

ON COHEN–MACAULAY MODULES ON SURFACE SINGULARITIES

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INTRODUCTION

During the study of Cohen–Macaulay modules on curve singularities (cf. [18, 12, 14, 9]) it was proved that these singularities split into three classes:

- *Cohen–Macaulay finite*, having only finitely many indecomposable Cohen–Macaulay modules;
- *Cohen–Macaulay tame*, i.e., such that, for each fixed r , the indecomposable Cohen–Macaulay modules of rank r form a finite set of 1-parameter families;
- *Cohen–Macaulay wild*, that can be characterized in two ways:
 - “geometrically” as those having n -parameter families of non-isomorphic indecomposable Cohen–Macaulay modules for arbitrary large n ,
 - “algebraically” as such that for every finitely generated algebra Λ there is an exact functor from the category of finite dimensional Λ -modules to the category of Cohen–Macaulay modules over this singularity, which maps non-isomorphic modules to non-isomorphic ones and indecomposable to indecomposable.

The latter property shows that the study of modules in the wild case is extremely complicated and needs an essentially new and highly non-trivial approach.

Moreover, it turned out that the above “trichotomy” is closely related to the position of a curve singularity in the well-known Arnold’s list of singularities having good deformation properties (cf. [1]). Namely:

- a singularity is Cohen–Macaulay finite if and only if it dominates one of the simple plane curves singularities A_n, D_n, E_6, E_7, E_8 ;
- a singularity is Cohen–Macaulay tame if and only if it dominates one of the “serial” unimodal singularities T_{pq} .

(Recall that a singularity (X, x) *dominates* (Y, y) if there is a birational surjection $(X, x) \rightarrow (Y, y)$.)

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In [13, 4] it was proved that a *normal surface singularity* (in characteristic 0) is Cohen–Macaulay finite if and only if it is a *quotient singularity*. In [19] it was shown that a *simple elliptic* singularity is always Cohen–Macaulay tame. In both cases a complete description of Cohen–Macaulay modules over such singularities was obtained. Moreover, in [19] Kahn elaborated a general method relating Cohen–Macaulay modules over normal surface singularities to vector bundles over some projective curves (usually singular and even non-reduced).

In this paper we use the Kahn correspondence and the results of [10] to prove the following main theorems:

- every cusp singularity (cf. [17, 20]) is Cohen–Macaulay tame;
- a minimally elliptic singularity (cf. [23]) that is neither simple elliptic nor a cusp one is Cohen–Macaulay wild.

As a corollary we get that any *log-canonical* surface singularity (cf. [21]) is Cohen–Macaulay tame (or finite, as quotient singularities are also log-canonical). There is some evidence that any other normal surface singularity is Cohen–Macaulay wild. We also give a description of Cohen–Macaulay modules over cusp singularities and use it to provide a description of Cohen–Macaulay modules over curve singularities of type T_{pq} (there was no such description in [9], the proof of their tameness was indirect).

1. GENERALITIES

In what follows we use the following

Definitions and Notations 1.1. • A *surface singularity* means a spectrum $X = \operatorname{Spec} A$, where A is a local, complete¹ noetherian ring of Krull dimension 2. We denote by $\mathfrak{m} = \mathfrak{m}_A$ the maximal ideal of A and by $\mathbf{k} = A/\mathfrak{m}$ the residue field. We always suppose that the field \mathbf{k} is algebraically closed.

- Such a singularity is called *normal* if the ring A is normal, i.e., integral and integrally closed in its field of fractions Q . In what follows we only consider normal surface singularities. Recall that any normal surface singularity is *isolated*, that is \mathfrak{m} is the unique singular point of it.
- A *resolution* of a normal surface singularity is a birational proper map $\pi : \tilde{X} \rightarrow X$, where \tilde{X} is smooth, which induces an isomorphism $\tilde{X} \setminus E \xrightarrow{\sim} X \setminus \{\mathfrak{m}\}$, where $E = \pi^{-1}(\mathfrak{m})_{\text{red}}$. E is called the *exceptional curve* on \tilde{X} (it is indeed a projective curve over \mathbf{k}). Denote by E_i ($i = 1, \dots, s$) the irreducible components of E . Put also $\tilde{X} = X \setminus \{\mathfrak{m}\}$.

¹For the analytic case, see Remark 8.3 at the end of the paper.

- A resolution π as above is called *minimal* if it cannot be decomposed as

$$\tilde{X} \xrightarrow{\pi_1} X' \xrightarrow{\pi_2} X,$$

where X' is also smooth. It is known (cf., e.g., [24]) that π is minimal if and only if neither of the components E_i is a smooth rational curve with self-intersection index $E_i \cdot E_i = -1$.

- A *cycle* on \tilde{X} is a divisor $Z = \sum_{i=1}^s k_i E_i$ with $k_i \in \mathbb{Z}$. Such a cycle is called *positive* if all $k_i > 0$. We write $Z \leq Z'$ if $Z' = \sum_{i=1}^s k'_i E_i$ with $k_i \leq k'_i$ for all i .
- The *fundamental cycle*, denoted by Z_0 , is the minimal positive cycle (with respect to the above defined partial order) such that $Z_0 \cdot E_i \leq 0$ for all i .
- A positive cycle Z is called a *weak reduction cycle* if the sheaf $\mathcal{O}_Z(-Z)$ is *generically generated* by its global section (i.e. generated outside a finite set) and $H^1(E, \mathcal{O}_Z(-Z)) = 0$. Here, as usually, we write $\mathcal{O}_Z = \mathcal{O}_{\tilde{X}}/\mathcal{O}_{\tilde{X}}(-Z)$ and consider Z as a closed subscheme of \tilde{X} .
- A *reduction cycle* is a weak reduction cycle Z such that the sheaf ω_Z^\vee is generically generated by global sections, where $\omega_Z = \omega_{\tilde{X}} \otimes_{\mathcal{O}_{\tilde{X}}} \mathcal{O}_Z(-Z)$ is the dualizing sheaf for Z and \mathcal{M}^\vee denotes $\mathcal{H}om_{\mathcal{O}_S}(\mathcal{M}, \mathcal{O}_S)$, where \mathcal{M} is a coherent sheaf on a scheme S .
- One calls a surface singularity *rational* if $H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) = 0$ and *minimally elliptic* if it is Gorenstein (i.e. $\omega_X \simeq \mathcal{O}_X$) and $H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) \simeq \mathbf{k}$.
- X is called a *simple elliptic singularity* if E is a smooth curve of genus 1.
- We call the exceptional curve E a *cyclic configuration* in the following cases:
 - (i) $s = 1$, E is rational and has a unique singular point that is a simple node;
 - (ii) $s = 2$, $E_1 \simeq E_2 \simeq \mathbb{P}^1$ and they intersect transversally in (exactly) 2 points;
 - (iii) $s > 2$, $E_i \simeq \mathbb{P}^1$ for all i , $E_i \cdot E_{i+1} = 1$, $E_s \cdot E_1 = 1$, and $E_i \cdot E_j = 0$ otherwise.
 For a cyclic configuration we set $E_{s+k} = E_k$ for all integers k .
- X is called a *cusp singularity* if E is a cyclic configuration. Note that simple elliptic and cusp singularities are both minimally elliptic. Moreover, for all of them $Z_0 = E$.
- Denote by $\text{Coh}(S)$ the category of coherent sheaves on a scheme S and by $\text{VB}(S)$ the category of *vector bundles*, that is locally free coherent sheaves on S .
- For a surface singularity $X = \text{Spec } A$ denote by $\text{CM}(X)$, or by $\text{CM}(A)$, the category of (maximal) Cohen–Macaulay modules over the ring A . If A is normal, Cohen–Macaulay modules

coincide with *reflexive* modules, i.e. such A -modules M that $M^{\vee\vee} \simeq M$, where $M^\vee = \text{Hom}_A(M, A)$. We always identify an A -module M with its “sheafification” \widetilde{M} , which is a quasi-coherent sheaf on X .

Recall the main result of Kahn [19, Theorem 1.4] concerning the relations between the vector bundles on a reduction cycle and the Cohen–Macaulay modules over a normal surface singularity.

Theorem 1.2. *Let $\pi : \widetilde{X} \rightarrow X$ be a resolution of a normal surface singularity with exceptional curve E and a reduction cycle Z . Denote by $R_Z : \text{CM}(X) \rightarrow \text{VB}(Z)$ the functor that is the composition of the functors*

$$\text{CM}(X) \rightarrow \text{VB}(\widetilde{X}) : M \mapsto (\pi^* M)^{\vee\vee}$$

and

$$\text{VB}(\widetilde{X}) \rightarrow \text{VB}(Z) : \mathcal{F} \mapsto \mathcal{F} \otimes_{\mathcal{O}_{\widetilde{X}}} \mathcal{O}_Z.$$

This functor maps non-isomorphic objects to non-isomorphic ones, and a vector bundle $\mathcal{V} \in \text{VB}(Z)$ is isomorphic to $R_Z M$ for some M if and only if it is generically generated by global sections and there is an extension of \mathcal{V} to a vector bundle \mathcal{V}_2 on $2Z$ such that the exact sequence

$$0 \longrightarrow \mathcal{V}(-Z) \longrightarrow \mathcal{V}_2 \longrightarrow \mathcal{V} \longrightarrow 0,$$

induces a monomorphism $H^0(E, \mathcal{V}(Z)) \rightarrow H^1(E, \mathcal{V})$.

We denote by $\text{VB}^K(Z)$ the full subcategory of $\text{VB}(Z)$ consisting of the vector bundles satisfying the latter Kahn’s conditions. Note that obviously $\text{rk } R_Z M = \text{rk } M$ for any Cohen–Macaulay module M .

For minimally elliptic singularities one can give a simpler description of the latter category (cf. [19, Theorem 2.1]).

Theorem 1.3. *Let $\pi : \widetilde{X} \rightarrow X$ be a minimal resolution of a minimal elliptic singularity. Then the fundamental cycle $Z = Z_0$ is a reduction cycle and the category $\text{VB}^K(Z)$ consists of all vector bundles of the form $n\mathcal{O}_Z \oplus \mathcal{G}$, where \mathcal{G} satisfies the following conditions:*

- (1) \mathcal{G} is generically generated by global sections,
- (2) $H^1(E, \mathcal{G}) = 0$,
- (3) $\dim_{\mathbf{k}} H^0(E, \mathcal{G}(Z)) \leq n$.

In particular, indecomposable objects of $\text{VB}^K(Z)$ are the following:

- the trivial line bundle \mathcal{O}_Z ,
- $n\mathcal{O}_Z \oplus \mathcal{G}$, where \mathcal{G} is an indecomposable vector bundle satisfying the above conditions (1),(2), and $n = \dim_{\mathbf{k}} H^0(E, \mathcal{G}(Z))$.

Remark. In the paper [19], Theorems 1.2 and 1.3 are proved in the “geometric” case, when A is a \mathbf{k} -algebra and $\text{char } \mathbf{k} = 0$, but one easily

verifies that all proofs can be directly extended to the general situation. The results of Laufer [23] used in [19] are also valid in this “abstract” context. Using the Grauert–Riemenschneider vanishing theorem to show that $H^1(E, \mathcal{O}_Z(-Z)) = 0$ for the fundamental cycle of a minimally elliptic singularity, is actually superfluous. Indeed, in this case $\omega_{\tilde{X}} = \mathcal{O}_{\tilde{X}}(-Z)$ [23, Theorem 3.4], so $\omega_Z \simeq \mathcal{O}_Z$. By the Serre’s duality [16, Corollary III.7.7], $H^1(E, \mathcal{O}_Z(-Z))$ is dual to $H^0(E, \omega_Z(Z)) = H^0(E, \mathcal{O}_Z(Z))$. The latter is zero since $\deg \mathcal{O}_Z(Z) = Z \cdot Z < 0$.

2. COHEN–MACAULAY MODULES ON CUSP SINGULARITIES

In this section E denotes a cyclic configuration, $\nu : \tilde{E} \rightarrow E$ the normalization of E and E_i ($i = 1, \dots, s$) the irreducible components of \tilde{E} (if $s > 1$, they can be identified with the irreducible components of E). We put $E_{s+k} = E_k$ for all integers k , $\mathcal{O} = \mathcal{O}_E$, $\tilde{\mathcal{O}} = \mathcal{O}_{\tilde{E}}$, $\mathcal{O}_i = \mathcal{O}_{E_i}$, and we always identify $\tilde{\mathcal{O}}$ and \mathcal{O}_i with their images under ν_* . Denote by $S = \{p_1, p_2, \dots, p_s\}$ the set of singular points of E and choose the indices in such a way that $\nu^{-1}(p_i) \subset E_i \cup E_{i+1}$. Note that the latter preimage always consists of 2 points, which we denote by p'_i, p''_{i+1} so that $p'_i \in E_i$ and $p''_{i+1} \in E_{i+1}$. Let \mathcal{J} be the *conductor* of $\tilde{\mathcal{O}}$ in \mathcal{O} , i.e. the biggest $\tilde{\mathcal{O}}$ -ideal contained in \mathcal{O} . Then $\mathcal{O}_i/\mathcal{J}\mathcal{O}_i = \mathbf{k}(p'_i) \oplus \mathbf{k}(p''_i)$.

Recall the description of vector bundles on a cyclic configuration given in [10]. Call an *s-sequence* any sequence of integers of length a multiple of s : $\mathbf{d} = (d_1, \dots, d_{rs})$. Call this sequence *aperiodic* if it cannot be obtained by repetition of a shorter s -sequence. Call a *shift* of \mathbf{d} any sequence $\mathbf{d}^k = (d_{k+1}, \dots, d_{rs}, d_1, \dots, d_k)$; if k is a multiple of s , we call such a shift an *s-shift*.

Theorem 2.1. *Indecomposable vector bundles on a cyclic configuration are in one-to-one correspondence with the triples $\mathbf{B} = (\mathbf{d}, m, \lambda)$, where m is a positive integer, $\lambda \in \mathbf{k}^* = \mathbf{k} \setminus \{0\}$ and \mathbf{d} is a aperiodic s -sequence defined up to an s -shift.*

We denote the vector bundle corresponding to the triple \mathbf{B} by $\mathcal{G}(\mathbf{B})$. The precise construction of this sheaf is the following:

- (1) Let $\mathbf{d} = (d_1, \dots, d_{rs})$. Put $\mathcal{G}_i = m\mathcal{O}_i(d_i)$ and $\tilde{\mathcal{G}}(\mathbf{B}) = \bigoplus_{i=1}^{rs} \mathcal{G}_i$. Note that $\mathcal{G}_i/\mathcal{J}\mathcal{G}_i \simeq m\mathbf{k}(p'_i) \oplus m\mathbf{k}(p''_i)$. Put also

$$F(\mathbf{B}) = \tilde{\mathcal{G}}(\mathbf{B})/\mathcal{J}\tilde{\mathcal{G}}(\mathbf{B}) \simeq \bigoplus_{i=1}^{rs} (m\mathbf{k}(p'_i) \oplus m\mathbf{k}(p''_i)).$$

Since $F(\mathbf{B})$ is a sky-scraper sheaf, we identify it with the vector space of its global sections.

- (2) Choose a \mathbf{k} -bases $\{\mathbf{e}'_{ik}, \mathbf{e}''_{ik}\}$ ($k = 1, \dots, m$) of $F(\mathbf{B})$ in such a way that $\{\mathbf{e}'_{ik}\}$ form a basis of $m\mathbf{k}(p'_i)$ and $\{\mathbf{e}''_{ik}\}$ form a basis of $m\mathbf{k}(p''_i)$.

- (3) Define the elements $\mathbf{e}_{ik} \in F(\mathbf{B})$ ($i = 1, \dots, rs, k = 1, \dots, m$) in the following way:

$$\mathbf{e}_{ik} = \begin{cases} \mathbf{e}'_{ik} + \mathbf{e}''_{i+1,k} & \text{if } i \neq rs, \\ \mathbf{e}'_{rs,k} + \lambda \mathbf{e}''_{1k} + \mathbf{e}''_{1,k-1} & \text{if } i = rs, k \neq 1, \\ \mathbf{e}'_{rs,1} + \lambda \mathbf{e}''_{11} & \text{if } i = rs, k = 1. \end{cases}$$

Denote by $G(\mathbf{B})$ the subspace of $F(\mathbf{B})$ with the basis

$$\{ \mathbf{e}_{ik} \mid i = 1, \dots, rs, k = 1, \dots, m \}.$$

- (4) Now $\mathcal{G}(\mathbf{B})$ is defined as the preimage of $G(\mathbf{B})$ in $\tilde{\mathcal{G}}(\mathbf{B})$ under the epimorphism $\mathcal{G}(\mathbf{B}) \rightarrow F(\mathbf{B})$. Note that $\text{rk } \mathcal{G}(\mathbf{B}) = mr$.

To use Kahn's Theorem we have to calculate the cohomologies of the sheaves $\mathcal{G}(\mathbf{B})$. Let $H(\mathbf{B}) = \tilde{\mathcal{G}}(\mathbf{B})/\mathcal{G}(\mathbf{B}) = F(\mathbf{B})/G(\mathbf{B})$. The exact sequence $0 \rightarrow \mathcal{G}(\mathbf{B}) \rightarrow \tilde{\mathcal{G}}(\mathbf{B}) \rightarrow H(\mathbf{B}) \rightarrow 0$ of sheaves over E gives the exact sequence of cohomology groups

$$0 \rightarrow H^0(\mathcal{G}(\mathbf{B})) \rightarrow H^0(\tilde{\mathcal{G}}(\mathbf{B})) \xrightarrow{h(\mathbf{B})} H(\mathbf{B}) \rightarrow H^1(\mathcal{G}(\mathbf{B})) \rightarrow H^1(\tilde{\mathcal{G}}(\mathbf{B})) \rightarrow 0$$

(we identify $H(\mathbf{B})$ with the space of its global sections and omit “ E ” in the notation of cohomology). One knows that $\dim_{\mathbf{k}} H^0(\tilde{\mathcal{G}}(\mathbf{B})) = m \sum_{i=1}^{rs} (d_i + 1)^+$ and $\dim_{\mathbf{k}} H^1(\tilde{\mathcal{G}}(\mathbf{B})) = m \sum_{i=1}^{rs} (d_i + 1)^-$, where

$$k^+ = \begin{cases} k & \text{if } k > 0, \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad k^- = \begin{cases} -k & \text{if } k < 0, \\ 0 & \text{otherwise,} \end{cases}$$

while $\dim_{\mathbf{k}} H(\mathbf{B}) = mrs$. So we only have to calculate $\text{rk } h(\mathbf{B})$. Denote by f the natural homomorphism $H^0(\tilde{\mathcal{G}}(\mathbf{B})) \rightarrow F(\mathbf{B})$. Then $\text{rk } h(\mathbf{B}) = \dim_{\mathbf{k}}(\text{Im } f + G(\mathbf{B})) - \dim_{\mathbf{k}} G(\mathbf{B})$.

Identifying E_i with \mathbb{P}^1 we can suppose that $p'_i = (1 : 0)$ and $p''_i = (0 : 1)$. If $d_i > 0$, a basis $\{g_0, \dots, g_{d_i}\}$ in $H^0(\mathcal{O}_i(d_i))$ can be chosen in such a way that $g_0(p'_i) = 1$, $g_1(p''_i) = 1$, while all other values $g_j(p'_i)$ and $g_j(p''_i)$ equal 0, so the basis elements \mathbf{e}'_{ik} and \mathbf{e}''_{ik} can be chosen as the images of g_0 and g_1 respectively. Hence $\text{Im } f$ is generated by the set $\{ \mathbf{e}'_{ik}, \mathbf{e}''_{ik} \mid d_i > 0 \} \cup \{ \mathbf{e}'_{ik} + \mathbf{e}''_{ik} \mid d_i = 0 \}$.

Call a subsequence $\mathbf{p} = (d_{k+1}, \dots, d_{k+l})$, where $0 \leq k < rs$ and $1 \leq l \leq rs$, a *positive part* of \mathbf{d} if $d_{k+j} \geq 0$ for $j = 1, \dots, l$ and either $l = rs$ or both $d_k < 0$ and $d_{k+l+1} < 0$. For such a positive part put $\theta(\mathbf{p}) = l$ if either $l = rs$ or $\mathbf{p} = (0, \dots, 0)$ and $\theta(\mathbf{p}) = l + 1$ otherwise. Denote by \mathcal{P} the set of all positive parts of \mathbf{d} and put $\theta(\mathbf{d}) = \sum_{\mathbf{p} \in \mathcal{P}} \theta(\mathbf{p})$. Put also $\delta(\mathbf{d}, \lambda) = 1$ if $\mathbf{d} = (0, \dots, 0)$ and $\lambda = 1$ and $\delta(\mathbf{d}, \lambda) = 0$ otherwise. The preceding consideration and a rather simple straightforward calculations show that, for $\mathbf{B} = (\mathbf{d}, m, \lambda)$,

$$\text{rk } H(\mathbf{B}) = m\theta(\mathbf{d}) - \delta(\mathbf{d}, \lambda),$$

whence the following:

Theorem 2.2. *For the vector bundle $\mathcal{G} = \mathcal{G}(\mathbf{d}, m, \lambda)$,*

$$\dim_{\mathbf{k}} H^0(E, \mathcal{G}) = m \left(\sum_{i=1}^{rs} (d_i + 1)^+ - \theta(\mathbf{d}) \right) + \delta(\mathbf{d}, \lambda)$$

$$\dim_{\mathbf{k}} H^1(E, \mathcal{G}) = m \left(\sum_{i=1}^{rs} (d_i + 1)^- + rs - \theta(\mathbf{d}) \right) + \delta(\mathbf{d}, \lambda).$$

We write $\mathbf{d} > 0$ if $d_i \geq 0$ for all $i = 1, \dots, rs$ and at least one of them is strictly positive. Denote by $\mathbf{0}$ the zero sequence $(0, \dots, 0)$ of length s (the unique aperiodic zero s -sequence) and set $|\mathbf{d}| = \sum_{i=1}^{rs} d_i$. Actually the rank of the vector bundle $\mathcal{G}(\mathbf{d}, m, \lambda)$ equals mr and its degree equals $m|\mathbf{d}|$.

Definition 2.3. We call an aperiodic s -sequence $\mathbf{d} = (d_1, d_2, \dots, d_{rs})$ *suitable* if $\mathbf{d} > \mathbf{0}$ and, for any shift \mathbf{d}^k of \mathbf{d} , \mathbf{d}^k does not contain a subsequence $(0, 1, 1, \dots, 1, 0)$ (in particular $(0, 0)$) and $\mathbf{d}^k \neq (0, 1, 1, \dots, 1)$.

Note that if \mathbf{d} is suitable, then $|\mathbf{d}| \geq r$ and $\dim H^0(E, \mathcal{G}) = m|\mathbf{d}|$, $\dim H^1(E, \mathcal{G}) = 0$ for $\mathcal{G} = \mathcal{G}(\mathbf{d}, m, \lambda)$.

Corollary 2.4. *A vector bundle $\mathcal{G} = \mathcal{G}(\mathbf{d}, m, \lambda)$ satisfies Kahn's conditions (1),(2) of Theorem 1.3 if and only if either it is suitable or $\mathbf{d} = \mathbf{0}$, $m = 1$ and $\lambda = 1$ (i.e. $\mathcal{G} \simeq \mathcal{O}$).*

Suppose now that the cycle configuration E is actually the exceptional curve of a cusp singularity $X = \text{Spec } A$. If $s > 1$, set $b_i = -E \cdot E_i = -E_i \cdot E_i - 2$. Note that the intersection indices $E_i \cdot E_i \leq -2$ and at least one of them is smaller than -2 . If $s = 1$, set $b_1 = -E \cdot E$. Let $\mathbf{b}^r = (b_1, \dots, b_s, \dots, b_1, \dots, b_s)$ (r times). We write \mathbf{b} for \mathbf{b}^1 . Put also $n(\mathcal{G}) = \dim_{\mathbf{k}} H^0(E, \mathcal{G}(E))$. Recall that in this case one can choose E itself as a reduction cycle.

Corollary 2.5. (1) *Suppose that a $\mathcal{G} = \mathcal{G}(\mathbf{d}, m, \lambda)$, where \mathbf{d} is a suitable sequence. Then*

$$n(\mathcal{G}) = m \left(\sum_{i=1}^{rs} (d_i - b_i + 1)^+ - \theta(\mathbf{d} - \mathbf{b}^r) \right) + \delta(\mathbf{d} - \mathbf{b}^r, \lambda),$$

so $n(\mathcal{G})\mathcal{O} \oplus \mathcal{G} \simeq R_E M(\mathbf{d}, m, \lambda)$, where $M(\mathbf{d}, m, \lambda)$ is a Cohen-Macaulay A -module of rank $rs + n(\mathcal{G})$.

(2) *All Cohen-Macaulay modules $M(\mathbf{d}, m, \lambda)$ are indecomposable and every indecomposable Cohen-Macaulay A -module is isomorphic either to A or to one of the modules $M(\mathbf{d}, m, \lambda)$, where \mathbf{d} is a suitable sequence.*

3. LIFTING FAMILIES

Vector bundles on cyclic configurations form natural 1-parametric families. We are going to show that these families can be lifted to 1-parametric families of Cohen-Macaulay modules on cusp singularities.

To do it, we first prove analogues of Kahn’s results for families of modules and vector bundles. For the sake of simplicity, we suppose from now on that A is a \mathbf{k} -algebra and all other algebras and schemes are \mathbf{k} -algebras and \mathbf{k} -schemes. Moreover, we only consider \mathbf{k} -schemes S such that $\mathbf{k}(x) = \mathbf{k}$ for each closed point $x \in S$ (for instance, schemes of finite type over \mathbf{k}). We write \otimes , \dim , \times , etc., instead of $\otimes_{\mathbf{k}}$, $\dim_{\mathbf{k}}$, $\times_{\mathrm{Spec} \mathbf{k}}$, etc. Especially “finite dimensional” means “finite dimensional over \mathbf{k} .”

We need also families with non-commutative base, so we give the corresponding definitions.

Definition 3.1. Let S be a \mathbf{k} -scheme and Λ be a \mathbf{k} -algebra (maybe non-commutative).

- A *family of \mathcal{O}_S -modules based on Λ* is, by definition, a coherent sheaf of $\mathcal{O}_S \otimes \Lambda$ -modules \mathcal{F} on S , flat over Λ .
- Such a family is called a *family of (maximal) Cohen–Macaulay modules* (respectively *vector bundles*) if for every finite dimensional Λ -module L the sheaf of \mathcal{O}_S -modules $\mathcal{F} \otimes_{\Lambda} L$ is a sheaf of Cohen–Macaulay \mathcal{O}_S -modules (respectively a locally free sheaf of \mathcal{O} -modules).
(Evidently, these properties have only to be verified for simple Λ -modules.)
- We denote $\mathcal{F}(L) = \mathcal{F} \otimes_{\Lambda} L$.
- We say that a family \mathcal{F} is *generically generated over S* if there is an open subset $U \subseteq S$ such that $S \setminus U$ is a finite set of closed points and the restriction of \mathcal{F} onto U is generated by the image of $\Gamma(S, \mathcal{F})$ in $\Gamma(U, \mathcal{F})$.

In the commutative situation we can also globalize the latter definition.

Definition 3.2. Let S, T be \mathbf{k} -schemes, p be the projection of $S \times T$ onto T and g_t , where t is a closed point of T , be the embedding $S \simeq S \times t \rightarrow S \times T$.

- A *family of \mathcal{O}_S -modules based on T* is, by definition, a coherent sheaf over $S \times T$ flat over T .
- If \mathcal{M} is a family of sheaves over S based on T and t is a closed point of T , denote by $\mathcal{M}(t)$ the sheaf $g_t^* \mathcal{M}$.
- Such a family is called a *family of vector bundles* (respectively *of (maximal) Cohen–Macaulay modules*) if $\mathcal{M}(t)$ is locally free (respectively maximal Cohen–Macaulay module) for each closed point $t \in T$.
- We say that a family \mathcal{F} is *generically generated over S* if there is an open subset $U \subseteq S$ such that $S \setminus U$ is a finite set of closed points and the restriction onto $U \times T$ of the natural homomorphism $p^* p_* \mathcal{F} \rightarrow \mathcal{F}$ is an epimorphism.

General properties of flat families [15, Sections 6.1–6.3] imply the following.

- Proposition 3.3.** (1) *Suppose that T is a Cohen–Macaulay \mathbf{k} -scheme. A coherent sheaf \mathcal{M} on $S \times T$, flat over T , is a family of (maximal) Cohen–Macaulay modules if and only if \mathcal{M} is a sheaf of (maximal) Cohen–Macaulay $\mathcal{O}_{S \times T}$ -modules.*
- (2) *Suppose that T is a smooth \mathbf{k} -scheme. A coherent sheaf \mathcal{M} on $S \times T$, flat over T , is a family of vector bundles if and only if \mathcal{M} is a vector bundle over $S \times T$ (i.e. a locally free sheaf of $\mathcal{O}_{S \times T}$ -modules).*

We return to Definitions and Notations 1.1 and prove some results analogous to those of Kahn about the relations between Cohen–Macaulay modules and vector bundles. In this case we call families of sheaves on $X = \text{Spec } A$ *families of A -modules*. Since we deal mostly with affine or even non-commutative base, we restrict our considerations to this case. The globalization of the obtained results is more or less evident.

A family of vector bundles over \tilde{X} based on an algebra Λ is called *full* if it is isomorphic to $(\pi^* \mathcal{M})^{\vee\vee}$, where \mathcal{M} is a family of Cohen–Macaulay A -modules based on Λ .

Recall that an algebra Λ is said to be *hereditary* if $\text{gl.dim } \Lambda \leq 1$ (both left and right). Especially the local rings of points of any smooth curve, as well as free non-commutative algebras, and, more generally, path algebras of (oriented) graphs (cf. [11, Section III.6]) are always hereditary.

Proposition 3.4. (Cf. [19, Proposition 1.2].) *Suppose that a family \mathcal{F} of vector bundles over \tilde{X} based on a hereditary algebra Λ (or on a smooth curve) satisfies the following conditions:*

- (1) *\mathcal{F} is generically generated over \tilde{X} .*
- (2) *The restriction map $H^0(\tilde{X}, \mathcal{F}) \rightarrow H^0(\tilde{X}, \mathcal{F}')$ is an epimorphism (or, equivalently, the map $H_E^1(\tilde{X}, \mathcal{F}) \rightarrow H^1(\tilde{X}, \mathcal{F})$ is a monomorphism).*

Then \mathcal{F} is full.

Proof. The claim is evidently local, so we only have to prove it in the case of algebras. Set $\mathcal{M} = \pi_* \mathcal{F}$, $\mathcal{F}' = \pi^* \mathcal{M}/(\text{torsion})$ and consider \mathcal{F}' as a subsheaf of \mathcal{F} in a natural way. Note that if $U \subseteq X$ is an open subset and $\pi^{-1}(U) = \bigcup U_i$ is an affine covering of $\pi^{-1}(U)$, $\mathcal{M}(U)$ is a submodule of $\bigoplus_i \mathcal{F}(U_i)$. Since Λ is hereditary, a submodule of a flat Λ -module is flat, so \mathcal{M} is flat over Λ . It is coherent since π is proper and $\mathcal{M}(L) \simeq \pi_* \mathcal{F}(L)$ is Cohen–Macaulay by [15, Proposition 6.3.1], so \mathcal{M} is a family of Cohen–Macaulay modules based on Λ . Moreover, since \mathcal{F} is generically generated over \tilde{X} , $\text{codim Supp } \mathcal{F}/\mathcal{F}' \geq 2$. But \tilde{X} is normal, hence any sheaf of the form \mathcal{N}^\vee is completely defined by its

stalks at the points of height 1. In particular, $(\pi^*\mathcal{M})^\vee \simeq (\mathcal{F}')^\vee \simeq \mathcal{F}^\vee$, so $\mathcal{F} \simeq \mathcal{F}^{\vee\vee} \simeq (\pi^*\mathcal{M})^{\vee\vee}$. \square

Proposition 3.5. (Cf. [19, Proposition 1.6].) *Suppose that Z is a weak reduction cycle on \tilde{X} , \mathcal{F} is a family of vector bundles on \tilde{X} based on a hereditary algebra Λ (or on a smooth curve) and $\mathcal{V} = \mathcal{F} \otimes_{\mathcal{O}_{\tilde{X}}} \mathcal{O}_Z$. If \mathcal{V} is generically generated by global sections and the map $\mathrm{H}^0(E, \mathcal{V}(Z)) \rightarrow \mathrm{H}^1(E, \mathcal{V})$ induced by the exact sequence*

$$0 \longrightarrow \mathcal{V}(-Z) \longrightarrow \mathcal{V}_2 \longrightarrow \mathcal{V} \longrightarrow 0,$$

where $\mathcal{V}_2 = \mathcal{F} \otimes_{\mathcal{O}_{\tilde{X}}} \mathcal{O}_{2Z}$, is a monomorphism, then \mathcal{F} is full.

Proof. Again one only has to consider the case of algebras; then one simply has to follow literally the proof of Kahn using proposition 3.4. \square

Proposition 3.6. (Cf. [19, Proposition 1.9].) *Suppose that Z is a weak reduction cycle on \tilde{X} and a family \mathcal{V} of vector bundles over Z based on a hereditary algebra Λ (for instance, on a smooth affine curve) satisfies the following conditions:*

- (1) \mathcal{V} is generically generated over Z .
- (2) There exists an extension of \mathcal{V} to a family of vector bundles \mathcal{V}_2 on $2Z$ such that the induced map $\mathrm{H}^0(E, \mathcal{V}(Z)) \rightarrow \mathrm{H}^1(E, \mathcal{V})$ is injective.

Then there is a full family of vector bundles \mathcal{F} over \tilde{X} such that $\mathcal{V} \simeq \mathcal{F} \otimes_{\mathcal{O}_{\tilde{X}}} \mathcal{O}_Z$. If Λ is finite dimensional, such a family is unique.

Proof. Since $\mathrm{H}^2(E, \mathcal{N}) = 0$ for each quasicohherent sheaf over Z , any family of vector bundles over $2Z$ can be lifted to a family of vector bundles over nZ for every n , hence to a family of vector bundles over \tilde{X} . Proposition 3.5 implies that \mathcal{F} is full. If Λ is finite dimensional, the last assertion follows directly from [19, Proposition 1.9]. \square

For minimally elliptic singularities one gets a simpler version.

Proposition 3.7. (Cf. [19, Theorem 2.1].) *Let $\pi : \tilde{X} \rightarrow X$ be the minimal resolution of a minimally elliptic singularity and Z be the fundamental cycle on \tilde{X} . Suppose that a family \mathcal{V} of vector bundles on Z based on a hereditary algebra Λ is of the form $\mathcal{V} = \mathcal{G} \oplus \mathcal{O}_Z \otimes \mathcal{P}$, where $\mathcal{P} = \mathrm{H}^0(E, \mathcal{G}(Z))$ and \mathcal{G} satisfies the following conditions:*

- (1) \mathcal{G} is generically generated over Z ;
- (2) $\mathrm{H}^1(E, \mathcal{G}) = 0$;
- (3) $\mathrm{H}^1(E, \mathcal{G}(Z))$ is flat as Λ -module.

Then there is a full family of vector bundles \mathcal{F} over \tilde{X} such that $\mathcal{F} \otimes_{\mathcal{O}_{\tilde{X}}} \mathcal{O}_Z \simeq \mathcal{V}$. If Λ is finite dimensional, such a family is unique.

Proof. Note first that the Künneth formulae [7] imply that for each Λ -module L

$$H^1(E, \mathcal{G} \otimes_{\Lambda} L) \simeq H^1(E, \mathcal{G}) \otimes_{\Lambda} L = 0 \quad (\text{since } H^2(E, \mathcal{G}) = 0)$$

and

$$H^0(E, \mathcal{G} \otimes_{\Lambda} L) \simeq H^0(E, \mathcal{G}) \otimes_{\Lambda} L \quad (\text{since } H^1(E, \mathcal{G}) \text{ is flat}).$$

Since the ring Λ is hereditary, the Λ -module \mathcal{P} is flat as a submodule of $\bigoplus_i \mathcal{V}(U_i)$ for an affine covering $E = \bigcup_i U_i$. It follows that in this situation we can again repeat the arguments of the corresponding proof from [19, Page 151] to show that the family $\mathcal{V} = \mathcal{G} \oplus \mathcal{O}_Z \otimes \mathcal{P}$ satisfies the conditions of Proposition 3.6. \square

Remark 3.8. Suppose that in Propositions 3.6 and 3.7 we consider families based on a smooth affine curve T . Then the uniqueness assertion from [19, Proposition 1.9] can be applied to the generic fibres. It implies that for any two liftings $\mathcal{F}, \mathcal{F}'$ there is an open dense subset $U \subseteq T$ such that $\mathcal{F}|_U \simeq \mathcal{F}'|_U$. On the other hand, if the base is a projective smooth curve T , we can only lift a family \mathcal{V} of vector bundles over Z satisfying the corresponding conditions to a family of vector bundles over \tilde{X} on an open subset $U \subseteq T$. Moreover, on a smaller open subset such a lifting is unique. This consideration implies that there is an étale covering $\theta : T' \rightarrow T$ such that $\theta^* \mathcal{V}$ can be lifted to a full family of vector bundles over \tilde{X} , hence gives rise to a family of Cohen–Macaulay A -modules with the base T' .

For cusp singularities we can now precise Corollary 2.5 in the following way. (We use the notations of Section 2.)

Corollary 3.9. *Let $X = \text{Spec } A$ be a cusp singularity. For each suitable aperiodic s -sequence \mathbf{d} there is a family of Cohen–Macaulay A -modules $\mathcal{M}(\mathbf{d})$ based on T , where $T = \mathbf{k}^* \setminus \{1\}$ if $\mathbf{d} = \mathbf{b}$, and $T = \mathbf{k}^*$ otherwise, such that $M(\mathbf{d}, m, \lambda) \simeq \mathcal{M}(\mathbf{d}) \otimes_{\mathcal{O}_T} L(m, \lambda)$ for every $m \in \mathbb{N}$ and every $\lambda \in T$, where $L(m, \lambda) = \mathcal{O}_T / \mathfrak{m}_{\lambda}^m$ (\mathfrak{m}_{λ} is the maximal ideal corresponding to the point $\lambda \in T$).*

Proof. Consider the family of vector bundles over \tilde{E} : $\tilde{\mathcal{G}} = \bigoplus_{i=1}^r s \mathcal{O}_i \otimes \mathcal{O}_T$. Define $\mathcal{G}(\mathbf{d})$ as the preimage in $\tilde{\mathcal{G}}$ of the sub-bundle in $\tilde{\mathcal{G}} / \mathcal{J} \tilde{\mathcal{G}}$ generated by the elements $\mathbf{e}'_i + \mathbf{e}''_{i+1}$ for $i \neq rs$ and the elements $\mathbf{e}'_{rs} + t \mathbf{e}''_1$, where t denotes the coordinate on T and $\mathbf{e}'_i, \mathbf{e}''_i$ are defined as on page 5, after Theorem 2.1. It is easy to see that $\mathcal{G}(\mathbf{d}) \otimes_{\mathcal{O}_T} L(m, \lambda) \simeq \mathcal{G}(\mathbf{d}, m, \lambda)$. Corollaries 2.4 and 2.5 show that such a family satisfies the conditions of Proposition 3.7 and $H^0(E, \mathcal{G}(E)) \simeq n(\mathbf{d}) \mathcal{O}_T$. Hence $\mathcal{G}(\mathbf{d}) \oplus n(\mathbf{d}) \mathcal{O}_{E \times T}$ can be lifted to a full family of vector bundles \mathcal{F} over \tilde{X} such that $(\mathcal{F} \otimes_{\mathcal{O}_T} L(m, \lambda)) \otimes_{\mathcal{O}_{\tilde{X}}} \mathcal{O}_E \simeq mn(\mathbf{d}) \mathcal{O}_E \oplus \mathcal{G}(\mathbf{d}, m, \lambda)$, where $n(\mathbf{d}) = \sum_{i=1}^{rs} (d_i - b_i + 1) - \theta(\mathbf{d} - \mathbf{b}^r)$. Therefore, we obtain a family $\mathcal{M}(\mathbf{d})$ of Cohen–Macaulay A -modules just as we need. \square

Note that among $\mathcal{M}(\mathbf{d}) \otimes_{\mathcal{O}_T} L(m, \lambda)$ one obtains all indecomposable Cohen–Macaulay A -modules except A and $M(\mathbf{b}, m, 1)$.

4. COHEN–MACAULAY TAME AND WILD SINGULARITIES

We provide now some formal definitions of “tame” and “wild” singularities (with respect to the classification of Cohen–Macaulay modules), analogous to the usual ones from the representation theory of algebras. Again we consider the case of \mathbf{k} -algebras.

Definition 4.1. (1) We call a surface singularity $X = \text{Spec } A$ *Cohen–Macaulay tame* if it is not Cohen–Macaulay finite and there is a set $\mathfrak{M} = \{ \mathcal{M}^\alpha \}$ of families of Cohen–Macaulay A -modules such that

- Each \mathcal{M}^α is based on a smooth connected curve C^α .
- The set $\mathfrak{M}_r = \{ \mathcal{M}^\alpha \mid \text{rk } \mathcal{M}^\alpha = r \}$ is finite; we denote the number of elements in it by $d(\mathfrak{M}, r)$.
- Almost all indecomposable Cohen–Macaulay A -modules of rank r (i.e. all except finitely many isomorphism classes) are isomorphic to $\mathcal{M}^\alpha(c)$ for some $\mathcal{M}^\alpha \in \mathfrak{M}_r$ and some closed point $c \in C^\alpha$.

We call a set \mathfrak{M} with these properties a *parametrizing set* for Cohen–Macaulay A -modules. We also denote by $d(r, X) = \min \{ d(\mathfrak{M}, r) \}$, where \mathfrak{M} runs through all parametrizing sets for Cohen–Macaulay A -modules.

- (2) We call a tame surface singularity *bounded* (or of *polynomial growth*) if there is a polynomial $\varphi(r)$ such that $d(r, X) \leq \varphi(r)$ for all r , and *unbounded* otherwise. We say that this singularity is of *exponential growth* if $d(r, X)$ grows exponentially when $r \rightarrow \infty$.

Example 4.2. (1) Corollary 3.9 shows immediately that every cusp singularity is Cohen–Macaulay tame. Moreover, it follows from [10] that it is of exponential growth.

- (2) Just in the same way, the results of [19] show that every simple elliptic singularity is also Cohen–Macaulay tame, bounded (the corresponding families are based either on $\text{Pic}^0 E$ or on $\text{Pic}^0 E \setminus \{0\}$).

Remark 4.3. Denote by $\text{Pic}^0 E$ the subgroup of the Picard group of a (singular) curve E consisting of the classes of all line bundles \mathcal{L} such that the degree of the restriction of \mathcal{L} onto every irreducible component of E equals 0. One can easily see that $\text{Pic}^0 E \simeq \mathbf{k}^*$ for every cusp singularity. Since the parametrizing families for cusp singularities are based either on \mathbf{k}^* or on $\mathbf{k}^* \setminus \{1\}$, both examples above have a lot of common features.

is generated by global sections and $H^1(E, \mathcal{G}(n)) = 0$. Since $\mathcal{G}(n)$ is obviously also strict, one can suppose that \mathcal{G} itself has these properties. By Proposition 3.7, there is a full family \mathcal{F} of vector bundles over \tilde{X} based on $\mathbf{k}\Gamma_3$ such that $\mathcal{F} \otimes_{\mathcal{O}_{\tilde{X}}} \mathcal{O}_Z \simeq \mathcal{G} \oplus \mathcal{O}_Z \otimes_{\mathbf{p}_*} \mathcal{G}$. Let $\mathcal{F} = (\pi^* \mathcal{M})^{\vee\vee}$, where \mathcal{M} is a family of Cohen–Macaulay modules. Since \mathcal{G} is strict, Kahn’s theorem (Theorem 1.3) implies that \mathcal{M} is also strict, hence X is Cohen–Macaulay wild. \square

Consider the following procedure that allows to obtain new Cohen–Macaulay tame singularities.

Let B be a local noetherian ring with maximal ideal \mathfrak{m} , $A \supseteq B$ be a finite B -algebra. We call this extension (or the corresponding morphism $\text{Spec } A \rightarrow \text{Spec } B$) *split* if the embedding $B \rightarrow A$ splits as monomorphism of B -modules, i.e. $A \simeq B \oplus (A/B)$. It is called *unramified* if $A/\mathfrak{m}A$ is a separable B/\mathfrak{m} -algebra, or, equivalently, the natural epimorphism $\varepsilon : A \otimes_B A \rightarrow A$ of A -bimodules splits [5]. A is called *unramified in codimension 1* if the extension $B_{\mathfrak{p}} \subseteq A_{\mathfrak{p}}$ is unramified for every prime ideal $\mathfrak{p} \subseteq B$ of height 1. We also denote by $A \boxtimes M$, where M is a B -module, the second dual $(A \otimes_B M)^{\vee\vee}$ of the A -module $A \otimes_B M$.

Lemma 4.7. *Let $B \subseteq A$ be a finite extension of normal rings. This extension is unramified in codimension 1 if and only if the epimorphism of A -bimodules $\varepsilon^{\vee\vee} : A \boxtimes A \rightarrow A$ splits.*

Proof. If $\varepsilon^{\vee\vee}$ splits, $\varepsilon_{\mathfrak{p}}^{\vee\vee}$ also splits for every prime \mathfrak{p} . If $\text{ht } \mathfrak{p} = 1$, $\varepsilon_{\mathfrak{p}}^{\vee\vee}$ coincides with $\varepsilon_{\mathfrak{p}}$, thus the extension $B_{\mathfrak{p}} \subseteq A_{\mathfrak{p}}$ is unramified. Suppose now that $B \subseteq A$ is unramified in codimension 1. Then the extension $K \subseteq L$ is separable, hence the exact sequence

$$0 \longrightarrow J \longrightarrow L \otimes_K L \xrightarrow{\phi} L \longrightarrow 0$$

splits and $L \otimes_K L$ is a semi-simple ring, so $L \otimes_K L = J \oplus J'$, where $J' = \text{Ann}_{L \otimes_K L} J$, $\phi(J') = L$, and the restriction of ϕ onto J' is an isomorphism. Obviously, $A \boxtimes A$ is a B -subalgebra in $L \otimes_K L$ and $\varepsilon^{\vee\vee}$ is the restriction of ϕ onto $A \boxtimes A$. Set $I = J \cap (A \boxtimes A)$, $I' = J' \cap (A \boxtimes A)$. Then ϕ induces an epimorphism of L -bimodules $\phi_* : \text{Hom}_{L-L}(L, L \otimes_K L) \rightarrow \text{Hom}_{L-L}(L, L)$. Since $L \simeq (L \otimes_K L)/J$, ϕ_* can be identified with a mapping $J' \rightarrow L$, so it is an isomorphism. Moreover, $\phi_*(I'_{\mathfrak{p}}) = A_{\mathfrak{p}}$ for every prime ideal \mathfrak{p} of height 1, as the extension $B_{\mathfrak{p}} \subset A_{\mathfrak{p}}$ is unramified. But obviously $I' = \bigcap_{\text{ht } \mathfrak{p}=1} I'_{\mathfrak{p}}$ and $A = \bigcap_{\text{ht } \mathfrak{p}=1} A_{\mathfrak{p}}$, so $\phi_*(I') = A$ and $A \boxtimes A \rightarrow A$ splits. \square

Proposition 4.8. *Let $X = \text{Spec } A$, $Y = \text{Spec } B$ be normal surface singularities and $X \rightarrow Y$ be a finite surjective morphism given by an extension $B \rightarrow A$.*

- (1) *If $X \rightarrow Y$ is split and X is Cohen–Macaulay tame, so is also Y ; moreover, if X is bounded, so is Y .*

- (2) If $X \rightarrow Y$ is unramified in codimension 1 and Y is Cohen–Macaulay tame, so is also X ; moreover, if Y is bounded, so is X .

Proof. (1) Suppose that this extension is split and $\mathrm{rk}_B A = m$. If N is a Cohen–Macaulay B -module of rank r , it is a direct summand of $A \otimes_B N$, hence of $A \boxtimes N$ as B -module. Note that $\mathrm{rk}_A A \boxtimes N = r$ too, so $\mathrm{rk}_B A \boxtimes N = mr$. Suppose that A is Cohen–Macaulay tame. For every family \mathcal{M}^α from Definition 4.1 let t^α be the general point of the curve C^α and $\mathcal{M}^\alpha(t^\alpha) = \bigoplus_\beta \tilde{N}^{\alpha\beta}$, where $\tilde{N}^{\alpha\beta}$ are indecomposable $B \otimes \mathbf{k}(t^\alpha)$ -modules. There is an open subset $U^\alpha \subseteq C^\alpha$ and a decomposition $\mathcal{M}^\alpha|_{U^\alpha} \simeq \bigoplus_\beta \mathcal{N}^{\alpha\beta}$ (as family of B -modules) such that $\mathcal{N}^{\alpha\beta}(c)$ is indecomposable for all $c \in U^\alpha$. Denote by \mathfrak{N}_r the set of all $\mathcal{N}^{\alpha\beta}$ such that $\mathrm{rk} \mathcal{N}^{\alpha\beta} = r$, while $r/m \leq \mathrm{rk} \mathcal{M}^\alpha \leq r$. It is a finite set having at most $\sum_{j=[r/m]}^r [mj/r] d(\mathfrak{M}, j)$ elements. We also denote by \mathfrak{M}_s^* the set of indecomposable Cohen–Macaulay A -modules of rank s that are not of the form $\mathcal{M}^\alpha(c)$ ($c \in U_\alpha$) and by \mathfrak{N}_r^* the set of indecomposable B -modules of rank r that are direct summands (over B) of modules $M \in \mathfrak{M}_s^*$ with $r/m \leq s \leq r$. Certainly \mathfrak{M}_s^* and \mathfrak{N}_r^* are both finite.

Let N be any indecomposable Cohen–Macaulay B -module of rank r . It is a direct summand of an indecomposable Cohen–Macaulay A -module M of rank s with $r/m \leq s \leq r$. If $\mathrm{rk} M = s$, either $M \in \mathfrak{M}_s^*$ or $M \simeq \mathcal{M}^\alpha(c)$ for some $\mathcal{M}_\alpha \in \mathfrak{M}_s$ and some $c \in C^\alpha$. In the former case $N \in \mathfrak{N}_r^*$, while in the latter case $N \simeq \mathcal{N}^{\alpha\beta}(c)$ for some $\mathcal{N}^{\alpha\beta} \in \mathfrak{N}_r$ and some $c \in U^\alpha$. Therefore Y is Cohen–Macaulay tame. Moreover, $d(r, Y) \leq \sum_{j=[r/m]}^r [mj/r] d(j, X)$, hence if X is bounded, so is Y .

(2) Suppose that this extension is unramified in codimension 1. If M is any Cohen–Macaulay A -module, one easily verifies that $(A \boxtimes M)^\vee \simeq ((A \boxtimes A) \otimes_A M)^\vee$. Hence M^\vee is isomorphic to a direct summand of $(A \boxtimes M)^\vee$ and M is isomorphic to a direct summand of $A \boxtimes M$. So, there is an indecomposable Cohen–Macaulay B -module N such that M is isomorphic to a direct summand of $A \boxtimes N$ and $\mathrm{rk}_B N \leq \mathrm{rk}_A M \leq m \mathrm{rk}_B N$. Now the same considerations as above show that if B is Cohen–Macaulay tame (bounded), so is A . \square

Remark 4.9. Suppose that $X \rightarrow Y$ is both split and unramified in codimension 1. Then the proof of proposition 4.8 implies that if X is of exponential growth, so is Y .

Important examples arise from group actions.

Proposition 4.10. *Let G be a finite group of automorphisms of a normal surface singularity A , and let $B = A^G$ be the subalgebra of G -invariants. For a prime ideal $\mathfrak{p} \subset \mathbf{A}$ set*

$$G_{\mathfrak{p}} = \{ g \in G \mid g\mathfrak{p} = \mathfrak{p} \text{ and } g \text{ acts trivially on } A/\mathfrak{p} \}.$$

- (1) If $\text{char } \mathbf{k}$ does not divide $n = \text{card } G$, the extension $B \subseteq A$ splits.
- (2) If $G_{\mathfrak{p}} = \{1\}$ for every prime ideal $\mathfrak{p} \subset A$ of height 1, this extension is unramified in codimension 1.

In the latter case we say that G acts freely in codimension 1.

Proof. (1) The mapping $A \rightarrow B$, $a \mapsto \frac{1}{n} \sum_{g \in G} ga$, splits this extension.

(2) Let $\mathfrak{q} \subset B$ be any prime ideal of height 1, $\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_s$ be all minimal prime ideals of A containing \mathfrak{q} . All of them are of height 1, and G acts transitively on the set $\{\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_s\}$ [25, Section III.3]. Denote $K = B_{\mathfrak{q}}/\mathfrak{q}B_{\mathfrak{q}}$, $L = A_{\mathfrak{q}}/\mathfrak{q}A_{\mathfrak{q}}$ and $L_i = A_{\mathfrak{p}_i}/\mathfrak{p}_iA_{\mathfrak{p}_i}$. Since $\text{rk}_B A = \text{card } G = n$, L is an n -dimensional K -algebra, and $\prod_{i=1}^s L_i$ is a factor-algebra of L . Set $G_i = \{g \in G \mid g\mathfrak{p} = \mathfrak{p}\}$. These subgroups are all of the same cardinality m and $ms = n$. The subgroup G_i acts as a group of automorphisms of L_i over K , so $m \leq \dim_K L_i$ and $n = ms \leq \dim_K \prod_{i=1}^s L_i$. Hence, $L = \prod_{i=1}^s L_i$ and $m = \dim_K L_i$, so L_i is a Galois extension of K and L is separable over K . \square

Corollary 4.11. We keep the notations of the preceding proposition.

- (1) If $\text{char } \mathbf{k}$ does not divide $\text{card } G$ and A is Cohen–Macaulay tame (bounded), so is B .
- (2) If G acts freely in codimension 1 and B is tame (bounded), so is A .
- (3) If both conditions hold, A is of exponential growth if and only if so is B .

Call a surface singularity Y a *simple elliptic-quotient* (respectively a *cuspidal-quotient*) if it is a quotient of a simple elliptic singularity (respectively of a cusp) by a finite group G such that $\text{char } \mathbf{k}$ does not divide $\text{card } G$. Both simple elliptic-quotient and cuspidal-quotient singularities will be called *elliptic-quotient* singularities. If $\text{char } \mathbf{k} = 0$, elliptic-quotient singularities coincide with those *log-canonical* ones which are not quotient singularities (cf. [21]).

Corollary 4.12. Every elliptic-quotient surface singularity is Cohen–Macaulay tame. Among them, simple elliptic-quotient are bounded, while cuspidal-quotient are of exponential growth.

A special case is that of Q -Gorenstein singularities and their Gorenstein coverings defined as follows.

Definition 4.13. Let $\omega = \omega_B$ be a dualizing ideal of a normal singularity B . Denote by $\omega^{[k]} = \bigcap_{\text{ht } \mathfrak{p}=1} \omega_{\mathfrak{p}}^k$ ($k \in \mathbb{Z}$) (note that each $\omega_{\mathfrak{p}}$ is a principal ideal, since $B_{\mathfrak{p}}$ is a discrete valuation ring). Call B Q -Gorenstein if there is $n > 0$, prime to $\text{char } \mathbf{k}$, such that the ideal $\omega^{[n]}$ is principal.

Proposition 4.14. Suppose that B is Q -Gorenstein. Let n be the smallest positive integer such that $\omega^{[n]}$ is principal, $\omega^{[n]} = \theta B$. Denote

$A = \bigoplus_{k=0}^{n-1} \omega^{[k]}$ and consider it as B -algebra by setting $a \cdot b = ab/\theta$ for $k+l \geq n$, $a \in \omega^{[k]}$, $b \in \omega^{[l]}$. Then A is a normal Gorenstein singularity and $B = A^G$, where G is a cyclic group of order n that acts on A freely in codimension 1. We call A the Gorenstein covering of B .

Proof. If $\mathfrak{p} \subset B$ is a prime ideal of height 1, $\omega_{\mathfrak{p}} = \gamma B_{\mathfrak{p}}$ for some γ and $\theta = \gamma^n \zeta$ for an invertible element $\zeta \in B_{\mathfrak{p}}$. Then $A_{\mathfrak{p}} \simeq B_{\mathfrak{p}}[t]/(t^n - \zeta)$, so it is unramified in codimension 1, especially $A_{\mathfrak{p}}$ is normal. Since A is Cohen–Macaulay, $A = \bigcap_{\text{ht } \mathfrak{p}=1} A_{\mathfrak{p}}$, so A is normal itself. Moreover, $\text{Hom}_B(\omega^{[k]}, \omega) \simeq \omega^{[1-k]}$, so $\omega_A \simeq \text{Hom}_B(A, \omega_B) \simeq A$, thus A is Gorenstein. If K is the field of fractions of B , L is the field of fractions of A , then $L \simeq K[t]/(t^n - \zeta)$ is a Galois extension of K with cyclic Galois group of order n . Therefore $B = A^G$ and G acts freely in codimension 1. \square

Proposition 4.15. *Let B be a Q -Gorenstein surface singularity, A be its Gorenstein covering. If A is Cohen–Macaulay tame (bounded, of exponential growth), so is B and vice versa.*

Remark. If $\text{char } \mathbf{k} = 0$, the log-canonical singularities are just those Q -Gorenstein that their Gorenstein coverings are either rational double points (in the case of quotient singularities), or simple elliptic, or cusp singularities.

We call a normal surface singularity Q -elliptic if it is Q -Gorenstein and its Gorenstein covering is minimally elliptic.

Corollary 4.16. *A Q -elliptic singularity is Cohen–Macaulay tame if and only if it is elliptic-quotient; otherwise it is Cohen–Macaulay wild.*

5. CURVE SINGULARITIES T_{pq}

Important examples of cusp singularities for $\text{char } \mathbf{k} = 0$ are the “serial” unimodal singularities [1] T_{pqr} : $A = \mathbf{k}[[x, y, z]]/(x^p + y^q + z^r + \alpha xyz)$, where $r \leq p \leq q$ and $1/p + 1/q + 1/r < 1$, $\alpha \in \mathbf{k}^*$ (in this case all values of α lead to isomorphic algebras). Note that if $1/p + 1/q + 1/r = 1$, the corresponding singularity is simple elliptic except for finitely many special values of α . If $r = 2$, Cohen–Macaulay modules over this singularity are closely related to those over the curve singularity T_{pq} . Indeed, in this case the singularity T_{pq2} can be rewritten in the form: $A = \mathbf{k}[[x, y, z]]/(z^2 + x^p + y^q + \beta x^2 y^2)$ for $\beta = -\alpha^2/4$. Therefore, one can use the *Knörrer’s correspondence* (cf. [22, 27]) described in the following proposition to relate Cohen–Macaulay modules over A and over the curve singularity $A/(z^2) = \mathbf{k}[[x, y]]/(x^p + y^q + \beta x^2 y^2)$, denoted T_{pq} .

Proposition 5.1. *Let $f \in \mathbf{k}[[x_1, x_2, \dots, x_n]]$ be a non-invertible element, $A = \mathbf{k}[[x_0, x_1, \dots, x_n]]/(x_0^2 + f)$ and $A' = \mathbf{k}[[x_1, x_2, \dots, x_n]]/(f)$. For every Cohen–Macaulay module M over A let $\text{rest } M = M/x_0 M$*

and M^σ be the A -module that coincide with M as a group, while the action of the elements of A is given by the rule: $a \cdot_{M^\sigma} v = a^\sigma \cdot_M v$, where $g(x_0, x_1, \dots, x_n)^\sigma = g(-x_0, x_1, x_2, \dots, x_n)$. Then:

- every indecomposable Cohen–Macaulay A' -module is a direct summand of $\text{rest } M$ for an indecomposable Cohen–Macaulay A -module M and $M \simeq A$ if and only if $\text{rest } M \simeq A'$;
- if M is an indecomposable Cohen–Macaulay A -module such that $M \not\simeq M^\sigma$, $\text{rest } M$ is also indecomposable;
- if $M \not\simeq A$ is an indecomposable Cohen–Macaulay A -module such that $M \simeq M^\sigma$, then $\text{rest } M$ is a direct sum of two non-isomorphic indecomposable Cohen–Macaulay A' -modules, which we denote by $\text{rest}_1 M$ and $\text{rest}_2 M$.
- if M, N are non-isomorphic indecomposable Cohen–Macaulay A -modules and $M \not\simeq N^\sigma$, then the indecomposable A' -modules obtained from M and from N as described above are also non-isomorphic.

Obviously, always $\text{rest } M \simeq \text{rest}(M^\sigma)$ and $A^\sigma \simeq A$.

So, to describe the Cohen–Macaulay modules over the curve singularity T_{pq} we have to know when $M^\sigma \simeq M$ for a Cohen–Macaulay module M over the surface singularity T_{pqr} . The automorphism $g \rightarrow g^\sigma$ induces an automorphism of the minimal resolution \tilde{X} , hence an automorphism of the exceptional curve E and of the category of vector bundles over E . We denote all these automorphisms by σ too. The following result is immediate (cf. the definition of R_E from Theorem 1.2).

Proposition 5.2. $R_E M^\sigma \simeq (R_E M)^\sigma$ for every Cohen–Macaulay A -module M .

From the description of the minimal resolutions of T_{pqr} given, for instance, in [20, 23], one can deduce the following shape of the exceptional curve E for the minimal resolution of T_{pq2} and for the action of the automorphism σ on it.

Proposition 5.3. (We use the notations from Section 2.)

- (1) If $p = 3$ (hence $q \geq 7$), then $s = q - 6$. If $q > 7$, E has 1 component E_1 with self-intersection -3 and $s - 1$ components E_2, \dots, E_s with self-intersection -2 ; $E_1^\sigma = E_1$ and $E_i^\sigma = E_{s+2-i}$. If $q = 7$, E is irreducible with one node and $E \cdot E = -1$. In both cases $\sigma(p_1'') = p_1'$.
- (2) If $p = 4$ (hence $q \geq 5$), then $s = q - 4$. If $q > 5$, E has 1 component E_1 with $E_1 \cdot E_1 = -4$ and $s - 1$ components E_2, \dots, E_s with $E_i \cdot E_i = -2$. If $q = 5$, E is irreducible with one node and $E \cdot E = -2$. The action of σ is the same as in the previous case.
- (3) If $p \geq 5$, then $s = p + q - 8$. Put $t = p - 3$. E has 2 components E_1 and E_t with $E_i \cdot E_i = -3$ and $s - 2$ components with $E_i \cdot E_i =$

-2 (no one if $p = q = 5$). $E_i^\sigma = E_{t+1-i}$ if $1 \leq s \leq t$ and $E_i^\sigma = E_{s+t+1-i}$ if $t < i \leq s$; $\sigma(p_1'') = p_t'$.

Given an s -sequence $\mathbf{d} = (d_1, d_2, \dots, d_{rs})$, denote by \mathbf{d}^σ the sequence $(d_1', d_2', \dots, d_{rs}')$ obtained from \mathbf{d} by the following procedure (we keep the notations of Proposition 5.3):

- if $p = 3$ or $p = 4$, then $d_1' = d_1$ and $d_i' = d_{rs+2-i}$ for $1 < i \leq rs$;
- if $p \geq 5$, then $d_i' = d_{t+1-i}$ for $1 \leq i \leq t$ and $d_i' = d_{rs+t+1-i}$ for $t < i \leq rs$.

We call an s -sequence \mathbf{d} σ -symmetric if $\mathbf{d}^\sigma = \mathbf{d}^k$, where \mathbf{d}^k is an s -shift of \mathbf{d} (cf. page 5). Then the description of Cohen–Macaulay modules from Section 2 implies the following results.

Corollary 5.4. *Let $M = M(\mathbf{d}, m, \lambda)$ be an indecomposable Cohen–Macaulay module over the surface singularity T_{pq2} (cf. Corollary 2.5). Then $M^\sigma \simeq M(\mathbf{d}^\sigma, m, 1/\lambda)$. In particular, $M \simeq M^\sigma$ if and only if the sequence \mathbf{d} is σ -symmetric and $\lambda = \pm 1$.*

Corollary 5.5. *The indecomposable Cohen–Macaulay modules over the curve singularity $A' = \mathbf{k}[[x, y]]/(x^p + y^q + \beta x^2 y^2)$ of type T_{pq} ($q \geq p$, $1/p + 1/q < 1/2$) are the following:*

- $N(\mathbf{d}, m, \lambda) = \text{rest } M(\mathbf{d}, m, \lambda)$, where either the sequence \mathbf{d} is not σ -symmetric or $\lambda \neq \pm 1$;
- $N_i(\mathbf{d}, m, \pm 1) = \text{rest}_i M(\mathbf{d}, m, \pm 1)$ ($i = 1, 2$), where \mathbf{d} is σ -symmetric;
- the regular module A' .

The only isomorphisms between so defined Cohen–Macaulay modules are $N(\mathbf{d}, m, \lambda) \simeq N(\mathbf{d}^\sigma, m, 1/\lambda)$.

Note that the modules $N(\mathbf{d}, m, \lambda)$ with $\lambda \neq \pm 1$ form rational families based on $(\mathbf{k}^* \setminus \{1, -1\})/\mathbb{Z}_2$, where the cyclic group \mathbb{Z}_2 acts on \mathbf{k} mapping λ to $1/\lambda$. One easily sees that this factor is isomorphic to $\mathbf{k}^* \setminus \{1, -1\}$ itself.

6. HYPERSURFACE SINGULARITIES

We also can use the previous results and Knörrer’s correspondence to describe Cohen–Macaulay modules on hypersurface singularities of type T_{pqr} in any dimension. Such a hypersurface is given by the equation $x^p + y^q + z^r + \alpha xyz + \sum_{j=1}^k v_j^2$ for some k (maybe, 0). Recall the main results concerning Knörrer’s correspondence [22, 27].

Proposition 6.1. *Let $f \in \mathbf{k}[[x_1, x_2, \dots, x_n]]$ be a non-invertible element, $A = \mathbf{k}[[x_1, x_2, \dots, x_n]]/(f)$ and $\tilde{A} = \mathbf{k}[[x_0, x_1, \dots, x_n]]/(f + x_0^2)$. For every Cohen–Macaulay module M over A let $\text{syz } M$ denote the first syzygy module of M considered as \tilde{A} -module via the natural epimorphism $\tilde{A} \rightarrow A \simeq \tilde{A}/(x_0)$. If M has no free direct summands, let ΩM denote its first syzygy as A -module. Then:*

- every indecomposable Cohen–Macaulay \tilde{A} -module is a direct summand of $\text{syz } M$ for an indecomposable Cohen–Macaulay A -module M and $M \simeq A$ if and only if $\text{syz } M \simeq \tilde{A}$;
- if M is an indecomposable Cohen–Macaulay A -module such that $M \not\cong \Omega M$, $\text{syz } M$ is also indecomposable;
- if $M \not\cong A$ is an indecomposable Cohen–Macaulay A -module such that $M \simeq \Omega M$, then $\text{syz } M$ is a direct sum of two non-isomorphic indecomposable Cohen–Macaulay \tilde{A} -modules, which we denote by $\text{syz}_1 M$ and $\text{syz}_2 M$.
- if M, N are non-isomorphic indecomposable Cohen–Macaulay A -modules and $M \not\cong \Omega N$, then the indecomposable \tilde{A} -modules obtained from M and from N as described above are also non-isomorphic.
- if M has no free direct summands, then $\text{syz}(\Omega M) \simeq \text{syz } M$.

Obviously, this proposition implies immediately that if A is Cohen–Macaulay tame, so is \tilde{A} (and vice versa in view of Proposition 5.1), just as in Proposition 4.8. In particular, as we have already mentioned, for $k = 0$, the (surface) singularity of type T_{pqr} is simple elliptic for $1/p + 1/q + 1/r = 1$ and a cusp for $1/p + 1/q + 1/r < 1$ [23]. Hence the results of Section 3 imply

Corollary 6.2. *Every hypersurface singularity of type T_{pqr} is Cohen–Macaulay tame.*

7. AUSLANDER–REITEN QUIVERS

In this section we calculate Auslander–Reiten quivers for Cohen–Macaulay modules and related vector bundles. Recall (cf. e.g. [4, 6]) that an Auslander–Reiten sequence in a category of modules or of vector bundles is a non-split exact sequence

$$0 \longrightarrow M' \xrightarrow{g} N \xrightarrow{f} M \longrightarrow 0,$$

where M, M' are indecomposable, such that any homomorphism $N' \rightarrow M$ that is not a split epimorphism factors through f (equivalently, any homomorphism $M' \rightarrow N'$ that is not a split monomorphism factors through g). Each of the modules M, M' uniquely defines the second one. Usually M' is denoted by τM and called the *Auslander–Reiten translate* of M . In [6] it was proved that for any indecomposable vector bundle \mathcal{G} on a projective curve E there is an Auslander–Reiten sequence

$$0 \longrightarrow \omega_E \otimes_{\mathcal{O}_E} \mathcal{G} \longrightarrow \mathcal{E} \longrightarrow \mathcal{G} \longrightarrow 0.$$

In particular, if E is a cyclic configuration, then, using the notation of Section 2, $\omega_E \simeq \mathcal{O}_E \simeq \mathcal{G}(\mathbf{0}, 1, 1)$ and if $\mathcal{G} = \mathcal{G}(\mathbf{d}, m, \lambda)$, there is an exact sequence

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{E} \longrightarrow \mathcal{G} \longrightarrow 0,$$

where

$$\mathcal{E} \simeq \begin{cases} \mathcal{G}(\mathbf{d}, 2, \lambda) & \text{if } m = 1, \\ \mathcal{G}(\mathbf{d}, m + 1, \lambda) \oplus \mathcal{G}(\mathbf{d}, m - 1, \lambda) & \text{if } m > 1, \end{cases}$$

that is easily recognized as Auslander–Reiten sequence (if $\text{char } \mathbf{k} = 0$ it follows immediately from [6], and the combinatoric description of $\text{VB}(X)$ in [10] does not depend on the characteristic). In particular, the Auslander–Reiten quiver of the category of vector bundles over a cyclic configuration is a disjoint union of *homogeneous tubes*

$$\mathcal{T}(\mathbf{d}, \lambda) : \quad \mathcal{G}(\mathbf{d}, 1, \lambda) \rightleftarrows \mathcal{G}(\mathbf{d}, 2, \lambda) \rightleftarrows \mathcal{G}(\mathbf{d}, 3, \lambda) \rightleftarrows \dots$$

with $\tau\mathcal{G} = \mathcal{G}$ for every indecomposable vector bundle \mathcal{G} . Here \mathbf{d} runs through aperiodic s -sequences and $\lambda \in \mathbf{k}^*$.

Passing from vector bundles to Cohen–Macaulay modules, we use the following result of Kahn:

Proposition 7.1. [19, Theorem 3.1] *If X is a minimally elliptic singularity, Z its fundamental cycle and $0 \rightarrow M \rightarrow N \rightarrow M \rightarrow 0$ an Auslander–Reiten sequence in $\text{CM}(X)$, then the exact sequence $0 \rightarrow R_Z M \rightarrow R_Z N \rightarrow R_Z M \rightarrow 0$ is the direct sum of an Auslander–Reiten sequence $0 \rightarrow \mathcal{G} \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow 0$ and a split sequence $0 \rightarrow n\mathcal{O}_Z \rightarrow 2n\mathcal{O}_Z \rightarrow n\mathcal{O}_Z \rightarrow 0$.*

(Note that, as a minimally elliptic singularity is Gorenstein, any Auslander–Reiten sequence in $\text{CM}(X)$ is of the form $0 \rightarrow M \rightarrow N \rightarrow M \rightarrow 0$, cf. [6, Theorem I.3.1].)

Corollary 7.2. *All Auslander–Reiten sequences of Cohen–Macaulay modules over a cusp singularity are the following:*

$$\begin{aligned} 0 \rightarrow M(\mathbf{d}, 1, \lambda) \rightarrow M(\mathbf{d}, 2, \lambda) \rightarrow M(\mathbf{d}, 1, \lambda) \rightarrow 0 & \text{ if } \mathbf{d} \neq \mathbf{b} \text{ or } \lambda \neq 1; \\ 0 \rightarrow M(\mathbf{b}, 1, 1) \rightarrow A \oplus M(\mathbf{b}, 2, 1) \rightarrow M(\mathbf{b}, 1, 1) \rightarrow 0; \\ 0 \rightarrow M(\mathbf{d}, m, \lambda) \rightarrow M(\mathbf{d}, m + 1, \lambda) \oplus M(\mathbf{d}, m - 1, \lambda) \rightarrow M(\mathbf{d}, m, \lambda) \rightarrow 0 & \text{ if } m > 1. \end{aligned}$$

In particular, the Auslander–Reiten quiver of the category $\text{CM}(X)$, where X is a cusp singularity, is a disjoint union of homogeneous tubes

$$\mathfrak{T}(\mathbf{d}, \lambda) : \quad M(\mathbf{d}, 1, \lambda) \rightleftarrows M(\mathbf{d}, 2, \lambda) \rightleftarrows M(\mathbf{d}, 3, \lambda) \rightleftarrows \dots$$

where $\mathbf{d} \neq \mathbf{b}$ or $\lambda \neq 1$, and one “special” tube

$$\mathfrak{T}(\mathbf{b}, 1) : \quad A \rightleftarrows M(\mathbf{b}, 1, 1) \rightleftarrows M(\mathbf{b}, 2, 1) \rightleftarrows M(\mathbf{b}, 3, 1) \rightleftarrows \dots$$

enlarged by the regular module A that is both projective and ext-injective in this category (the latter means that $\text{Ext}_A^1(M, A) = 0$ for any Cohen–Macaulay module M). Again $\tau M = M$ for all indecomposable modules $M \not\cong A$ (obviously, τA does not exist).

Passing from the singularities T_{pq2} to T_{pq} , one only has to note that Knörrer’s correspondence maps an Auslander–Reiten sequence $0 \rightarrow$

$M \rightarrow N \rightarrow M \rightarrow 0$ either to an Auslander–Reiten sequence $0 \rightarrow \text{rest } M \rightarrow \text{rest } N \rightarrow \text{rest } M \rightarrow 0$ (if $M^\sigma \not\cong M$) or to the direct sum of Auslander–Reiten sequences $0 \rightarrow \text{rest}_2 M \rightarrow N' \rightarrow \text{rest}_1 M \rightarrow 0$ and $0 \rightarrow \text{rest}_1 M \rightarrow N'' \rightarrow \text{rest}_2 M \rightarrow 0$ if $M \simeq M^\sigma$ (cf. [22, Corollary 2.10]). In particular, considering the Auslander–Reiten sequence $0 \rightarrow M(\mathbf{b}, 1, 1) \rightarrow A \oplus M(\mathbf{b}, 2, 1) \rightarrow M(\mathbf{b}, 1, 1) \rightarrow 0$, we see that A' must be in the middle term of the Auslander–Reiten sequence starting either from $N_1(\mathbf{b}, 1, 1)$ or from $N_2(\mathbf{b}, 1, 1)$. We choose the second possibility (it only depends on the notation). We also choose the notations in such a way that $N_i(\mathbf{d}, m+1, \pm 1)$ is a direct summand of the middle term of the Auslander–Reiten sequence starting from $N_i(\mathbf{d}, m, \pm 1)$ ($i = 1, 2$). Then the Auslander–Reiten quiver is in this case a disjoint union of homogeneous tubes

$$\tilde{\mathfrak{T}}(\mathbf{d}, \lambda) : N(\mathbf{d}, 1, \lambda) \rightleftarrows N(\mathbf{d}, 2, \lambda) \rightleftarrows N(\mathbf{d}, 3, \lambda) \rightleftarrows \dots$$

for non- σ -symmetric \mathbf{d} or for $\lambda \neq 1$; regular tubes $\tilde{\mathfrak{T}}(\mathbf{d}, \pm 1)$ of period 2 for σ -symmetric \mathbf{d} ($\mathbf{d} \neq \mathbf{b}$ or $\lambda = -1$):

$$\begin{array}{ccccccc} N_1(\mathbf{d}, 1, \pm 1) & \xrightarrow{\quad} & N_1(\mathbf{d}, 2, \pm 1) & \xrightarrow{\quad} & N_1(\mathbf{d}, 3, \pm 1) & \xrightarrow{\quad} & \dots \\ & \searrow & \swarrow & \searrow & \swarrow & \searrow & \\ N_2(\mathbf{d}, 1, \pm 1) & \xrightarrow{\quad} & N_2(\mathbf{d}, 2, \pm 1) & \xrightarrow{\quad} & N_2(\mathbf{d}, 3, \pm 1) & \xrightarrow{\quad} & \dots \end{array}$$

and one “special” tube $\tilde{\mathfrak{T}}(\mathbf{b}, 1)$ of period 2 enlarged by the regular module A' :

$$\begin{array}{ccccccc} & & N_1(\mathbf{b}, 1, 1) & \xrightarrow{\quad} & N_1(\mathbf{b}, 2, 1) & \xrightarrow{\quad} & N_1(\mathbf{b}, 3, 1) & \xrightarrow{\quad} & \dots \\ & \nearrow & & \searrow & \nearrow & \searrow & & \nearrow & \\ A' & & N_2(\mathbf{b}, 1, 1) & \xrightarrow{\quad} & N_2(\mathbf{b}, 2, 1) & \xrightarrow{\quad} & N_2(\mathbf{b}, 3, 1) & \xrightarrow{\quad} & \dots \\ & \searrow & & \swarrow & \searrow & \swarrow & & \searrow & \end{array}$$

The action of the Auslander–Reiten translate τ is trivial in homogeneous tubes and coincides with the obvious axis symmetry in those of period 2.

The procedure is quite the same when passing to hypersurface singularities T_{pqr} of higher dimensions. Namely, again the mapping syz transforms an Auslander–Reiten sequence $0 \rightarrow M \rightarrow N \rightarrow M \rightarrow 0$ either to an Auslander–Reiten sequence $0 \rightarrow \text{syz } M \rightarrow \text{syz } N \rightarrow \text{syz } M \rightarrow 0$ (if $M \not\cong \Omega M$) or to the direct sum of Auslander–Reiten sequences $0 \rightarrow \text{syz}_2 M \rightarrow N' \rightarrow \text{syz}_1 M \rightarrow 0$ and $0 \rightarrow \text{syz}_1 M \rightarrow N'' \rightarrow \text{syz}_2 M \rightarrow 0$.

For a hypersurface singularity of type T_{pqr} ($1/p + 1/q + 1/r < 1$), one can use *Knörrer’s periodicity* [22, 27] to describe the part of the Auslander–Reiten quiver not containing the free module and the result of

Sølberg [26] on the position of the free module in an Auslander–Reiten sequence. Then one obtains that if the dimension $n = k + 3$ is even, the Auslander–Reiten sequence is the same as for the corresponding surface singularity, with the only exception that the tube containing the free module begins not from two arrows but from $2^{n/2}$ ones (half of them starting and half of them ending at A). If n is odd, the Auslander–Reiten sequence looks like that for the curve case, except of the tube containing the free module, where to each arrow starting or ending at A should be added $2^{\lfloor \frac{n-1}{2} \rfloor}$ ones such that each next arrow goes to the opposite direction with respect to the preceding one.

8. SUMMARY AND CONJECTURES

We can now summarize the known results on Cohen–Macaulay types of isolated Cohen–Macaulay singularities. This is done in Table 1. To make it more uniform, we call a curve singularity of type T_{pq} *simple elliptic* if $1/p + 1/q = 1/2$ and a *cuspidal* if $1/p + 1/q < 1/2$. In the same way, we call a hypersurface singularity of type T_{pqr} *simple elliptic* if $1/p + 1/q + 1/r = 1$ and a *cuspidal* if $1/p + 1/q + 1/r < 1$, though it seems not to be the usual practice.

Moreover, there is some evidence that these results are *complete*, that is, all the remaining singularities are Cohen–Macaulay wild. We formulate the corresponding conjectures as well as one related to non-isolated singularities. Here (X, x) denotes *any* Cohen–Macaulay singularity of an algebraic variety over a field of characteristic 0 and Cohen–Macaulay type refers to its complete local ring A .

Conjecture 8.1. *In the following cases the ring $A = \hat{\mathcal{O}}_{X,x}$ is Cohen–Macaulay wild:*

- (1) (X, x) is a surface singularity which is neither a quotient nor an elliptic-quotient.
- (2) (X, x) is a hypersurface singularity, which is neither a simple one nor of type T_{pqr} .
- (3) (X, x) is a non-isolated singularity with the dimension of the singular locus greater than 1.

If this conjecture is true, it yields a complete description of isolated Cohen–Macaulay tame surface and hypersurface singularities together with a classification of their indecomposable Cohen–Macaulay modules. At the moment we have neither further conjectures, nor even examples concerning Cohen–Macaulay types of non-isolated singularities, even in hypersurface case, though it seems that very few of them can be tame.

Remark 8.2. All known examples of Cohen–Macaulay tame unbounded singularities, in particular those from Table are actually of exponential growth. It seems very plausible that it is always so. Nevertheless, just as in the case of finite dimensional algebras, it can only be shown *a*

Table 1.
Cohen–Macaulay types of singularities

CM type	curves	surfaces	hypersurfaces
finite	dominate simple	quotient	simple
tame bounded	dominate simple elliptic	simple elliptic- quotient (no other ?)	simple elliptic (no other ?)
tame unbounded	dominate cusp	cusp- quotient (no other ?)	cusp (no other ?)
wild	all other	all other ?	all other ?

posteriori, when one has a description of modules. We do not see any “natural” way to prove this conjecture without such calculations.

Remark 8.3. In the complex analytic case, Artin’s Approximation Theorem [2] implies that the list of Cohen–Macaulay modules (Corollary 2.5) remains the same if A denotes the ring of germs of analytic functions on a cusp singularity. The lifting of families (Propositions 3.6 and 3.7) is more cumbersome. We do only claim that, for each point $t \in T$, a lifting is possible over a neighbourhood U of t in T . Combined with the uniqueness assertion, just as in Remark 3.8, it gives a lifting of an appropriate family to the universal covering \tilde{T} of T . If T is a smooth curve, so is \tilde{T} , therefore the results on tameness from Section 4 remain valid. On the other hand, in the case of cusps it seems credible that the families $\mathcal{G}(\mathbf{d})$ from the proof of Corollary 3.9 can actually be lifted over T , just as in [19] for simple elliptic case. Moreover, in the case (iv) of Remark 4.5, where the base is a finite dimensional algebra, Artin’s Approximation Theorem can also be applied, so Theorem 4.6 remains valid in the analytic case too.

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