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s1

1. Categories of complexes

Let $\mathscr{A}$ be an additive category. Recall that a *complex* $A^\bullet = \{\, A^n, d^n \mid n \in \mathbb{Z} \,\}$ in $\mathscr{A}$ is a sequence of morphisms

$$\cdots \to A^{n-1} \xrightarrow{d^{n-1}} A^n \xrightarrow{d^n} A^{n+1} \to \ldots$$

such that $d^n d^{n-1} = 0$ for every $in \mathbb{Z}$. If necessary, we write d^n_A instead of d. The morphisms d^n are called the *differential* of the complex $A^\bullet$. A *morphism of complexes* $f^\bullet : A^\bullet \to B^\bullet$ is a set of morphisms $f^n : A^n \to B^n$ such that the diagram

$$\begin{tikzcd}
\ldots \arrow[r] & A^{n-1} \arrow[r, "d_A^{n-1}n-1"] \arrow[d, "f^{n-1}"'] & A^n \arrow[r, "d_A^n"] \arrow[d, "f^n"'] & A^{n+1} \arrow[r] \arrow[d, "f^{n+1}"'] & \ldots \\
\ldots \arrow[r] & B^{n-1} \arrow[r, "d_B^{n-1}n-1"'] & B^n \arrow[r, "d_B^n"'] & B^{n+1} \arrow[r] & \ldots
\end{tikzcd}$$

is commutative, i.e. $d^n_B f^n = f^{n+1} d^n_A$ for every $n \in \mathbb{Z}$. We denote by $\mathsf{Kom}(\mathscr{A})$ the category of complexes over $\mathscr{A}$. It is also an additive category. Moreover, if $\mathscr{A}$ is abelian, so is $\mathsf{Kom}(\mathscr{A})$. For any complex $A^\bullet$ and an integer k we denote by $A^\bullet[k]$ the *shifted complex* such that $A[k]^n = A^{n+k}$ and $d^n_{A[k]} = d^{n+k}_A$. If $f^\bullet : A^\bullet \to B^\bullet$ is a morphism of complexes, it induces a morphism $f^\bullet] : A^\bullet[k] \to B^\bullet[k]$ such that $f[k]^n = f^{n+k}$. Thus we get a set of autoequivalences $[k] : \mathsf{Kom}(\mathscr{A}) \to \mathsf{Kom}(\mathscr{A})$. Every (additive) functor $F : \mathscr{A} \to \mathscr{B}$ induces a functor $\mathsf{Kom}(\mathscr{A}) \to \mathsf{Kom}(\mathscr{B})$, which we also denote by F.

Given a morphism of complexes $f^\bullet : A^\bullet \to B^\bullet$, we define its *cone* as the complex $Cf^\bullet = C^\bullet$, where

$$C^n = A^{n+1} \oplus B^n \quad \text{and} \quad d^n_C = \begin{pmatrix} -d^{n+1}_A & 0 \\ f^{n+1} & d^n_B \end{pmatrix} : A^{n+1} \oplus B^n \to A^{n+2} \oplus B^{n+1}.$$

There are natural morphisms of complexes $\iota f : B^\bullet \to Cf^\bullet$ and $\pi f : Cf^\bullet \to A^\bullet[1]$, where

$$\iota f^n = \begin{pmatrix} 0 \\ 1_{B^n} \end{pmatrix} : B^n \to A^{n+1} \oplus B^n,$$
$$\pi f^n = \begin{pmatrix} 1_{A^{n+1}} & 0 \end{pmatrix} : A^{n+1} \oplus B^n \to A^{n+1}.$$

If $\mathscr{A}$ is an abelian category, the sequence of complexes

e11 (1.1) $$\operatorname{cone} f : 0 \to B^\bullet \xrightarrow{\iota f} Cf^\bullet \xrightarrow{\pi f} A^\bullet[1] \to 0$$

is obviously exact.

Suppose that the category $\mathscr{A}$ is abelian (for instance, the category R-Mod of modules over a rings R or the category $\mathrm{Qcoh}(X)$ of quasicoherent sheaves over a scheme or an algebraic variety X). Then, for any complex $A^\bullet$ in $\mathscr{A}$, there are embeddings $\operatorname{Im} d^{n-1} \subseteq \operatorname{Ker} d^n$, so one can define the *cohomologies* $\mathrm{H}^\bullet(A^\bullet)$ of this complex setting $\mathrm{H}^n(A^\bullet) = \operatorname{Ker} d^n / \operatorname{Im} d^{n-1}$. It is convenient to consider $\mathrm{H}^\bullet(A^\bullet)$ as a complex with zero differentials. If $f^\bullet : A^\bullet \to B^\bullet$ is a morphism of complexes, it induces a morphism of homologies $\mathrm{H}^\bullet(f^\bullet)$. So we get a functor $\mathrm{H}^\bullet$ on the category of complexes. Itis compatible withy the shifts in the sence that $\mathrm{H}^\bullet(A^\bullet[k]) = \mathrm{H}^\bullet(A^\bullet)[k]$.

It is known (and easily checked) that any exact sequence of complexes $\xi : 0 \to A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet \to 0$ induces an exact sequence of cohomologies

$$\cdots \to \mathrm{H}^n(A^\bullet) \xrightarrow{\mathrm{H}^n(f^\bullet)} \mathrm{H}^n(B^\bullet) \xrightarrow{\mathrm{H}^n(g^\bullet)} \mathrm{H}^n(C^\bullet) \xrightarrow{\delta^n_\xi}$$
$$\xrightarrow{\delta^n_\xi} \mathrm{H}^{n+1}(A^\bullet) \xrightarrow{\mathrm{H}^{n+1}(f^\bullet)} \mathrm{H}^{n+1}(B^\bullet) \xrightarrow{\mathrm{H}^{n+1}(g^\bullet)} \mathrm{H}^{n+1}(C^\bullet) \to \dots$$

The morphisms δ^n_ξ are called the *connecting morphisms* for the exact sequence ξ. We recall the construction of δ_ξ for the case of modules.[1] Given a class $\bar{c} \in \mathrm{H}^n(C^\bullet)$, we choose its representative $c \in \operatorname{Ker} d^n$ and a preimage b of c under g^n. Then $g^{n+1}(d^n b) = d^n(g^n b) = d_n c = 0$, so $d^n b = f^{n+1} a$ for some $a \in A^{n+1}$. Obviously $d^{n+1} a = 0$, and we define $\delta^n_\xi \bar{c}$ as the class of a in $\mathrm{H}^{n+1}(A^\bullet)$. To check that the resulting sequence of cohomologies is exact is an easy exercise. Together with the exact sequence for the cone, it gives the following important result.

t11 **Theorem 1.1.** *Let $f^\bullet : A^\bullet \to B^\bullet$ be a morphism of complexes in an abelian category $\mathscr{A}$. It induces an exact sequence of cohomologies*

$$\cdots \to \mathrm{H}^n(A^\bullet) \xrightarrow{\mathrm{H}^n(f^\bullet)} \mathrm{H}^n(B^\bullet) \xrightarrow{\mathrm{H}^n(\iota f^\bullet)} \mathrm{H}^n(Cf^\bullet) \xrightarrow{\mathrm{H}^n(\pi f^\bullet)}$$
$$\xrightarrow{\mathrm{H}^n(\pi f^\bullet)} \mathrm{H}^{n+1}(A^\bullet) \xrightarrow{\mathrm{H}^{n+1}(f^\bullet)} \mathrm{H}^{n+1}(B^\bullet) \xrightarrow{\mathrm{H}^{n+1}(\iota f^\bullet)} \mathrm{H}^n(Cf^\bullet) \to \dots$$

Proof. We only have to verify that the conecting morphisms for the exact sequence (1.1) coinside with the morphisms from $\mathrm{H}^\bullet(f^\bullet)$. It is an easy exercise, which we leave to the reader. $\square$

Let $f^\bullet : A^\bullet \to B^\bullet$ be a morphism of complexes. We say that $f^\bullet$ is *homotopy trivial* and write $f^\bullet \sim 0$ if there are morphisms $s^n : A^n \to B^{n-1}$ such that $f^n = d^{n-1}s^n + s^{n+1}d^n$ for all $n \in \mathbb{Z}$. One easily check (an exercise!) that if $f^\bullet \sim 0$, then $\mathrm{H}^\bullet(f^\bullet) = 0$. The homotopy trivial morphisms form an ideal $\mathscr{T}$ in $\mathsf{Kom}(\mathscr{A})$. It means that $f^\bullet \sim 0$ and $g^\bullet \sim 0$ implies that $f^\bullet + g^\bullet \sim$

[1] In general case, one can use the fact that for any small abelian category there is a full exact embedding into a category of modules, see [2, Chapter IV].

0 as well as $f^\bullet \phi^\bullet \sim 0$ and $\psi^\bullet f^\bullet \sim 0$ whenever these sums and products are defined. Therefore we can consider the *homotopy category* $\mathcal{K}(\mathscr{A}) = \mathsf{Kom}(\mathscr{A})/\mathscr{T}$. It has the same objects as $\mathsf{Kom}(\mathscr{A})$ but $\mathrm{Hom}_{\mathcal{K}(\mathscr{A})}(A^\bullet, B^\bullet) = \mathrm{Hom}_{\mathsf{Kom}(cA)}(A^\bullet, B^\bullet)/\mathscr{T}(A^\bullet, B^\bullet)$. The complexes isomorphic in $\mathcal{K}(\mathscr{A})$ are said to be *homotopic*. We write $f^\bullet \sim g^\bullet$ if $f^\bullet - g^\bullet \sim 0$, that is $f^\bullet$ and $g^\bullet$ are in the same class in the factor-category $\mathcal{K}(\mathscr{A})$. In this case $\mathrm{H}^\bullet(f^\bullet) = \mathrm{H}^\bullet(g^\bullet)$, so we can consider $\mathrm{H}^\bullet$ as a functor defined on the homotopy category $\mathcal{K}(\mathscr{A})$.

A *triangle* in the category of complexes or in the homotopy category is, by definition, a sequence of morhisms

e12 (1.2)
$$A^\bullet \xrightarrow{f^\bullet} B^\bullet \xrightarrow{g^\bullet} C^\bullet \xrightarrow{h^\bullet} A^\bullet[1].$$

Morphisms of triangles are just commutative diagrams

$$\begin{tikzcd}
A^\bullet \arrow[r, "f^\bullet"] \arrow[d, "\alpha^\bullet"'] & B^\bullet \arrow[r, "g^\bullet"] \arrow[d, "\beta^\bullet"'] & C^\bullet \arrow[r, "h^\bullet"] \arrow[d, "\gamma^\bullet"'] & A^\bullet[1] \arrow[d, "\alpha^\bullet[1]"'] \\
A'^\bullet \arrow[r, "f'^\bullet"'] & B'^\bullet \arrow[r, "g'^\bullet"'] & C'^\bullet \arrow[r, "h'^\bullet"'] & A'^\bullet[1].
\end{tikzcd}$$

A triangle in $\mathcal{K}(\mathscr{A})$ is said to be *exact* if it is isomorphic to a *cone triangle*

(1.3)
$$A^\bullet \xrightarrow{f^\bullet} B^\bullet \xrightarrow{\iota f^\bullet} Cf^\bullet \xrightarrow{\pi f^\bullet} A^u l[1].$$

Theorem 1.1 implies an immediate consequence.

t12 **Corollary 1.2.** *Let a triangle* (1.2) *be exact in* $\mathcal{K}(\mathscr{A})$. *It induces an exact sequence of cohomologies*

$$\cdots \to \mathrm{H}^n(A^\bullet) \xrightarrow{\mathrm{H}^n(f^\bullet)} \mathrm{H}^n(B^\bullet) \xrightarrow{\mathrm{H}^n(g^\bullet)} \mathrm{H}^n(C^\bullet) \xrightarrow{\mathrm{H}^n(h^\bullet)}$$
$$\xrightarrow{\mathrm{H}^n(h^\bullet)} \mathrm{H}^{n+1}(A^\bullet) \xrightarrow{\mathrm{H}^{n+1}(f^\bullet)} \mathrm{H}^{n+1}(B^\bullet) \xrightarrow{\mathrm{H}^{n+1}(g^\bullet)} \mathrm{H}^n(Cf^\bullet) \to \dots$$

If $F : \mathscr{A} \to \mathscr{B}$ is a functor, its extension to complexes maps homotopy trivial morphisms to homotopy trivial ones, so it induces a functor $\mathcal{K}(F) : \mathcal{K}(\mathscr{A}) \to \mathcal{K}(\mathscr{B})$.

The category $\mathcal{K}(\mathscr{A})$ with the shifts and the class of exact trinagles is an example of a *triangulated category* in the sence of [1], though we will not precise now the latter notion. Nevertheless, the following properties are important.

t13 **Proposition 1.3.** (1) *Every morphism* $f^\bullet$ *can be embedded in an exact triangle* (1.2).

(2) *Given a commutative diagram*

$$\begin{tikzcd}
A^\bullet \arrow[r, "f^\bullet"] \arrow[d, "\alpha^\bullet"'] & B^\bullet \arrow[r, "g^\bullet"] \arrow[d, "\beta^\bullet"'] & C^\bullet \arrow[r, "h^\bullet"] & A^\bullet[1] \\
A'^\bullet \arrow[r, "f'^\bullet"'] & B'^\bullet \arrow[r, "g'^\bullet"'] & C'^\bullet \arrow[r, "h'^\bullet"'] & A'^\bullet[1],
\end{tikzcd}$$

where both rows are exact triangles, there is a morphism $\gamma^\bullet : C^\bullet \to C'^\bullet$ such that the whole diagram

$$\begin{tikzcd}
A^\bullet \arrow[r, "f^\bullet"] \arrow[d, "\alpha^\bullet"'] & B^\bullet \arrow[r, "g^\bullet"] \arrow[d, "\beta^\bullet"'] & C^\bullet \arrow[r, "h^\bullet"] \arrow[d, "\gamma^\bullet"'] & A^\bullet[1] \arrow[d, "\alpha^\bullet[1]"'] \\
A'^\bullet \arrow[r, "f'^\bullet"'] & B'^\bullet \arrow[r, "g'^\bullet"'] & C'^\bullet \arrow[r, "h'^\bullet"'] & A'^\bullet[1].
\end{tikzcd}$$

is commutative. (Note that $\gamma^\bullet$ need not be unique.)

(3) *The triangle* (1.2) *is exact if and only if so is the triangle*

$$B^\bullet \xrightarrow{g^\bullet} C^\bullet \xrightarrow{h^\bullet} A^\bullet[1] \xrightarrow{-f^\bullet[1]} B^\bullet[1].$$

Proof. Left to the reader. Note that we can always replace an exact triangle by a cone triangle. It makes obvious (1) and (2) and makes easier (3). □

We often use special subcategories of the categories of complexes. Namely, we denote by $\mathsf{Kom}^+(\mathscr{A})$ (respectively, by $\mathsf{Kom}^-(\mathscr{A})$) the full subcategory of *left bounded* (respectively, right bounded) complexes $A^\bullet$, i.e. such that $A^n = 0$ for $n < n_0$ (respectively, for $n > n_0$) where n_0 is an integer. We set $\mathsf{Kom}^b(\mathscr{A}) = \mathsf{Kom}^+(\mathscr{A}) \cap \mathsf{Kom}^-(\mathscr{A})$, the subcategory of (two-sided) *bounded complexes.* We denote by $\mathcal{K}^{\mathrm{s}}(\mathscr{A})$, where $\mathrm{s} \in \{+, -, b\}$, the full subcategory of $\mathcal{K}(\mathscr{A})$ consisting of complexes homotopic (i.e. isomorphic in $\mathcal{K}(\mathscr{A})$) to complexes from $\mathsf{Kom}^{\mathrm{s}}(\mathscr{A})$. For the category $\mathcal{K}^-(\mathscr{A})$ one often uses "lower indices" and "homology" language instead of "cohomolgy." Namely, for any complex $A^\bullet$ one sets $A_n = A^{-n}$ and $d_n = d^{-n} : A_n \to A_{n-1}$. Respectively, one write $\mathrm{H}_n(A_\bullet)$ instead of $\mathrm{H}^{-n}(A^\bullet)$. Then, in $\mathcal{K}^-(\mathscr{A})$ one has $A_n = 0$ for $n < n_0$ for some integer n_0.

There is a natural functor $\mathscr{A} \to \mathsf{Kom}(\mathscr{A})$ mapping every object $A \in \mathscr{A}$ to the complex $A^\bullet$, where $A^0 = A$, $A^n = 0$ for $n \neq 0$. Obviously, this functor is full and faithful. Moreover, the induced functor $\mathscr{A} \to \mathcal{K}(\mathscr{A})$ is also full and faithful. So we usually identify $\mathscr{A}$ with its image in $\mathsf{Kom}(\mathscr{A})$ or in $\mathcal{K}(\mathscr{A})$. In particular, we can consider shifts $A[k]$ of objects from $\mathscr{A}$.

s2

2. Derived categories

We have seen that homotopic morphisms induces the same morphisms of cohomologies. In particular, homotopic compexes have isomorphic cohomologies. The converse is not true. For instance, the complex from $\mathsf{Kom}(\mathbb{Z}\text{-Mod})$

$$A^\bullet : \dots \to \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \to \dots$$

is an exact sequence, so has zero cohomologies. Nevertheless, applying the functor $\otimes_{\mathbb{Z}}\mathbb{Z}/2$, we get the complex

$$\dots \to \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \to \dots$$

which is not exact; its cohomologies are $\mathbb{Z}/2$ at every place. Therefore the complex $A^\bullet$ is not homotopic to the zero complex. To improve this situation, consider the following construction. In what follows we consider complexes over a fixed abeliam category $\mathscr{A}$.

A morphism $s^\bullet : A^\bullet \to B^\bullet$ is said to be a *quasi-isomorphism* if all induced morphisms $\mathrm{H}^n(s^\bullet)$ are isomophisms. We denote by $\mathcal{S}$ the class of all quasi-isomorphisms in $\mathsf{Kom}(\mathscr{A})$ as well as its image in $\mathcal{K}(\mathscr{A})$. We define the *derived category* $\mathcal{D}(\mathscr{A})$ as follows. First note that the class of quasi-isomorphisms has the following properties.

t21 **Proposition 2.1.** (1) *If $s^\bullet, t^\bullet \in \mathcal{S}$ and the product $s^\bullet t^\bullet$ is defined, it is also in $\mathcal{S}$.*

(2) *Every pair $f^\bullet, s^\bullet$, where $s^\bullet \in \mathcal{S}$, with a common source or target can be complemented to a commutative diagram*

$$\begin{tikzcd} A^\bullet \arrow[r, "f^\bullet"] \arrow[d, "s^\bullet"'] & B^\bullet \arrow[d, "t^\bullet"'] \\ C^\bullet \arrow[r, "g^\bullet"'] & D^\bullet \end{tikzcd} \quad \text{or} \quad \begin{tikzcd} D^\bullet \arrow[r, "g^\bullet"] \arrow[d, "t^\bullet"'] & B^\bullet \arrow[d, "s^\bullet"'] \\ C^\bullet \arrow[r, "f^\bullet"'] & A^\bullet \end{tikzcd}$$

with $t^\bullet \in \mathcal{S}$.

(3) *There is a morphism $s^\bullet \in \mathcal{S}$ such that $s^\bullet f^\bullet \sim 0$ if and only if there is a morphism $t^\bullet \in \mathcal{S}$ such that $f^\bullet t^\bullet \sim 0$.*

Proof. (1) is evident.

(2) The cone sequence for $s^\bullet$ induces the exact triangle

$$Cs^\bullet[-1] \xrightarrow{\tau^\bullet} A^\bullet \xrightarrow{s^\bullet} C^\bullet \xrightarrow{\iota s^\bullet} Cs^\bullet,$$

where $\tau^\bullet = -\pi s^\bullet[-1]$. By Proposition 1.3(2), it gives a commutative diagram

$$\begin{tikzcd} Cs^\bullet[-1] \arrow[r, "\tau^\bullet"] \arrow[d, equal] & A^\bullet \arrow[r, "s^\bullet"] \arrow[d, "f"'] & C^\bullet \arrow[r, "\iota s^\bullet"] \arrow[d, "g"'] & Cs^\bullet \arrow[d, equal] \\ Cf^\bullet[-1] \arrow[r, "f^\bullet\tau^\bullet"] & B^\bullet \arrow[r, "t^\bullet"] & C(f^\bullet\tau^\bullet)^\bullet \arrow[r] & Cf^\bullet \end{tikzcd}$$

Since $s^\bullet$ is a quasi-isomorphism, the complex $Cs^\bullet$ is acyclic. Therefore $t^\bullet$ is a quasi-isomorphism. Analogously the case of common target is considered (exercise!).

(3) Let $s^\bullet f^\bullet \sim 0$, where $s^\bullet : A^\bullet \to C^\bullet$, $f^\bullet : B^\bullet \to A^\bullet$, and $\{\, h^n \,\}$ be the set of morphisms $B^n \to C^{n-1}$ defining this homotopy. As before, consider the cone sequence for $s^\bullet$ and define $g^\bullet : B^\bullet \to Cs^\bullet[-1]$ setting $g^n = \begin{pmatrix} f^n & -h^n \end{pmatrix}$. Then $f^\bullet = \tau^\bullet g^\bullet$. Let now $t = \pi g^\bullet[-1]$. Then $g^\bullet t^\bullet = 0$, hence $f^\bullet t^\bullet = 0$. Since $Cs^\bullet$ is acyclic, $t^\bullet$ is a quasi-isomorphism. We leave the converse statement as an exercise. $\square$

(1) The *objects* of $\mathcal{D}(\mathscr{A})$ are complexes in $\mathscr{A}$.

(2) A *roof* from $A^\bullet$ to $B^\bullet$ is a pair of morphisms from $\mathcal{K}(\mathscr{A})$

$$\begin{tikzcd} & X^\bullet \arrow[dl, "s"'] \arrow[dr, "f"] & \\ A^\bullet & & B^\bullet, \end{tikzcd}$$

where $s \in \mathcal{S}$.

(3) Two roofs

$$\begin{tikzcd} & X^\bullet \arrow[dl, "s"'] \arrow[dr, "f"] & \\ A^\bullet & & B^\bullet \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & Y^\bullet \arrow[dl, "t"'] \arrow[dr, "g"] & \\ A^\bullet & & B^\bullet \end{tikzcd}$$

are said to be *equivalent* if there is a roof

$$\begin{tikzcd} & Z^\bullet \arrow[dl, "r"'] \arrow[dr, "h"] & \\ X^\bullet & & Y^\bullet \end{tikzcd}$$

such that the diagram

$$\begin{tikzcd} & & Z^\bullet \arrow[dl, "r"'] \arrow[dr, "h"] & & \\ & X^\bullet \arrow[dl, "s"'] \arrow[drrr, "f"' near start] & & Y^\bullet \arrow[dlll, "t" near start] \arrow[dr, "g"] & \\ A^\bullet & & & & B^\bullet \end{tikzcd}$$

is commutative.

(4) A *morphism* $A^\bullet \to B^\bullet$ in the category $\mathcal{D}(\mathscr{A})$ is a class of equivalent roofs from $A^\bullet$ to $B^\bullet$.

(5) The *product* of morphisms presented by the roofs

$$\begin{tikzcd} & X^\bullet \arrow[dl, "s"'] \arrow[dr, "f"] & \\ A^\bullet & & B^\bullet \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & Y^\bullet \arrow[dl, "t"'] \arrow[dr, "g"] & \\ B^\bullet & & C^\bullet \end{tikzcd}$$

is the morphism presented by the roof

$$\begin{tikzcd} & Z^\bullet \arrow[dl, "st'"'] \arrow[dr, "gf'"] & \\ A^\bullet & & C^\bullet, \end{tikzcd}$$

where t' and f' are defined from the commutative diagram

$$\begin{tikzcd} Z^\bullet \arrow[r, "f'^\bullet"] \arrow[d, "t'^\bullet"'] & Y^\bullet \arrow[d, "t^\bullet"'] \\ X^\bullet \arrow[r, "f^\bullet"'] & B^\bullet \end{tikzcd}$$

obtained by Proposition 2.1(2).

(6) The *sum* of morphisms presented by the roofs

$$\begin{tikzcd} & X^\bullet \arrow[dl, "s"'] \arrow[dr, "f"] & \\ A^\bullet & & B^\bullet \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & Y^\bullet \arrow[dl, "t"'] \arrow[dr, "g"] & \\ A^\bullet & & B^\bullet \end{tikzcd}$$

is the morphism presented by the roof

$$\begin{tikzcd} & Z^\bullet \arrow[dl, "st'"'] \arrow[dr, "ft'+gs'"] & \\ A^\bullet & & B^\bullet, \end{tikzcd}$$

where s' and t' are defined from the commutative diagram

$$\begin{tikzcd} Z^\bullet \arrow[r, "s'^\bullet"] \arrow[d, "t'^\bullet"'] & Y^\bullet \arrow[d, "t^\bullet"'] \\ X^\bullet \arrow[r, "s^\bullet"'] & A^\bullet \end{tikzcd}$$

obtained by Proposition 2.1(2).

Certainly, one has to verify that these definitions are compatible, that is that the equivalence of roofs is indeed an equivalence relations, that products and sums do not depend on the choice of representative roofs, etc. We refer to [1, Chapter 3] for these details, which are not difficult, but somewhat cumbersome.

We have a natural functor $Q : \mathcal{K}(\mathscr{A}) \to \mathcal{D}(\mathscr{A})$ mapping a morphism $f^\bullet : A^\bullet \to B^\bullet$ of complexes to the class of the roof

$$\begin{tikzcd} & A^\bullet \arrow[dl, "1_A"'] \arrow[dr, "f"] & \\ A^\bullet & & B^\bullet, \end{tikzcd}$$

The following statement is immediate.

22 **Theorem 2.2.** *The functor Q maps quasi-isomorphisms to isomorphisms and is universal with respect to this property. The latter means that if $T : \mathcal{K}(\mathscr{A}) \to \mathscr{C}$ is a functor mapping quasi-isomorphisms to isomorphisms, there is a unique functor $\bar{T} : \mathcal{D}(\mathscr{A}) \to \mathscr{C}$ such that $T \simeq \bar{T}Q$.*

Again we can consider the cohomology functors as defined on $\mathcal{D}(\mathscr{A})$. Namely, given a morphism ϕ presented by a roof

$$\begin{tikzcd} & X^\bullet \arrow[dl, "s"'] \arrow[dr, "f"] & \\ A^\bullet & & B^\bullet, \end{tikzcd}$$

one can define $\mathrm{H}^\bullet(\phi)$ as $\mathrm{H}^\bullet(f^\bullet)\mathrm{H}^\bullet(s^\bullet)^{-1}$. Exercise: to verify that this definition is consistent.

Note that the full embedding $\mathscr{A} \to \mathcal{K}(\mathscr{A})$ followed by the functor Q remains a full embedding. Indeed, one can easily check (exercise) that any roof

$$\begin{tikzcd} & C^\bullet \arrow[dl, "s"'] \arrow[dr, "f"] & \\ A & & B, \end{tikzcd}$$

where A, B are objects from $\mathscr{A}$, is equivalent to the roof

$$\begin{tikzcd} & A \arrow[dl, "1_A"'] \arrow[dr, "\phi"] & \\ A & & B, \end{tikzcd}$$

where $\phi = \mathrm{H}^0(f)\mathrm{H}^0(s)^{-1}$.

Just as for $\mathcal{K}(\mathscr{A})$ We call a triangle in $\mathcal{D}(\mathscr{A})$ *exact* if it is isomorphic in $\mathcal{D}(\mathscr{A})$ to a cone triangle. Since isomorphisms in $\mathcal{D}(cA)$ preserves cohomologies, the following statement holds.

t23 **Corollary 2.3.** *Let a triangle* (1.2) *be exact in* $\mathcal{D}(\mathscr{A})$. *It induces an exact sequence of cohomologies*

$$\cdots \to \mathrm{H}^n(A^\bullet) \xrightarrow{\mathrm{H}^n(f^\bullet)} \mathrm{H}^n(B^\bullet) \xrightarrow{\mathrm{H}^n(g^\bullet)} \mathrm{H}^n(C^\bullet) \xrightarrow{\mathrm{H}^n(h^\bullet)}$$
$$\xrightarrow{\mathrm{H}^n(h^\bullet)} \mathrm{H}^{n+1}(A^\bullet) \xrightarrow{\mathrm{H}^{n+1}(f^\bullet)} \mathrm{H}^{n+1}(B^\bullet) \xrightarrow{\mathrm{H}^{n+1}(g^\bullet)} \mathrm{H}^n(Cf^\bullet) \to \dots$$

Just as in case of the homotopy category, we can consider the subcategories $\mathcal{D}^{\mathrm{s}}(\mathscr{A})$, where $\mathrm{s} \in \{+, -, b\}$. In these lectures we will be mostly interested in the category $\mathcal{D}^b(\mathscr{A})$.

Exercise 2.4. t24 Prove that if $\mathrm{H}^n(A^\bullet) = 0$ for $n > n_0$ (respectively, $\mathrm{H}^n(A^\bullet) = 0$ for $n < n_0$), the complex $A^\bullet$ is quasi-isomorphic to a right bounded complex) (respectively, to a left bounded complex). If both conditions hold, $A^\bullet$ is isomorphic to a bounded complex.

It means that we can define $\mathcal{D}^{\mathrm{s}}(\mathscr{A})$ by the cohomology properties of complexes.

Suppose that an abelian category $\mathscr{A}$ *has enough projecive objects*, i.e. for every object $A \in \mathscr{A}$ there is an epimorphism $P \to A$, where P is projective.

(It is the case for the category R-Mod.) Let $\mathcal{P}_{\mathscr{A}}$ be the full subcategory of projective objects. It is an additive category, so the homotopy category $\mathcal{K}(\mathcal{P}_{\mathscr{A}})$ is defined. The following result gives a very convenient description of the right bounded derived category in this case.

t25 **Theorem 2.5.** *If the category $\mathscr{A}$ has enough projective objects, the natural embedding $\mathcal{K}^-(\mathcal{P}_{\mathscr{A}}) \to \mathcal{K}^-(\mathscr{A})$ induces an equivalence $\mathcal{K}^-(\mathcal{P}_{\mathscr{A}}) \simeq \mathcal{D}^-(\mathscr{A})$.*

Proof. Let $A^\bullet$ be a complex from $\mathsf{Kom}(\mathscr{A})$. Its *projective resolution* is, by definition, a complex $P^\bullet$ from $\mathsf{Kom}(\mathcal{P}_{\mathscr{A}})$ together with a quasi-isomorphim $\phi^\bullet : P^\bullet \to A^\bullet$. To prove the theorem, we have to prove the following facts:

(1) Every right bounded complex has a right bounded projective resolution.
(2) If $s^\bullet : P^\bullet \to Q^\bullet$ is a quasi-isomorphism of complexes from $\mathsf{Kom}^-(\mathcal{P}_{\mathscr{A}})$, there is a morphism $t^\bullet : Q^\bullet \to P^\bullet$ such that $s^\bullet t^\bullet \sim 1_Q$ and $t^\bullet s^\bullet \sim 1_P$, i.e. $s^\bullet$ is an isomorphism in $\mathcal{K}(\mathscr{A})$.

The construction of a projective resolution of a complex is analogous to the construction of a projective resolution of a module. First recall that for every pair of morphisms α, β with a common target there is their "pullback," i.e a universal commutative diagram

$$\begin{tikzcd} D \arrow[r, "\tilde\alpha"] \arrow[d, "\tilde\beta"'] & C \arrow[d, "\beta"] \\ B \arrow[r, "\alpha"] & A. \end{tikzcd}$$

"Universal" means that if a diagram

$$\begin{tikzcd} D' \arrow[r, "v"] \arrow[d, "u"'] & C \arrow[d, "\beta"] \\ B \arrow[r, "\alpha"] & D' \end{tikzcd}$$

is commutative, there is a unique map $\gamma : D' \to D$ such that $u = \tilde\beta\gamma$ and $v = \tilde\alpha\gamma$. Namely, one can set $D = \operatorname{Ker}\begin{pmatrix}\alpha \\ -\beta\end{pmatrix} \subseteq B \oplus C$; then $\tilde\alpha$ and $\tilde\beta$ arises from the projections of $B \oplus C$ onto B and C. We denote $D = B \oplus_{\alpha,\beta} C$ or, somewhat ambiguously, $D = B \oplus_A C$.

Suppose that $A^\bullet \in \mathsf{Kom}^-(\mathscr{A})$; we may suppose that $A^n = 0$ for $n > 0$. Choose an epimorphism $\phi^0 : P^0 \to A^0$ and consider the pullback

$$\begin{tikzcd} D \arrow[r, "u"] \arrow[d, "v"'] & P^0 \arrow[d, "\phi^0"] \\ A^{-1} \arrow[r, "d^1"] & A^0 \end{tikzcd}$$

(We write d for d_A.) Since ϕ^0 is an epimorphism, so is v, hence $\operatorname{Im}\phi^0 u = \operatorname{Im} d^{-1}$. Choose an epimorphism $p : P^{-1} \to D$ and set $\delta^{-1} = up,\ \phi^{-1} = vp$.

Then $P^0/\operatorname{Im} d_P^{-1} \simeq A^0/\operatorname{Im} d^{-1}$. So we can take $P^{-1} \xrightarrow{\delta^{-1}} P^0$ as the beginning of a projective resolution for $A^\bullet$. Note also that $\operatorname{Ker} d^{-1} \subseteq \operatorname{Im}\phi^{-1} \cap \operatorname{Ker}\delta{-1}$. Now we use a recursive procedure. Suppose at we have already constructed a complex

$$\Pi^\bullet : 0 \to P^{-n} \xrightarrow{\delta^{-n}} P^{-n+1} \to \cdots \to P^0 \to 0$$

and a morphism $\phi^\bullet : \Pi \to A^\bullet$ such that $\mathrm{H}^{-i}(\phi^\bullet)$ are isomorphisms for $i < n$ and epimorphism for $i = n$. Consider the pullback

$$\begin{tikzcd} D \arrow[r, "u"] \arrow[d, "v"'] & K \arrow[d, "\phi^{-n}e"] \\ A^{-n-1} \arrow[r, "d^{-n-1}"] & A^{-n} \end{tikzcd}$$

where $K = \operatorname{Ker}\delta^{-n}$ and e is the embedding $K \to P^{-n}$. One easily see that $(\phi^{-n}e)(\operatorname{Im} u) = \operatorname{Im} d^{-n-1}$. Choose an epimorphism $\pi : P^{-n-1} \to D$ and set $\phi^{-n-1} = v\pi$, $\delta^{-n-1} = eu\pi$. Then we get a complex $\Pi'^\bullet$ and a morphsim $\phi^\bullet : \Pi'^\bullet \to A^\bullet$ such that $\mathrm{H}^{-i}(\phi^\bullet)$ are isomorphisms for $i < n+1$ and epimorphism for $i = n+1$. As a result of this recursive construction, we obtain a projective resolution for the complex $A^\bullet$.

To prove (2), we use the following lemma.

t26 **Lemma 2.6.** *Suppose that a complex $P^\bullet \in \mathsf{Kom}^-(\mathscr{A})$ is acyclic. Then $1_P \sim 0$.*

Proof. We may suppose that $P^n = 0$ for $n > 0$. Let $d = d_P$. Then d^{-1} is an epimorphism. Since P^0 is projective, there is a morphism $s^0 : P^0 \to P^1$ such that $d^{-1}s^0 = 1_{P^0}$. It implies that $P^{-1} = \operatorname{Ker} d^{-1} \oplus \operatorname{Im} s^0$. Then $\operatorname{Ker} d^{-1}$ is projective and d^{-2} induces an epimorphism $P^{-2} \to \operatorname{Ker} d^{-1}$. Thus there is a morphism $s^{-1} : P^{-1} \to P^{-2}$ zero on $\operatorname{Im} s^0$ and such that $d^{-2}s^{-1}$ is identical on $\operatorname{Ker} d^{-1}$, wherefrom $d^{-2}s^{-1} + s^0 d^{-1} = 1_{P^{-2}}$. Going on, we get a homotopy $1_P \sim 0$. (We leave the obvious details to the readers.) $\square$

Let now $P^\bullet$ and $Q^\bullet$ be complexes from $\mathsf{Kom}^-(\mathscr{A})$ and $f^\bullet : P^\bullet \to Q^\bullet$ be a quasi-isomorphism. Then the cone $C^\bullet = Cf^\bullet$ is acyclic, so $1_C \sim 0$. Let this homotopy be given by a set of morphisms s^n, where

$$s^n = \begin{pmatrix} -x^{n+1} & y^n \\ z^{n+1} & t^n \end{pmatrix}.$$

(The indices correspond to the origins of the morphisms, for instance, $x^{n+1} : P^{n+1} \to P^n$.) Recalling the formula for $d = d_C$, we get from the equalities $s^{n-1}d^n + d^{n+1}s^n = 1_C$ the equalities

$$\begin{aligned} &x^{n+2}d_P^{n+1} + y^{n+1}f^{n+1} + d_P^n x^{n+1} = 1_{P_{n+1}}, \\ &t^{n+1}d_Q^n + f^n y^n + d_Q^n t^n = 1_{Q_n}, \\ &y^{n+1}d_Q^n - d_P^n y^n = 0. \end{aligned}$$

The last equality means that $y^\bullet$ is a morphism $Q^\bullet \to P^\bullet$, while the other two ones show that $y^\bullet f^\bullet \sim 1_P$ and $f^\bullet y^\bullet \sim 1_Q$. $\square$

Just in the same way we can procede if $\mathscr{A}$ has enough injective objects, i.e. for every object A there is a monomorphism $A \to I$, where I is injective. It is the case both if $\mathscr{A} = R$-Mod and if $\mathscr{A} = \mathrm{Qcoh}(X)$. Let $\mathcal{I}_{\mathscr{A}}$ be the full subcategory of injective objects. Then we have the dual to Theorem 2.5.

Theorem 2.5°. *If the category $\mathscr{A}$ has enough injective objects, the natural embedding $\mathcal{K}^+(\mathcal{I}_{\mathscr{A}}) \to \mathcal{K}^+(\mathscr{A})$ induces an equivalence $\mathcal{K}^+(\mathcal{I}_{\mathscr{A}}) \simeq \mathcal{D}^+(\mathscr{A})$.*

Note that Theorems 2.5 and 2.5° *do not imply* that $\mathcal{K}^b(\mathcal{P}_{\mathscr{A}}) \simeq \mathcal{D}^b(\mathscr{A})$ or $\mathcal{K}^b(\mathcal{I}_{\mathscr{A}}) \simeq \mathcal{D}^b(\mathscr{A})$. The reason is that a projective (or injective) resolution of a bounded complex need not be bounded from the left (respectively, from the right). For instance, if $\mathscr{A} = \mathbb{Z}/4$-Mod, the module $\mathbb{Z}/2$ does not have a finite projective resolution. It follows, for instance, from the fact that $\mathrm{Ext}^n(\mathbb{Z}/2, mZ/2) = \mathbb{Z}/2$ for all $n \geqslant 0$. Actually, it is the only obstacle. Namely, call a category $\mathscr{A}$ *left regular* (respectively, *right regular*) if every objects from $\mathscr{A}$ has a finite projetive resolution (respectively, a finite injective resolution. One can show that if $\mathscr{A}$ has both enough projective and enough injective modules, these conditions are equivalent.

t27 **Theorem 2.7.** *Suppose that the category $\mathscr{A}$ has enough projective modules. The follwing conditions are equivalent.*

(1) *$\mathscr{A}$ is left regular.*
(2) *Every bounded complex has a bounded projective resolution.*
(3) *The natural functor $\mathcal{K}^b(\mathcal{P}_{\mathscr{A}}) \to \mathcal{D}^b(\mathscr{A})$ is an equivalence.*

Proof. (2)⇒(3)⇒(1) is evident. Suppose $\mathscr{A}$ is left regular, $A^\bullet \in \mathsf{Kom}^b(\mathscr{A})$ and $P^\bullet$ is a right bounded projective resolution of $A^\bullet$. If $A^n = 0$ for $n \leqslant m$, then $\mathrm{H}^n(P^\bullet) = 0$ for $n \leqslant m$. Consider a finite projective resolution of $\mathrm{Im}\, d^m$:

$$0 \to Q^r \to Q^{r+1} \to \dots \to Q^0 \to 0.$$

Let $\Pi^\bullet$ be the complex

$$0 \to Q^r \to Q^{r+1} \to \dots \to Q^0 \xrightarrow{\alpha} P^{m+1} \to P^{m+2} \to \dots$$

where α is the composition of the projection $Q^0 \to \mathrm{Im}\, d^m$ and the embedding $\mathrm{Im}\, d_m \to P^{m+1}$. It is quasi-isomorphic to $P^\bullet$, so there is a quasi-isomorphim $\Pi^\bullet \to P^\bullet$. Therefore, $\Pi^\bullet$ is a bounded projective resolution of $A^\bullet$, so (1)⇒(2). $\square$

Certainly, the dual theorem for right regular categories is also true, but from now on we leave the formulation and proofs of all dual notions and results to the reader.

Suppose that $\mathscr{A}$ has enough projective objects. The image in $\mathcal{D}^b(\mathscr{A})$ of the homotopy category $\mathcal{K}^b(\mathscr{A})$ is called the *perfect derived category* of $\mathscr{A}$ and is denoted by $\mathrm{Perf}(\mathscr{A})$. A complex $A^\bullet$ is said to be *perfect* if it is quasi-isomorphic to a complex from $\mathrm{Perf}(\mathscr{A})$/ Equivalently, it has a bounded projective resolution.

The subcategory $\mathrm{Perf}(\mathscr{A}) \subseteq \mathscr{A}$ is *triangular*, i.e. $A^\bullet[1] \in \mathrm{Perf}(\mathscr{A})$ if and only if $A^\bullet \in \mathrm{Perf}(\mathscr{A})$. Moreover, it is *full* (or *Serre subcategory*), which means that if $A^\bullet \to B^\bullet \to C^\bullet \to A^\bullet[1]$ is an exact triangle and two of the objects A, B, C are in $\mathrm{Perf}(\mathscr{A})$, then the third one also belongs to $\mathrm{Perf}(\mathscr{A})$. It follows from the fact that a cone of a morphism of bounded complexes of projective objects is again a bounded complex of projective objects. It allows to define a quotient category, just as we have defined the derived category "inverting quasi-isomorphisms." Namely, let now $\mathcal{T}$ be the set of morphisms $f^\bullet : A^\bullet \to B^\bullet$ such that $Cf^\bullet \in \mathrm{Perf}(\mathscr{A})$. One can show that $\mathcal{T}$ has the properties (1)–(3) of Proposition 2.1. Then one can construct the new category using "roofs" with the left side from $\mathcal{T}$. The resulting category is called the *singular derived category* of $\mathscr{A}$ and denoted by $\mathcal{D}_{sing}(\mathscr{A})$. Certainly, if $\mathscr{A}$ is left regular (for instance, $\mathscr{A} = R$-Mod for a ring R of finite global dimension), $\mathcal{D}_{sing}(\mathscr{A}) = 0$, so this category indeed mesures the "singularity" of $\mathscr{A}$.

For two complexes $A^\bullet, B^\bullet$ we define

$$\mathrm{Ext}^k_{\mathscr{A}}(A^\bullet, B^\bullet) = \mathrm{Hom}_{\mathcal{D}(cA)}(A^\bullet, B^\bullet[k]).$$

Since the shift is an equivalence of categories, also

$$\mathrm{Ext}^k(A^\bullet, B^\bullet) \simeq \mathrm{Hom}_{\mathcal{D}(\mathscr{A})}(A^\bullet[-k], B^\bullet) \simeq \mathrm{Hom}_{\mathcal{D}(\mathscr{A})}(A^\bullet[j], B^\bullet[j+k])$$

for any $j \in \mathbb{Z}$. In particular, $\mathrm{Ext}^k_{\mathscr{A}}(A, B)$ is defined for objects $A, B \in \mathscr{A}$ considered as complexes as above. The following result explains this notion.

t28 **Theorem 2.8.** *For any two objects $A, B \in \mathscr{A}$*

(1) $\mathrm{Ext}^0_{\mathscr{A}}(A, B) \simeq \mathrm{Hom}_{\mathscr{A}}(A, B)$.
(2) $\mathrm{Ext}^k_{\mathscr{A}}(A, B) = 0$ *if* $k < 0$.
(3) *If $k > 0$, the elements of $\mathrm{Ext}^k(A, B)$ can be presented by the roofs $A \xleftarrow{s} C^\bullet f B$, where $C^\bullet$ is the complex of the form*

e21 (2.1) $$C^\bullet : 0 \to B = C^{-n} \to C^{-n+1} \to \cdots \to C^{-1} \to C^0 \to 0,$$

such that $\mathrm{H}^i(C^\bullet) = 0$ for $i \neq 0$, $\mathrm{H}^0(C^\bullet) \simeq A$, the quasi-isomorphism s is defined by the projection $C^0 \to \mathrm{H}^0(C^\bullet) \simeq A$ and f is given by the identity map $C^{-n} \to B$.

Note that the complexes of the form (2.1) can be identified with the exact sequences

$$0 \to B \to A_n \to A_{n-1} \to \cdots \to A_2 \to A_1 \to A \to 0 :$$

just set $A_k = C^{-k+1}$ and define the map $A_1 = C^0 \to A$ as the surjection $C^0 \to \mathrm{H}^0(C^\bullet) \simeq A$.

Proof. (1) is already known, since the functor $\mathscr{A} \to \mathcal{D}(\mathscr{A})$ is a full embedding.

(2) Consider ane roof $A \xleftarrow{s} X^\bullet \xrightarrow{f} B[k]$, where $k < 0$ and s is a quasi-isomorphism. Then the complex $X^\bullet$ is exact at all terms X^n except X^0, in particular, the sequence $X^{k-1} \xrightarrow{d^{k-1}} X^k \xrightarrow{d_k} X^{k+1}$ is exact. Therefore, the morphism of complexes

$$\begin{array}{ccccccccccc}
\widetilde{X}^\bullet : \dots & \longrightarrow & X^{-1} & \longrightarrow & X^0 & \longrightarrow & \operatorname{Im} d^0 & \longrightarrow & 0 & \longrightarrow & \dots \\
& & \Big\| & & \Big\| & & {\scriptstyle\iota}\Big\downarrow & & {\scriptstyle 0}\Big\downarrow & & \\
\dots & \longrightarrow & X^{-1} & \longrightarrow & X^0 & \longrightarrow & X^1 & \longrightarrow & X^2 & \longrightarrow & \dots
\end{array}$$

where ι is the natural embedding, is a quasi-isomorphism, so we can replace the original roof by the roof $A \xleftarrow{s} \widetilde{X}^\bullet \xrightarrow{f} B[k]$. But $B[k]$ has a unique nobzero component at the place $-k$, while $\widetilde{X}^{-k} = 0$, so there are no nozero morphisms $\widetilde{X}^\bullet \to B[k]$.

(3) Let $A \xleftarrow{s} X^\bullet \xrightarrow{f} B[k]$ be a roof, where $k > 0$. Then, as above, $\operatorname{Im} d_X^{n-1} = \operatorname{Ker} d_X^n$ if $n \neq 0$ and $f^\bullet$ is given by a morphism $f : X^{-k} \to B$ such that $fd^{-k-1} = 0$, so f factors through $X^{-k}/\operatorname{Ker} d^{-k}$. Then we can replace $X^\bullet$ by the quisi-isomorphic complex, replacing X^0 by $\operatorname{Ker} d^0$, X^{-k} by $X^{-k}/\operatorname{Ker} d^{-k}$ and X^n by 0 for $n > 0$ and for $n < -k$. So we may suppose that $X^\bullet$ is of the form

$$\dots\ 0 \to X^{-k} \xrightarrow{d} X^{-k+1} \xrightarrow{d} \dots \xrightarrow{d} X^{-1} \xrightarrow{d} X0\ \dots$$

We can define a new complex $\widetilde{X}$ and a morphism $t : X \to \widetilde{X}$ as follows:

$$\begin{array}{ccccccccc}
X : 0 & \longrightarrow & X^{-k} & \xrightarrow{\quad d \quad} & X^{-k+1} & \xrightarrow{\quad d \quad} & X^{-k+2} & \longrightarrow & \dots \\
& & \begin{pmatrix} -1 \\ f \end{pmatrix}\Big\downarrow & & \begin{pmatrix} 1 \\ 0 \end{pmatrix}\Big\downarrow & & \Big\| & & \\
\widetilde{X} : 0 & \longrightarrow & X^{-k} \oplus B & \xrightarrow[\begin{pmatrix} -d & 0 \\ f & 1 \end{pmatrix}]{} & X^{-k+1} \oplus B & \xrightarrow[\begin{pmatrix} d & 0 \end{pmatrix}]{} & X^{-k+2} & \longrightarrow & \dots
\end{array}$$

One easily sees (exercise) that $\widetilde{X}$ is exact ourside the 0-th term and t is a quasi-isomorphism. Moreover, f factors as $f't$, where $f' = \begin{pmatrix} 0 & 1 \end{pmatrix} : X^{-k} \oplus B \to B$. Hence, we can replace X by $\widetilde{X}$. Finally, there is an embedding of complexes

$$\begin{array}{ccccccccc}
\bar{X} : 0 & \longrightarrow & X^{-k} & \xrightarrow{\quad 1 \quad} & X^{-k} & \longrightarrow & 0 & & \\
& & \begin{pmatrix} 1 \\ 0 \end{pmatrix}\Big\downarrow & & \begin{pmatrix} -d \\ f \end{pmatrix}\Big\downarrow & & & & \\
\widetilde{X} : 0 & \longrightarrow & X^{-k} \oplus B & \xrightarrow[\begin{pmatrix} -d & 0 \\ f & 1 \end{pmatrix}]{} & X^{-k+1} \oplus B & \xrightarrow[\begin{pmatrix} d & 0 \end{pmatrix}]{} & X^{-k+2} & \longrightarrow & \dots
\end{array}$$

Since the first row is acyclic, the projection $\widetilde{X} \to \widetilde{X}/\bar{X}$ is a quasi-isomorphism. Since $f' \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0$, f' factors through the quotient $C^\bullet = \widetilde{X}/\bar{X}$, which is just of the form (2.1). $\square$

s3

3. Derived functors

Let $F : \mathscr{A} \to \mathscr{B}$ be a functor between additive categories. It induces a functor $\mathsf{Kom}(\mathscr{A}) \to \mathsf{Kom}(\mathscr{B})$ (componentwise), whichy we also denote by F. Moreover, the latter maps homotopy trivial morphisms of complexes to homotopy trivial, so induces a functor $\mathcal{K}F : \mathcal{K}(\mathscr{A}) \to \mathcal{K}(\mathscr{B})$. On the contrary, if the categories $\mathscr{A}, \mathscr{B}$ are abelian and the functor F is not exact, it does not map acyclic complexes to acyclic, so it does not map quasi-isomorphisms to quasi-isomorphisms. Therefore, it does not induce a functor between derived categories. We are going to "improve" this situation. It is possible if $\mathscr{A}$ contains enough projective or injective objects and we consider right bounded or, respectively, left bounded complexes. Recall that $Q_{\mathscr{A}}$ denote the natural functor $\mathcal{K}(\mathscr{A}) \to \mathcal{D}(\mathscr{A})$, which is identity on complexes and maps a morphism $f : A^\bullet \to B^\bullet$ to the ass of the roof $A^\bullet \xleftarrow{1_A} A^\bullet \xrightarrow{f} B$.

So suppose that $\mathscr{A}$ has enough projective objects. Then for every complex $A^\bullet \in \mathcal{K}^-(\mathscr{A})$ is quasi-isomorphic to a complex from $\mathcal{K}^-(\mathcal{P}_{\mathscr{A}})$. We fix a quasi-isomorphism $s_A : P^\bullet_A \to A^\bullet$, where $P^\bullet \in \mathcal{K}^-(\mathscr{A})$ and define $LF(A^\bullet)$ as $(Q_{\mathscr{B}} \circ \mathcal{K}F)(P^\bullet_A)$, that is as the image of $\mathcal{K}F(P^\bullet_A)$ in $\mathcal{D}^-(\mathscr{B})$. If $f : A^\bullet \to B^\bullet$ is a morphism of complexes, it can be embedded into a commutative diagram

$$\begin{tikzcd} P^\bullet_A \arrow[r, "\widetilde{f}"] \arrow[d, "s_A"'] & P^\bullet_B \arrow[d, "s_B"'] \\ A^\bullet \arrow[r, "f"] & B^\bullet \end{tikzcd}$$

and we define $LF(f)$ as the image of $F(\widetilde{F})$ in $\mathcal{D}(\mathscr{B})$. Recall that $\widetilde{f}$ is defined up to a homotopy, so its choice does not imply $LF(f)$. In this way we obtain a functor $LF : \mathcal{D}^-(\mathscr{A}) \to \mathcal{D}^-(\mathscr{B})$ called a *left adjoint functor* of F. We define a morhism of functors $\varepsilon^-_{\mathscr{A}} : LF \circ Q_{\mathscr{A}} \to Q_{\mathscr{B}} \circ \mathcal{K}F$ setting

$$\varepsilon^-_{\mathscr{A}}(A^\bullet) = Q_{\mathscr{B}}(s_A) : LF(Q_{\mathscr{A}}(A^\bullet)) = Q_{\mathscr{B}}(\mathcal{K}F(P^\bullet)) \to Q_{\mathscr{B}}(\mathcal{K}F(A^\bullet)).$$

If $\mathscr{A}$ has enough injective objects, one can construct a *right adjoint functor* $RF : \mathcal{D}^+(\mathscr{A}) \to \mathcal{D}^+(\mathscr{B})$ considering injective resolutions $t_A : A^\bullet \to I^\bullet_A$ and setting $RF(A^\bullet) = \mathcal{K}F(I^\bullet_A)$. Then a natural functor $\varepsilon^+_{\mathscr{A}} : Q_{\mathscr{B}} \circ \mathcal{K}F \to RF \circ Q_{\mathscr{A}}$ is defined. We leave the details to the reader.

These constructions depend on the choice of projective or injective resolutions. Nevertheless, one easily verifies that another choice of resolutions gives an equivalent functor, so one can speak on *"the" left* (or *right*) *adjoint functor.* Actually, they can be characterized by a unversal property as follows.

We say that a functor $G : \mathcal{D}(\mathscr{A}) \to \mathcal{D}(\mathscr{B})$ is *exact* if it maps exact triangels to exact triangles. Recall that $Q_{\mathscr{A}}$ denote the natural functor $\mathcal{K}(\mathscr{A}) \to \mathcal{D}(\mathscr{A})$.

t31 **Theorem 3.1** (See § III.6 of [1])**.**

(1) *Suppose that $\mathscr{A}$ has enough projective objects.*
  (a) *A left derived functor LF is exact.*
  (b) *If $G : \mathcal{D}^-(\mathscr{A}) \to \mathcal{D}^-(\mathscr{B})$ is exact and $\eta : G \circ Q_{\mathscr{A}} \to Q_{\mathscr{B}} \circ \mathcal{K}F$ is a morphism of functors, there is a unique moprhism $\theta : G \to LF$ such that $\eta = \varepsilon^-_{\mathscr{A}}(\theta \circ Q_{\mathscr{A}})$.*

(2) *Suppose that $\mathscr{A}$ has enough injective objects.*
  (a) *A right derived functor RF is exact.*
  (b) *If $G : \mathcal{D}^+(\mathscr{A}) \to \mathcal{D}^+(\mathscr{B})$ is exact and $\eta : Q_{\mathscr{B}} \circ \mathcal{K}F \to G \circ Q_{\mathscr{A}}$ is a morphism of functors, there is a unique moprhism $\theta : RF \to G$ such at $\eta = (Q_{\mathscr{A}} \circ \theta)\varepsilon^+_{\mathscr{A}}$.*

Since the values of LF and RF are complexes in the derived category, we can consider their cohomologies, thus defining the functors $\mathcal{D}^{\pm}(\mathscr{A}) \to \mathscr{B}$:

$$L_nF(A^\bullet) = \mathrm{H}^{-n}(LF(A^\bullet)),$$
$$R^nF(A^\bullet) = \mathrm{H}^n(RF(A^\bullet)).$$

Moreover, for any exact triangle

$$A^\bullet \to B^\bullet \to C^\bullet \to A^\bullet[1]$$

in $\mathcal{D}^{\pm}(\mathscr{A})$ we obtain exact sequences

$$\cdots \to L_{n+1}F(C^\bullet) \to L_nF(A^\bullet) \to L_nF(B^\bullet) \to L_nF(C^\bullet) \to L_{n-1}F(A^\bullet) \to \dots$$
$$\cdots \to R^{n-1}F(C^\bullet) \to R^nF(A^\bullet) \to R^nF(B^\bullet) \to R^nF(C^\bullet) \to R^{n+1}F(A^\bullet) \to \dots$$

t32 **Example 3.2.** Let A, B are objects of $\mathscr{A}$, $F = \mathrm{Hom}(\_, B)$. We calculate $R^nF(A)$. Note that F is a *contravariant functor*, that is a functor $\mathscr{A}^{\mathrm{op}} \to \mathsf{Ab}$, where $\mathscr{A}^{\mathrm{op}}$ is the category opposite to $\mathscr{A}$ and Ab is the category of abelian groups. To define RF we must construct injective resolutions in $\mathscr{A}^{\mathrm{op}}$ or, the same, projective resolutions in $\mathscr{A}$. If $P^\bullet$ is a projective resolution of A, then $RF(A) = F(P^\bullet)$, which is the image in $\mathscr{D}(\mathsf{Ab})$ of the complex whose n-th component is $\mathrm{Hom}_{\mathscr{A}}(P^{-n}, B)$ and the differential maps $\alpha : P^{-n} \to B$ to $\alpha d^{-n-1} : P^{-n-1}, B$. Therefore, for the cohomologies we obtain

$$R^n \mathrm{Hom}_{\mathscr{A}}(A, B) = \frac{\{\, \alpha : P^{-n} \to B \mid \alpha d^{-n-1} = 0 \,\}}{\{\, \beta d^{-n} \mid \beta : P^{-n+1} \to B \,\}}.$$

But this quotient coincide with $\mathrm{Hom}_{\mathcal{K}(\mathscr{A})}(P^\bullet, B[n])$ (exercise: explain this claim!). Since $P^\bullet$ is a complex of projective objects quasi-isomorphic to A, we know that

$$\mathrm{Hom}_{\mathcal{K}(\mathscr{A})}(P^\bullet, B[n]) \simeq \mathrm{Hom}_{\mathcal{D}(\mathscr{A})}(P^\bullet, B[n]) \simeq$$
$$\simeq \mathrm{Hom}_{\mathcal{K}(\mathscr{A})}(A, B[n]) = \mathrm{Ext}^n(A, B).$$

So $R^n \operatorname{Hom}_{\mathscr{A}}(A,B) \simeq \operatorname{Ext}^n_{\mathscr{A}}(A,B)$.

t33 **Exercise 3.3.** Obtain the same result for the functor $RF' = \operatorname{Hom}_{\mathscr{A}}(A, \_)$.

So we can calculate $\operatorname{Ext}^n_{\mathscr{A}}$ as the right derived functor of $\operatorname{Hom}_{\mathscr{A}}$ either with the fixed first argument or with the fixed second one.

In some cases one can use other resolutions to calculate derived functors.

t34 **Definition 3.4.** Let $F : \mathscr{A} \to \mathscr{B}$ be a functor. We call an object $C \in \mathscr{A}$ *right F-acyclic* if $R^nF(C) = 0$ for $n \neq 0$ and *left F-acyclic* for $n \neq 0$.

Note that in both cases we only have to consider $n > 0$, since always R^nF and $L_nF = 0$. Note also that every projective object is left F-acyclic and every injective object is right F-acyclic with respect to to any functor F.

From the exact sequence of derived functors it follows that if a sequence $0 \to A \to B \to C \to 0$ is exact and the objects B, C are left F-acyclic, so is also A. If A, B are right F-acyclic, so is C.

t35 **Theorem 3.5.** (1) *Suppose that the functor $F : \mathscr{A} \to \mathscr{B}$ is right exact and $C^\bullet$ is a right bounded complex of left F-acyclic objects. Then $LF(C^\bullet) \simeq F(C^\bullet)$ in $\mathcal{D}(\mathscr{B})$.*

(2) *Suppose that the functor $F : \mathscr{A} \to \mathscr{B}$ is left exact and $C^\bullet$ is a left bounded complex of right F-acyclic objects. Then $RF(C^\bullet) \simeq F(C^\bullet)$ in $\mathcal{D}(\mathscr{B})$.*

Proof. We only consider case (1), leaving (2) as an easy exercise. First note that if $P^\bullet$ is a projective resolution of an object A with $P^n = 0$ for $n > 0$, then we have an exact sequence $P^{-1} \to P^0 \to A \to 0$. Applying F we get an exact sequence $FP^{-1} \to FP^0 \to FA \to 0$. Therefore, $R^0F(A) = \operatorname{Coker}(FP^{-1} \to FP^0) \simeq F(A)$. Now the crucial role has the following lemma.

t36 **Lemma 3.6.** *If a right bounded complex $C^\bullet$ is acyclic and all objects C^n are left F-acyclic, then the complex $FC^\bullet$ is acyclic too.*

Proof. First consider the case when $C^\bullet$ is a short exact sequence $0 \to C_1 \to C_2 \to C_3 \to 0$ with left F-acyclic C_i. It gives rise to an exact triangle

$$C_1 \to C_2 \to C_3 \to C_1[1],$$

therefore for the derived functors we get an exact sequence

$$L_1F(C_3) = 0 \to L_0F(C_1) = FC_1 \to L_0F(C_2) = FC_2 \to L_0F(C_3) = FC_3 \to 0,$$

so $FC^\bullet isexact$ is exact. Now every acyclic right bounded complex $C^\bullet$ arises from short exact sequences $0 \to C_1^n \to C^n \to C_2^n \to 0$, where $C_1^n = \operatorname{Ker} d^n$ and $C_2^n = \operatorname{Im} d^n$; moreover, $C_2^n = C_1^{n+1}$. If $C^n = 0$ for $n > m$, $C_2^{m-1} = C_1^m = C^m$ is left F-acyclic, so C_1^{m-1} is left F-acyclic. An easy induction shows that all C_i^n $(i = 1, 2)$ are left F-acyclic. Therefore, applying F, we get ecaxt sequences $0 \to FC_1^n \to FC^n \to FC_2^n \to 0$. Gluing them together,we see that the whole complex $FC^\bullet$ is acyclic. $\square$

Let now $s : P^\bullet \to C^\bullet$ be a projective resolution of a right bounded complex consisting of left F-acyclic objects. Since s is a quasi-isomorphism, the cone $Cs^\bullet$ is acyclic. Its terms are direct sums $P^{n+1} \oplus C^n$, so are left F-acyclic. Therefore its image $F(Cs^\bullet)$ is acyclic too. The triangle

$$FP^\bullet \xrightarrow{Fs} FC^\bullet \to F(Cs^\bullet) \to FP^\bullet[1]$$

is also a cone triangle, hence exact. Since $F(Cs^\bullet)$ is acyclic, Fs is a quasi-isomorphism. Thus $LF(C^\bullet) = FP\bullet \simeq FC^\bullet$ in the derived category. □

From this theorem we obtain an important corollary concerning the compositions of derived functors.

t37 **Corollary 3.7.** *Let we have two functors $\mathscr{A} \xrightarrow{F} \mathscr{B} \xrightarrow{G} \mathscr{C}$.*

(1) *Suppose the categories $\mathscr{A}, \mathscr{B}$ have enough projective modules, G is right exact and F maps projective objects to left F-acyclic. Then $L(GF) \simeq LG \circ LF$.*
(2) *Suppose the categories $\mathscr{A}, \mathscr{B}$ have enough injective modules, G is left exact and F maps injective objects to right F-acyclic. Then $R(GF) \simeq RG \circ RF$.*

Proof. This time we prove (2). Let $s : A^\bullet \to I^\bullet$ be an injective resolution of a left bounded complex $A^\bullet$. Then, by definition, $R(GF)(A^\bullet) \simeq GF(I^\bullet)$ in $\mathcal{D}(\mathscr{C})$ and $RF(A^\bullet) \simeq F(I^\bullet)$ in $\mathcal{D}(\mathscr{B})$. Since $F(I^\bullet)$ is left bounded and consists of G-acyclic modules, $RG(F(I^\bullet)) \simeq GF(I^\bullet)$. It gives the necessary isomorphism $RG \circ RF(A^\bullet) \simeq R(GF)(A^\bullet)$. □

References

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