrow & & \big\downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

The morphism in the first column is a monomorphism with cokernel isomorphic to the skyscraper sheaf $\mathcal{O}_A \otimes \mathcal{O}_B$. As $\mathrm{H}^1(\mathcal{O}_A \otimes \mathcal{O}_B) = 0$, we get a commutative diagram of cohomologies

$$\begin{array}{ccccccc} \mathrm{H}^1(\mathcal{O}_A(-D) & \longrightarrow & \mathrm{H}^1(\mathcal{O}_N) & \longrightarrow & \mathrm{H}^1(\mathcal{O}_D) & \longrightarrow & 0 \\ \big\downarrow & & \big\downarrow & & \big\downarrow & & \\ \mathrm{H}^1(\mathcal{O}_C(-M) & \longrightarrow & \mathrm{H}^1(\mathcal{O}_C) & \longrightarrow & \mathrm{H}^1(\mathcal{O}_M) & \longrightarrow & 0 \\ \big\downarrow & & \big\downarrow & & \big\downarrow & & \\ 0 & & 0 & & 0 & & \end{array}$$

It induces an exact sequence

$$\mathrm{H}^1(\mathcal{O}_N) \longrightarrow \mathrm{H}^1(\mathcal{O}_C) \oplus \mathrm{H}^1(\mathcal{O}_D) \longrightarrow \mathrm{H}^1(\mathcal{O}_M) \to 0.$$

Thus $\mathrm{h}^1(\mathcal{O}_M) \geqslant \mathrm{h}^1(\mathcal{O}_C) + \mathrm{h}^1(\mathcal{O}_D) - \mathrm{h}^1(\mathcal{O}_N) \geqslant \mathrm{h}^1(\mathcal{O}_D)$, since $\mathrm{h}^1(\mathcal{O}_C) = \mathrm{h}^1(\mathcal{O}_D) \geqslant \mathrm{h}^1(\mathcal{O}_N)$. Therefore $\mathrm{h}^1(\mathcal{O}_M) = \mathrm{h}^1(\mathcal{O}_D)$. It evidently implies that the smallest divisor Z_h with this property exists.

By the theorem on formal functions [**Ha**, Theorem III.11.1] $\widehat{R^1\pi_*\mathcal{O}_X} \simeq \varprojlim_D \mathrm{H}^1(\mathcal{O}_D)$, where D runs through effective cycles. But the mappings $\mathrm{H}^1(\mathcal{O}_D) \to \mathrm{H}^1(\mathcal{O}_C)$ are bijective for $D > C \geqslant Z_h$, hence $R^1\pi_*\mathcal{O}_X \simeq \mathrm{H}^1(\mathcal{O}_{Z_h})$. (Since it is finite dimensional, no completion is needed.) □

Remark 4.3. It is possible that $\mathrm{H}^1(X, \mathcal{O}_X) = 0$; such singularities are called *rational.* Then $Z_h = 0$. The Laufer procedure (Proposition 3.1) shows that it is only possible if all components E_i are projective lines, i.e. $p(E_i) = 0$, and $(Z_j.E_{i_j}) = 1$ for all steps of this algorithm, in particular $(E_i.E_j) \leqslant 1$ for all $i \neq j$. On the other hand, if these conditions hold, $\mathrm{H}^1(\mathcal{O}_Z) = 0$. If, moreover, the resolution is minimal, so $(E_i.E_i) \leqslant -2$, the adjunction formula (Proposition 1.2(5)) gives $(K.E_i) \geqslant 0$. Thus $(Z.E_i) \leqslant 0 \leqslant (K.E_i)$, so the proof of Theorem 4.1 shows that $Z_h \leqslant Z$ and $\mathrm{H}^1(\mathcal{O}_X) = \mathrm{H}^1(\mathcal{O}_Z) = 0$, i.e. the singularity is rational. Note that Proposition 3.1(6) together with Proposition 3.3 shows that the value $\chi(Z) = -(K + Z.Z)/2$ does not change under a monoidal transformation, thus holds for each resolution if it holds for one of them. So a singularity is rational if and only if $p(Z) = 0$ for the fundamental cycle of some (then of any) resolution.

5. Examples

We consider several examples of surface singularities. All of them are indeed *hypersurface singularities*, i.e. those of surfaces embedded in $\mathbb{A}^3$, hence given by one equation $F(x_1, x_2, x_3) = 0$. We always suppose that $F(0,0,0) = 0$ and take for $\mathbf{A}$ the local ring of the point $p = (0,0,0)$. It is always Cohen–Macaulay [**Ha**, Proposition II.8.23], so it is normal if and only if p is an isolated singularity. Note that p is a singular point if and only if F contains no linear terms. We also suppose that $\operatorname{char} \mathbf{k} = 0$. Remind that the monoidal transformation at the point p replace $S = \operatorname{Spec} \mathbf{A}$ by the closure $Y \subset S \times \mathbb{P}^2$ of the sub-scheme $\widetilde{Y} \subseteq U \times \mathbb{P}^2$, where $U = S \setminus \{ p \}$ and $\widetilde{Y}$ is given by the equations $\xi_i x_j = \xi_j x_i$, $(\xi_1 : \xi_2 : \xi_3)$ being homogeneous coordinates in $\mathbb{P}^3$. Actually Y is covered by three affine sheets Y_j $(j = 1,2,3)$ respectively to three copies of $\mathbb{A}^2$ covering $\mathbb{P}^3$. Namely, Y_j is the closure in $S \times \mathbb{A}^2$ of the sub-scheme $\widetilde{Y}_j \subseteq U \times \mathbb{A}^2$ given by the equations $x_i = \lambda_i x_j$, where $i \in \{1,2,3\}$, $i \neq j$. Note that here U can be given by one inequality $x_j \neq 0$. The pre-image of p is given on the sheet Y_j by the equation $x_j = 0$. If S was an isolated singularity, all singularities of Y are sitting on this curve.

Example 5.1. The simplest surface singularity is the *ordinary double point* $x_1^2 + x_2^2 + x_3^2 = 0$. Perform the monoidal transformation at the point p. It gives:

$$\widetilde{Y}_1 : \quad x_2 = \lambda_2 x_1,\ x_3 = \lambda_3 x_1,\ x_1^2 + \lambda^2 x_1^2 + \lambda_3 x_1^2,\ x_1 \neq 0,$$

hence

$$Y_1 : \quad \lambda_2^2 + \lambda_3^2 + 1 = 0 \ \text{ (embedded in } \mathbb{A}^3 \text{ with coordinates } x_1, \lambda_2, \lambda_3).$$

So Y_1 is a quadratic cylinder and has no singular points. The same is for Y_j, $j = 2,3$. Thus $\tau : Y \to S$ is a (minimal) resolution of this singularity. The exceptional curve E (its part in Y_1) is given by the equation $x_1 = 0$; it is a conic.

To calculate the intersection number $(E.E)$ we use a simple property of the definitions from Section 1.

Proposition 5.2. *Let X be a smooth surface, $f \in K(X)$ be a rational function, (f) be its divisor, and E be a projective curve on X. Then $((f).E) = 0$.*

Proof. By definition, $((f).E) = \deg_E(\mathcal{O}_X((f)) \otimes \mathcal{O}_E) = \deg_E(\mathcal{O}_E) = 0$, because $\mathcal{O}_X((f)) \simeq \mathcal{O}_X$. □

In our example each of the functions x_j has a zero of the first degree on E. But, say, x_3 has two more zeros given on Y_1 by the equation $\lambda_3 = 0$, or $\lambda_2 = \pm\sqrt{-1}$. Hence $(x_3) = E + C_1 + C_2$. Moreover, $C_1 \cap C_2 = \emptyset$ and both of them intersect E transversally at one point. So $((x_3).E) = (E.E) + (C_1.E) + (C_2.E) = (E.E) + 2 = 0$ and $(E.E) = -2$. Since $E \simeq \mathbb{P}^1$, the dual graph of our singularity is just

$$\underset{2}{\bullet}$$

As $Z = E$ and $p(E) = 0$, this singularity is rational.

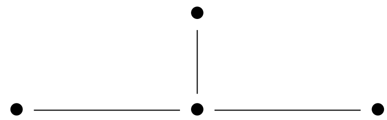
Example 5.3. The singularity *of type* D_4 is given by the equation $x_1^2 = x_2^3 - x_2 x_3^2$. Performing the monoidal transformation, get

$$\begin{aligned}
&Y_1: \quad 1 = x_1(\lambda_2^3 - \lambda_2\lambda_3^2), \quad \tau^{-1}(p) \cap Y_1 = \emptyset,\\
&Y_2: \quad \lambda_1^2 = x_2(1 - \lambda_3^2), \quad \tau^{-1}(p) \cap Y_2: \ x_2 = \lambda_1 = 0,\\
&Y_3: \quad \lambda_1^2 = x_3(\lambda_2^3 - \lambda_2), \quad \tau^{-1}(p) \cap Y_3: \ x_3 = \lambda_1 = 0.
\end{aligned}$$

In particular, Y_1 is smooth; the singular points on Y_3 are $p_1 = (0,0,0)$, $p_2 = (0,0,1)$, $p_3 = (0,0,-1)$; the singular points on Y_2 are the same p_2, p_3 (in a different presentation, of course). The pre-image of p consists of one component E_0 isomorphic to $\mathbb{P}^1$.

In a neighbourhood of p_1 we can consider $y_1 = \lambda_1$, $y_2 = \lambda_2^3 - \lambda_2$, $y_3 = \lambda_3$ as local coordinates on Y_3. So its equation becomes $y_1^2 = y_2 y_3$, that of an ordinary double point. Therefore a monoidal transformation at p_1 resolves it. The same is the case with the points p_2, p_3. If we perform all three monoidal transformation, we get a (minimal) resolution of our singularity. Each of them gives a new component E_k of the exceptional curve ($k = 1,2,3$). For instance, the equations of E_1 on the second sheet are $y_2 = 0,\ \lambda_1^2 = \lambda_3,$ (the latter is the equation of this sheet itself). The equations of the pre-image of E_0 on the same sheet are $\lambda_1 = \lambda_3 = 0$, so it intersects E_1 transversally. The same is true for E_2, E_3.

To calculate self-intersection numbers, consider the divisor (x_1). On Y it has zeros at E_0 and on the curves C_k $(k = 1,2,3)$ that have on Y_3 the equations $\lambda_1 = 0$ and, respectively, $\lambda_2 = 0, 1, -1$. They intersect E_0 transversally at the points, respectively, p_k. Hence after monoidal transformations at p_k the (strict) pre-images of E_0 and C_k do not meet at all, but both of them intersect E_k transversally. As x_1 becomes $y_1 y_3$ on Y_3, it has a zero of order 1 on each C_k. On the second sheet of he monoidal transformation at p_1, x_1 becomes $\lambda_1\lambda_3 y_2^2 = \lambda_1^3 y_2$, so it has a zero of order 2 on E_1 and a zero of order 3 on E_0. Thus $(x_1) = 3E_0 + 2(E_1 + E_2 + E_3) + (C_1 + C_2 + C_3)$, wherefrom one easily gets $(E_k.E_k) = -2$ for $k = 0,1,2,3$. Therefore the dual graph of this singularity is



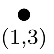
with all weights equal 2.

Find the fundamental cycle Z of this resolution using the Laufer procedure. Starting from $Z_1 = E_0$, we get

$$Z_2 = Z_1 + E_1,\ Z_3 = Z_2 + E_2,\ Z_4 = Z_3 + E_3,\ Z_5 = Z_4 + E_0,$$

and $Z = Z_5 = 2E_0 + E_1 + E_2 + E_3$ (in particular, $Z \neq E$ and is not reduced). Moreover, the formula (3.2) gives $p(Z) = 0$. So this singularity is also rational.

Example 5.4. Let $S:\ x_1^3+x_2^3+x_3^3 = 0$. The monoidal transformation at p gives for Y_1 the equation $\lambda_2^3 + \lambda_3^3 + 1 = 0$. It is smooth, as well as two other sheets, so $Y \to S$ is a minimal resolution. The exceptional curve E is a plane smooth cubic given by the intersection of Y_1 with $x_1 = 0$. The same curve we obtain on two other sheets too. All functions x_i have simple zeros on E. Other zeros, say, of x_2 on Y_1 are $\lambda_2 = 0,\ \lambda_3^3 = -1$. There are three of them, intersecting E transversally. Hence $(E.E) = -3$ and the dual graph is



Here $Z = E$, $p(E) = 1$ and $(E + K.E) = -2\chi(E) = 0$, thus the proof of Theorem 4.1 gives $Z_h = E$ and $\mathrm{h}^1(\mathcal{O}_X) = 1$. In particular, this singularity is not rational.

Example 5.5. Our last example is the singularity *of type* T_{237} given by the equation $x_1^2 = x_2^3+x_2^2x_3^2+x_3^7$. Blowing up at the point $p = (0,0,0)$ gives nothing on the first sheet. On the second sheet we have

$$\lambda_1^2 = x_2 + x_2^2\lambda_3^2 + x_2^5\lambda_3^7,$$

so $\tau^{-1}(p)$ is $x_2 = \lambda_1 = 0$, which contains no singular points. On the third sheet we have

$$\lambda_1^2 = \lambda_2^3 x_3 + \lambda_2^2 x_3^2 + x_3^5,$$

so $\tau^{-1}(p)$ is $E_1 : x_3 = \lambda_1 = 0$. The unique singular point is $q = (0,0,0)$. Rewrite it in new coordinates as $y_1^2 = y_2^3 y_3 + y_2^2 y_3^2 + y_3^5$. Blowing it up gives nothing on the first sheet again. On the second sheet we get

$$\lambda_1^2 = y_2^2(\lambda_3 + \lambda_3^2 + y_2\lambda_3^5).$$

Now one can see that thus obtained singularity is not normal: the function $\eta = \lambda_1/y_2$ belongs to the integral closure of its coordinate ring. Adding it, we obtain the equation

$$\eta^2 = \lambda_3 + \lambda_3^2 + y_2\lambda_3^5.$$

It defines a smooth surface. The strict pre-image of E_1 is $\eta = \lambda_3 = 0$, and the pre-image of q is E_2 : $y_2 = 0, \eta^2 = \lambda_3 + \lambda_3^2$. They intersect transversally at the point $(0,0,0)$. There are no singular points on this sheet.

On the third sheet we obtain

$$\lambda_1^2 = y_3^2(\lambda_2^3 + \lambda_2^2 + y_3),$$

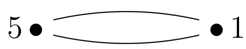
which is again non-normal. To normalize, add the function $\zeta = \lambda_1/y_3$ getting

$$\zeta^2 = \lambda_2^3 + \lambda_2^2 + y_3.$$

The exceptional curve, which coincide with E_2, is $y_3 = 0,\ \lambda_1^2 = \lambda_2^3 + \lambda_2^2$. There are no singular points on this sheet too, so we have got a resolution $\psi : Y \to S$. This time it is neither minimal nor transversal. Indeed, the curve E_2 is not smooth: on the third sheet it has a singular point $\lambda_2 = \lambda_3 = 0$ (an ordinary node, or double point). On the other hand, calculating the divisor (x_1) gives $(x_1) = 3E_1 + 3E_2 + A$, where A is the curve given, say, on the third sheet after the first blowing up by the equations $y_1 = 0 = y_2^3 + y_2^2 y_3 + y_3^4$. It intersects E_1 transversally at the point q, hence does not intersect it after the second blowing up. Its equations on the third sheet sheet after normalization become $\zeta = 0 = \lambda_2^3 + \lambda_2^2 + y_3$, Hence its intersection with E_2 consists of two points $(0,0,0)$ and $(0,-1,0)$; the first one being of multiplicity 2. Thus $(E_1.E_2) = 1,\ (A.E_2) = 3,\ (A.E_1) = 0$, wherefrom $(E_1.E_1) = -1,\ (E_2.E_2) = -2$. So E_1 is a contractible line and $\psi = \pi \circ \sigma$, where $\pi : X \to S$ is a minimal resolution and $\sigma : Y \to X$ is a blowing up with the exceptional line E_1. Denote by E the image of E_2 on X. Accordingly to Proposition 2.2(5), $(E.E) = -1$.

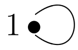
Just as in the preceding example, $Z = Z_h = E$, so $\mathrm{h}^1(\mathcal{O}_X) = 1$ and this singularity is also non-rational.

To get a minimal transversal resolution, we must blow up the singular point e of E (one blowing up is enough since it is an ordinary double point). After such a transformation we get $(E'.E') = -5$, where $E' = \sigma' E$ (again by Proposition 2.2(5)), so the dual graph of our singularity is



(the second vertex corresponds to the new exceptional line L, the pre-image of e). For this resolution one can easily check that the fundamental cycle is $Z = E' + 2L$. On the other hand, since $p(E') = p(L) = 1$, one can calculate $(K.E') = 3,\ (K.L) = -1$. The Laufer algorithm (Proposition 3.1) shows that $\mathrm{h}^1(E' + L) = 1$. Moreover, $(E' + L.L) = 1 = -(K.L)$ and $(E' + L.E') = -3 = -(K.E')$. Thus the proof of Theorem 4.1 shows that $Z_h \leqslant E' + L$, where Z_h is the cohomological cycle. As $\mathrm{h}^1(E') = \mathrm{h}^1(L) = 0$, $Z_h = E' + L$.

Note that sometimes one allows, on a transversal resolution, ordinary double points not only as intersections of components, but also as singular points of components of the exceptional curve, presenting them at the dual graph as *loops*. The genus that occurs in weights is the *geometric genus*, which equals $p(E_i) - \delta$, where δ is the number of singular points, and again genus 0 is omitted. Then the minimal resolution of our singularity, which satisfies this condition, has the dual graph



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