On Schappert’s characterization of strictly unimodal plane curve singularities

Yuri A. Drozd *  Gert-Martin Greuel

Introduction

The representation theory of curve singularities (more precisely, their local rings) turns out to be closely related to their “deformation” properties. Namely, as was shown in [6],[8],[7], such ring $\mathcal{R}$ is of finite type, that is has only finitely many torsion-free indecomposable modules (up to isomorphism), if and only if it dominates one of the so called simple plane curve singularities in the sense of [1]. In [4] the authors have shown that $\mathcal{R}$ is of tame type, that is has only 1-parametre families of indecomposable torsion-free modules, if and only if it dominates one of the unimodal plane curve singularities of type $T_{pq}$ (again in the classification of [1]).

These singularities form the “serial” part of the list of all unimodal plane curve singularities. There are also 14 “exceptional” ones, which happen to be wild, that is possesses $n$-parametre families of (non-isomorphic) indecomposable modules for arbitrary large $n$. Of course, the bimodal plane curve singularities in the sense of [1] are also wild. Nevertheless, in [10] was shown that all uni- and bimodal plane curve singularities possess only 1-parametre families of ideals. Remark that in [12] these singularities are called strictly unimodal and we prefer to use this terminology.

The aim of our paper is to show that the strictly unimodal plane curve singularities are in some sense “universal” among those having not more than 1-parametre families of ideals. Namely, we prove that a curve singularity has this property if and only if it dominates one of the strictly unimodal plane curve singularities.

*Supported by DFG and ISF, grant RKJ200, together with the Government of Ukraine.
Moreover, we prove this result for the curve singularities of arbitrary characteristic using, instead of the definition of such singularities by corresponding relations, their definition via parametrization given in [10]. Remark that it follows from [6],[8],[7] that a curve singularity has only finitely many non-isomorphic ideals if and only if it is of finite type (in contrast with 1-parametre case).

The proof of this theorem follows the same scheme as that of the main result on tameness from [4]. Namely, we first introduce some “overing conditions” for the ring \( R \) and show that whenever they do not hold, \( R \) possess 2-parametre families of non-isomorphic ideals. Then we show that these conditions imply that \( R \) dominates a strictly unimodal plane curve singularity. To accomplish the proof, we need also to show that any strictly unimodal plane curve singularity has not more than 1-parametre families of ideals. But indeed, one can calculate all ideals of these rings. It has been done in [10]. Though Schappert used the “definition via relations”, one can verify (and we do it here for three most complicated examples) that his calculations depend only on the parametrization of these rings. This calculation of ideals shows that all strictly unimodal plane curve singularities really have only 1-parametre families of ideals.

1. Preliminaries

**Notation 1.1.** Throughout this article we use the following notations:

1. \( R \) denotes a complete local noetherian ring without nilpotent elements.
2. \( Q \) its full ring of fractions.
3. \( m = \text{rad} R \) its unique maximal ideal.
4. \( k = R/m \), the residue field of \( R \).
5. \( R_0 \) its normalization, i.e. its integral closure in \( Q \).
6. \( R_i = m^i R_0 + R \).
7. \( m_i = m^i R_0 + m \) (obviously, it is the maximal ideal of \( R_i \) for \( i > 0 \)).
8. \( d (M) \) the minimal number of generators of an \( R \)-module \( M \) or, the same, \( \dim (M/mM) \) (over the residue field \( k \)).
9. \( d_i = d (R_i) \).
Later on we suppose \( k \) to be algebraically closed.

**Definition 1.2.** \( R \) is said to be a *curve singularity* provided it satisfies the following conditions:

1. \( R \) is \( k \)-algebra and \( R/m = k \).
2. \( R \) is of Krull dimension 1.

Really, such rings are just the completions of the local rings of points of arbitrary (reduced) algebraic curves over the field \( k \).

It is known (cf. [3]) that in this case \( d_0 \) is finite and, moreover, \( d(I) \leq d_0 \) for each \( R \)-ideal \( I \).

Remind the definitions related to the families of \( R \)-modules (cf. [5],[9]). Really, we will consider here only *full* \( R \)-ideals, i.e. such ideals \( I \), that \( QI = Q \) (later on we omit the epithet “full”).

**Definition 1.3.** Let \( X \) be an algebraic variety over \( k \) and \( \mathcal{I} \) an \( R \otimes \mathcal{O}_X \)-ideal, such that \( Q\mathcal{I} = Q \otimes \mathcal{O}_X \). Call \( \mathcal{I} \) a *family of ideals with the base* \( X \) if it is flat over \( \mathcal{O}_X \) and, moreover, \( \mathcal{I}/r\mathcal{I} \) is \( \mathcal{O}_X \)-flat for each non-zero-divisor \( r \in R \).

A series of such families, which are in some sense *universal*, can be constructed as follows. Consider the subvariety \( B(d) \) of the Grassmannian \( \text{Gr}(d, R_0/R) \), consisting of those subspaces, which are \( R \)-submodules in \( R_0/R \). The pre-image \( \mathcal{I}(d) \) in \( R_0 \otimes \mathcal{O}_{B(d)} \) of the canonical vector bundle on \( B(d) \) is then a family of \( R \)-ideals and any other family can be “glued” from the inverse images of the families \( \mathcal{I}(d) \) (cf. [5], Proposition 3.4 and Corollary 3.5). Hence, we are able to define, following [9], the *number of parameters for* \( R \)-ideals *par* \((1, R)\).

**Definition 1.4.** Denote \( B(d, i) \) the subset of \( B(d) \) consisting of such points \( x \) that the set (which is locally closed) \( \{ y \in B(d) \mid \mathcal{I}(d)(y) \simeq \mathcal{I}(d)(x) \} \) has the dimension \( i \) and define:

\[
\text{par} (1, R) = \max_{d,i} \{ \dim B(d, i) - i \}.
\]

(Remark that \( B(d, i) \) is also locally closed in \( B(d) \)).
They say that a ring $R_1$ dominates $R$ if $R \subseteq R_1 \subseteq R_0$. In this case, evidently, $\text{par}(1, R_1) \leq \text{par}(1, R)$. It follows from [7], [3] that $\text{par}(1, R) = 0$ if and only if $R$ dominates one of the so-called simple (or 0-modal) plane curve singularities in the sense of [1] (cf. also [12]). We are going to prove an analogous fact concerning the strictly unimodal plane curve singularities (cf. [1], [12])\footnote{In [1] these singularities are called “uni-” and “bimodal”, while in [12] they are called “strictly unimodal”. We use the latter terminology.}, whose list is given in Table 1. Remind some notions related to it.

As $k$ is algebraically closed, we may suppose, that $R_0 = \prod_{i=1}^{s} D_i$, where $D_i = k[[t_i]]$ (formal power series rings). The number $s$ is called the number of branches of $R$. Denote $t = (t_1, t_2, \ldots, t_s)$ and $v_i$ the standard valuation in $D_i$. For any element $r = (r_1, r_2, \ldots, r_s) \in R_0$ define its (multi-)valuation as the vector $v(r) = (v_1(r_1), v_2(r_2), \ldots, v_s(r_s))$. In Table 1 we prefer to present the plane curve singularities in a “parametrization” form, i.e. given by their generators $x, y$ as of complete subalgebras of $R_0$. Such presentation has an advantage that we may use it as a definition if $\text{char} k \neq 0$. Really, in the table the valuations $v(x)$ and $v(y)$ are given.

Later on we often write SUS instead of the words “strictly unimodal plane curve singularity”. We also use the following definition and notation.

Definition 1.5. Let $\{a_1, a_2, \ldots, a_d\}$ be a basis of $m/(m \cap mR_0)$, $v_j = v(a_j)$. The set $\{v_1, v_2, \ldots, v_d\}$ will be called a valuation type of $R$ and denoted by $\text{val}(R)$.

2. Main theorem

We pass now to the main theorem. In addition to the Notations 1.1, denote $\hat{I} = \hat{t}mR_0 + m$, $\hat{R} = \text{End } \hat{I}$ and $A_0$ the 4-dimensional $k$-algebra having a basis $\{1, a, b, ab\}$ with $a^2 = b^2 = 0$ (these notations will be used only in the case $\text{char} k = 2$).

Theorem 2.1. Let $R$ be a curve singularity. The following conditions are equivalent:

1. $\text{par}(1, R) \leq 1$.

2. $R$ dominates a simple or strictly unimodal plane curve singularity.
### Table 1.

<table>
<thead>
<tr>
<th>Type</th>
<th>$s$</th>
<th>$\mathbf{v}(x)$</th>
<th>$\mathbf{v}(y)$</th>
<th>Conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E$</td>
<td>1</td>
<td>(3) $l$</td>
<td>$(l)$</td>
<td>$l \in {7, 8, 10, 11}$</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>(1, 2) $\infty$, $l$</td>
<td>$(\infty, l)$</td>
<td>$4 \leq l \leq 7$</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>(1, 1, 1) $\infty$, $k$, $l$</td>
<td>$(\infty, k, l)$</td>
<td>$l = 2$ or $l = 3$</td>
</tr>
<tr>
<td>$T$</td>
<td>2</td>
<td>(2, $k$) $l$, $k$, $l$</td>
<td>$(l, 2)$</td>
<td>$k$, $l$ odd</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>(1, 1, $k$) $\infty$, $l$, $2$</td>
<td>$(\infty, l, 2)$</td>
<td>$k$ odd</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>(1, $\infty$, 1, $k$) $\infty$, $1$, $l$</td>
<td>$(\infty, 1, l, 1)$</td>
<td></td>
</tr>
<tr>
<td>$W$</td>
<td>1</td>
<td>(4) $l$</td>
<td>$(l)$</td>
<td>$5 \leq l \leq 7^*$</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>(1, 3) $\infty$, $l$</td>
<td>$(\infty, l)$</td>
<td>$l = 4$ or $l = 5$</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>(2, 2) $3$, $l$</td>
<td>$(3, l)$</td>
<td>$^*$</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>(1, 1, 2) $\infty$, $l$, $3$</td>
<td>$(\infty, l, 3)$</td>
<td></td>
</tr>
<tr>
<td>$Z$</td>
<td>2</td>
<td>(1, $l$) $\infty$, $3$</td>
<td>$(\infty, 3)$</td>
<td>$l \in {4, 5, 7, 8}$</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>(1, $\infty$, 2) $\infty$, $1$, $l$</td>
<td>$(\infty, 1, l)$</td>
<td>$2 \leq l \leq 5$</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>(1, $\infty$, 1, 1) $\infty$, $1$, 2, $l$</td>
<td>$(\infty, 1, 2, l)$</td>
<td></td>
</tr>
</tbody>
</table>

* If $\text{char} k = 2$, then, in case $W$, for $s = 1$, necessarily $l = 5$ and, for $s = 2$, if $\mathbf{v}(x) = (2, 2)$ and $l = 3$, necessarily $x^2 - y^3 \notin t^7 R_0$. 
3. (a) $d(R_0) \leq 4$;
(b) $d(R_1) \leq 3$;
(c) $d(R_2 + eR) \leq 3$ for any such idempotent $e \in R_0$, that $d(eR_0) = 1$
(provided it exists);
(d) if $d(R_0) = 3$, then $d(R_3) \leq 2$.
(e) if char $k = 2$, then $\mathcal{R}/\mathcal{I} \not\subseteq A_0$.

**Remark 2.2.** One can see, that the condition 3(c) of the theorem means that either $d(R_2) \leq 2$ or $eM \not\subseteq m + m^2R_0$.

**Proof.** $1 \Rightarrow 3$. Suppose first, that $d = d_0 \geq 5$. Consider the factorial algebra $A = R_0/mR_0$. If $V$ is any subspace in $A$, then its pre-image $M(V)$ in $R_0$ is an $R$-submodule. Moreover, if $V$ and $U$ are two subspaces in $A$ such that $AV = AU = A$, then, evidently, $M(V) \simeq M(U)$ if and only if $U = aV$ for some invertible element $a \in A$. Consider now the subset $Gr_0(m, A)$ of the Grassmannian $Gr(m, A)$, consisting of all such subspaces $V$ that $AV = A$. Obviously, it is an open subset, hence, algebraic variety over $k$ of the dimension $m(d - m)$. The group $G = A^*/k^*$ is acting on this variety, and different orbits of this action correspond to non-isomorphic $R$-ideals. But, as $d > 4$, $\dim G \leq \dim Gr_0(2, A) - 2$. In view of [5] (Corollary 3.8), this implies $\dim (1, R) \geq 2$.

Let now $d(R_0) \geq 4$. Remark that $\text{rad } R_1 = mR_0$ and $R_1/\text{rad } R_1 = k$. Thus, the algebra $A' = R_1/mR_1$ is local with the radical $J = mR_0/mR_1$. Moreover, $mR_1 \supseteq (mR_0)^2$, whence $J^2 = 0$. Then, for any subspace $W \subseteq J$, the subspace $V = k + W$ is a subalgebra in $A$ and its pre-image $M(V)$ is a subring of $R_0$. Hence, taking different subspaces $W \subseteq J$, we get non-isomorphic $R$-modules. As $\dim J \geq 3$, $\dim Gr(2, J) \geq 2$ and $\text{par}(1, R) \geq 2$, too.

**Remark 2.3.** The same observations show that $\text{par}(1, R) \geq \left\lceil \frac{d_0 - 2}{2} \right\rceil^2$ and if $d_0 = d_1$, then $\text{par}(1, R) \geq \left\lceil \frac{d_0 - 1}{2} \right\rceil^2$.

To complete the proof, we need two simple lemmas.

**Lemma 2.4.** Let $J \subseteq R$ be any ideal and $d = \dim (R_0/JR_0)$. Then $\dim (I/JI) \leq d$ for each $R$-ideal $I$.

**Proof.** Find an $R_0$-ideal $J_0 \subseteq J$. Then $\dim (I/JI) \leq \dim (R_0I/J_0I) \leq (R_0/J_0)$ as all $R_0$-ideals are principal. Hence, we can find an ideal $I$ with the maximal
value \( m_0 = \dim (I/JI) \). Now, if \( M \) is any torsion-free \( R \)-module of rank \( n \), we have \( \dim (M/JM) \leq nm_0 \). But \( R_0 \cap R_0 = R_0 \), hence, there exists an exact sequence of the form

\[
0 \rightarrow M \rightarrow nI \rightarrow R_0 \rightarrow 0
\]

for some \( n \) and some module \( M \) of rank \( n - 1 \). Taking factors modulo \( J \), we get:

\[
nm_0 = \dim (nI/nJI) \leq (n - 1)m_0 + d,
\]

whence \( m_0 \leq d \) q.e.d.

**Lemma 2.5.** Suppose that \( d_0 = d_1 = \ldots = d_k \). Then there exists an element \( r \in \mathfrak{m} \), such that:

1. Elements \( r, r^2, \ldots, r^k \) form a basis of \( \mathfrak{m}_{k+1}/\mathfrak{m}^{k+1}R_0 \).
2. \( rm_k = \mathfrak{m}m_k \).

**Proof.** To prove the first assertion, take \( r \in \mathfrak{m} \setminus \mathfrak{m}^2R_0 \), consider the dimensions

\[
c_j = \dim (\mathfrak{m}_j/\mathfrak{m}^2R_0)
\]

and remark, that

\[
d_j = 1 + \dim (\mathfrak{m}_j/\mathfrak{m}_{j+1}) = 1 + c_j + \dim (\mathfrak{m}^2R_0/\mathfrak{m}^{j+1}R_0) - c_{j+1}.
\]

Evidently, \( \dim (\mathfrak{m}^iR_0/\mathfrak{m}^{i+1}R_0) = d_0 \) for all \( j \). So, we have \( c_{j+1} = c_j + 1 \) for \( j \leq k \), whence, obviously, \( c_j = j - 1 \). In particular, \( \dim (\mathfrak{m}_{k+1}/\mathfrak{m}^{k+1}R_0) = k \).

As, of course, the elements \( r, r^2, \ldots, r^k \) are linear independent modulo \( \mathfrak{m}^{k+1}R_0 \), they really form a basis of this vector space.

Now remark, that \( r\mathfrak{m}_k \subseteq \mathfrak{m}m_k \) and \( \dim (\mathfrak{m}_k/r\mathfrak{m}_k) \leq \dim (R_0/\mathfrak{m}R_0) = d_0 \) in view of Lemma 2.4. But the just obtained result implies, that

\[
\dim (\mathfrak{m}_k/\mathfrak{m}m_k) = \dim (\mathfrak{m}_k/\mathfrak{m}_{k+1}) + \dim (\mathfrak{m}_{k+1}/\mathfrak{m}m_k) = d_0 - 1 + \dim (\mathfrak{m}_{k+1}/(\mathfrak{m}^2 + \mathfrak{m}^{k+1}R_0) = d_0.
\]

Therefore, \( r\mathfrak{m}_k = \mathfrak{m}m_k \) q.e.d.

Remark, that the proof of Lemma 2.5 implies also the following corollary.

**Corollary 2.6.** \( d_{j+1} \leq d_j \) for every \( j \geq 0 \).
Proof. We need to prove that \( c_{j+1} - c_j \leq c_{j+2} - c_{j+1} \). But \( c_j = \dim \left( \frac{m_j/m}{m/m \cap m R_0} \right) = \dim \left( \frac{m \cap m R_0}{m \cap m R_0 + m^{j+1} R_0/m^{j+1} R_0} \right) \). Therefore,

\[
(c_{j+2} - c_{j+1}) - (c_{j+1} - c_j) = \dim \left( \frac{m^{j+1} R_0/m^{j+2} R_0}{m^{j+1} R_0/m^{j+2} R_0} \right) - \dim \left( \frac{L_j/L_{j+1}}{L_j/L_{j+1}} \right),
\]

where \( L_j = m \cap m R_0 + m^{j+1} R_0 \). But \( \dim \left( \frac{m^{j+1} R_0/m^{j+2} R_0}{m^{j+1} R_0/m^{j+2} R_0} \right) = d_0 \) and \( m L_j \subseteq L_{j+1} \), whence \( \dim \left( \frac{L_j/L_{j+1}}{L_j/L_{j+1}} \right) \leq \dim \left( \frac{L_j/m L_j}{L_j/m L_j} \right) \leq d_0 \) \( \text{q.e.d.} \)

Now suppose, that \( d_0 = d_3 = 3 \). Consider the factoralgebra \( F = \frac{R_2/m^2 m_2}{m^2 m_2} \). Choosing \( r \) as in Lemma 2.5, we see, that \( m m_2 = r^2 m_2 \) and \( \dim (m_2/r^2 m_2) = 3 \). Of course, \( r \not\in r m_2 \), so we can choose \( r, u, v \in m_2 \) linear independent modulo \( r m_2 \). Then \( \{1, r, u, v, ru, rv\} \) is a basis of \( F \). Now \( F \) contains a 2-parameters family of subalgebras, containing the image of \( R \) (i.e. 1 and \( r \)), namely, the subalgebras \( A(\lambda, \mu) \) with the bases \( \{1, r, u + \lambda v + \mu, ru + \lambda rv\} \). Then their pre-images in \( R_2 \) form a 2-parameters family of coverings of \( R \), hence, of pairwise non-isomorphic \( R \)-ideals.

Remark 2.7. Really, as it follows from [3], in this case we could get only families of coverings, as there are at most two non-isomorphic ideals with a fixed endomorphism ring.

At last, suppose that \( d(\frac{R_2 + e R}{R_2}) = 4 \) for some idempotent \( e \in R_0 \), such that \( d(\frac{e R_0}{R_0}) = 1 \). As we have remarked, that means: \( d(\frac{R_2}{R_2}) = 3 \) and \( e m \subseteq m + m^3 R_0 \). Of course, the idempotent \( e \) is primitive and \( e R = e R_0 \). Denote \( R' = (1 - e) R; R'_k = (1 - e) R_k; d'_k = d(\frac{R'_k}{R'_k}) \). Then \( d'_0 = d'_1 = d'_2 = 3 \). Hence, we can apply Lemma 2.5 and choose an element \( r \in m \), such that \( \{r, r^2\} \) form a basis of \( \frac{m R'_2/m^3 R'_0}{m^3 R'_0} \). Consider the factoralgebra \( F = (e R + R_i)/(m^2 + m^3 R_0) \). If \( \{r, u, v\} \) is a basis of \( \frac{m R'_2/m^3 R'_0}{m^3 R'_0} \), then a basis of \( F \) can be chosen in the form: \( \{1, e, r, u, v, ru, rv\} \). The subspaces \( V(\lambda) = V(\lambda_0, \lambda_1, \lambda_2) \) with the bases \( \{1, e + \lambda_0 u + \lambda_1 v + \lambda_2 rv, r, \lambda_0 ru + \lambda_1 rv\} \), where \( \lambda_0 \neq 0 \), form a 3-dimensional family of \( R \)-submodules in \( F \). Thus, they define a 3-dimensional family \( M(\lambda) \) of \( R \)-ideals. But it follows from [5] (v. the proof of Proposition 3.6), that the ideals, isomorphic to some fixed \( M(\lambda) \), form the subvariety of dimension
\[
\dim \left( V(\lambda)/S(\lambda) \right), \text{ where } S(\lambda) = \{ a \in F \mid aV(\lambda) \subseteq V(\lambda) \}. \text{ As } \dim V(\lambda) = 4 \text{ and } S(\lambda) \supset \{1, r, \lambda_0 ru + \lambda_1 rv\}, \text{ this dimension is not more than (really, equals) 1. Hence, again } \par(1, R) \geq 2.
\]

If \(\text{char } k = 2\) and \(R/I \cong A_0\), consider the subspaces \(A(\lambda, \mu) \subset A_0\) with bases \(\{1, a + \lambda b + \mu ab\}\). They are subalgebras in \(A_0\) (as \(\text{char } k = 2\)), hence, their pre-images in \(R\) form a 2-parameters family of overrings of \(R\), hence, of non-isomorphic \(R\)-ideals \(\text{ q.e.d.}\).

3 \(\Rightarrow\) 2. Take any ring \(R\) satisfying the conditions 3(a–e). It is known (cf. [3],[7]), that if \(R\) has only finitely many non-isomorphic ideals then it dominates one of the simple plane curve singularities. So, we may suppose, that \(R\) has infinitely many non-isomorphic ideals, i.e. \(d_0 \geq 3\) and if \(d_0 = 3\), then also \(d_1 = 3\) (cf. ibid.). Suppose first that \(s = 1\), where \(s\) is the number of branches. Then the condition (a) implies that \(\text{val}(R) = \{3\} \text{ or } \{4\}\). In the first case the condition (d) easily implies that \(R\) contains also an element of the valuation \(l \in \{7, 8, 10, 11\}\). But then it dominates a SUS of type \(E\) (cf. the list). If \(\text{val}(R) = \{4\}\), then the condition (b) implies, that \(R\) contains an element of the valuation \(5 \leq l \leq 7\), hence, dominates a SUS of type \(W\). If \(\text{char } k = 2\), the condition (e) implies also that \(R\) contains an element of the valuation \(5\): otherwise \(I = t^6R_0 + k + kt^4\), hence \(R = t^2R_0 + k\) and \(R/I \cong A_0\).

Let now \(s = 2\). If \(d_0 = 3\) then \(\text{val}(R) = \{(1, 2)\}\) (up to the numbering of the branches; later on we omit this notice). Again the condition (d) implies that \(R\) contains an element of the valuation \((\infty, l)\) with \(4 \leq l \leq 7\), hence, dominates a SUS of type \(E\). Suppose that \(d_0 = 4\). Then the following cases can occur:

- \(\text{val}(R) = \{(1, 3)\}\). Then the condition (b) implies that \(R\) contains an element with the valuation either \((\infty, 4)\) or \((\infty, 5)\), hence, dominate a SUS of type \(W\).

- \(\text{val}(R) = \{(2, 2)\}\). Again (b) implies that \(R\) contains an element with the valuation \((3, l), \text{i.e.}\) dominates a SUS of type \(W\). Again, if \(\text{char } k = 2\) and \(x^3 - y^2 \in t^7R_0\), we get that \(R/I \cong A_0\).

- \(\text{val}(R) = \{(2, k), (l, 2)\}\). Then \(R\) dominates a SUS of type \(T\).

- \(\text{val}(R) = \{(1, l), (\infty, 3)\}\). Now the condition (c) obviously implies that \(l \leq 8\) (and not 6), hence, \(R\) dominates a SUS of type \(F\) .
If \( s = d_0 = 3 \), then \( \text{val}(R) = \{ (1,1,1) \} \) and the condition (d) implies that \( R \) contains an element of the valuation \((\infty, 3, l)\), i.e. dominates a SUS of type E. Let \( s = 3, \; d_0 = 4 \). If \( \text{val}(R) \) consists of only one vector, then it is \((1,1,2)\) and the condition (b) implies immediately that \( R \) contains also an element of the valuation \((\infty, 3, l)\), hence, dominate a SUS of type W. If \( \text{val}(R) \) consists of 2 vectors, then there are the following possibilities:

- \( \text{val}(R) = \{ (1,1,k), (\infty,l,2) \} \). Then \( R \) dominates a SUS of type T.
- \( \text{val}(R) = \{ (1,\infty,l), (\infty,1,2) \} \). Then the condition (c) implies that \( l \leq 5 \), hence, \( R \) dominates a SUS of type Z.

If \( \text{val}(R) \) consists of 3 vectors, they may be chosen as

\[
\{ (1,\infty,k), (\infty,1,l), (\infty,\infty,2) \}
\]

and \( R \) dominates a SUS of type T.

At last, let \( s = 4 \). The condition (b) implies that \( \text{val}(R) \) has at least 2 vectors. If there are really only 2 of them, then either \( \text{val}(R) = \{ (1,\infty,k,1), (\infty,1,1,l) \} \) or \( \text{val}(R) = \{ (1,\infty,1,1), (\infty,1,k,l) \} \). In the first case \( R \) dominates one of the SUS of type T, while in the latter case the condition (c) implies that \( k \leq 2 \), thus \( R \) dominates a SUS of type Z. Finally, if \( \text{val}(R) \) contains 3 vectors, one can easily see that \( R \) dominates a SUS of type T q.e.d.

3. Ideals of strictly unimodal plane curve singularities

Now we have to prove the implication \( 2 \Rightarrow 1 \), that is to show that any SUS from Table 1 has only 1-parametere families of ideas. Indeed, it has been done in the Schappert's work [10]. Though Schappert supposed that \( \text{char} \; k = 0 \) and used another definition of SUS, one can check that his calculations are valid for our list too, independently of the characteristics. To precise it, we show below examples of such calculations (in somewhat different form). Moreover, we have chosen the most complicated cases. Our calculations are based on the following simple observation (cf. [3]). Let \( R \) be a curve singularity, \( m \) its maximal ideal and \( S = \text{End}_R(m) = \{ a \in Q | am \subseteq m \} \). We keep these notations through the whole section and also put \( n = \text{rad} S, \; S' = \text{End}_S(n) \). For any \( R \)-ideal \( I, \; SI \) is an \( S \)-ideal and \( mI = mSI \subseteq I \subseteq SI \). Consider the factor \( V = V(I) = I/mI \). It is a generating subspace in \( W = SI/mSI \), i.e. such that \( FV = W \), where
\( F = S/\mathfrak{m} \), and one can easily check that \( I \simeq I' \) if and only if \( \gamma V(I) = V(I') \) for some map \( \gamma : FF \) induced by an automorphisms \( \gamma \) of \( SI \). Moreover, \( S \neq R \), whenever \( R \) is not discrete valuation ring. Therefore, we can calculate the ideals “inductively”, ascending by overrings. Remark also that any plane curve singularity \( R \) is Gorenstein [2], i.e. \( \text{inj.dim}_R R = 1 \). Hence, \( R \) has the only minimal overring \( R' \) and any \( R \)-ideal is either principal or \( R' \)-ideal. In particular, \( \text{par} (1, R) = \text{par} (1, R') \).

Remark that all SUS of type \( T \) are known to be tame, i.e. have at most 1-parametre families of indecomposable torsion-free modules of any rank. So we have to consider only the SUS of types \( W, Z \) and \( E \).

3.1. Ideals of singularities of type \( W \)

Here we consider the case, when \( s = 1 \) and \( R \) contains elements \( x, y \) with \( v(x) = 4 \), \( v(y) = 7 \) (“type \( W_{13} \)” in Arnold’s classification). It is convenient to suppose here that \( t = x^2/y \). Of course, we suppose also that \( \text{char} k \neq 2 \). It is easy to verify that then \( R \supset t^{13}R_0 \) and, if \( R \) is a plane curve singularity, its minimal overring contains even \( t^{14}R_0 \). Hence, we may restrict ourselves by the case, when \( R \) is the smallest subalgebra of \( R_0 \) containing \( t^{14}R_0 \) and generated modulo \( t^{14}R_0 \) by \( x \) and \( y \). Thus,

\[ R = \langle 1, x, y, x^2, xy, x^3 \rangle + t^{13}R_0, \]

where, as usual, we denote by \( \langle a_1, a_2, \ldots, a_m \rangle \) the \( k \)-subspace generated by \( a_1, a_2, \ldots, a_m \). An obvious calculation shows that in this case

\[ S = \langle 1, x, y, x^2 \rangle + t^{10}R_0 \]

and

\[ S' = \langle 1, z, x \rangle + t^6R_0, \]

where \( z = y/x \). As \( v(z) = 3 \), it follows from [6] or [8] that \( S' \) has only finitely many non-isomorphic ideals (it is the simple plane curve singularity of type \( E_6 \)). Moreover, in these articles the precise list of such ideals is given. Namely, they are, except \( R_0 \) and \( S' \) itself:

\[
A = S'\langle 1, t^5 \rangle, \\
A^* = S'\langle 1, t \rangle, \\
A' = S'\langle 1, t^2 \rangle.
\]
Here $A$ and $A'$ are overings of $S'$ and $A^*$ the module dual to $A^*$ with respect to the duality described e.g. in [3].

Now, for each of these ideals, say $M$, we have to calculate $M/nM$. Here is their list (we write “$r$” for the image of an element $r \in M$ in $M/nM$ too):

\[
S' = \langle 1, z, z^2, z^3 \rangle, \\
A = \langle 1, z, \bar{t^2}, \bar{t^6} \rangle, \\
A^* = \langle 1, t, z, \bar{t^6} \rangle, \\
A' = \langle 1, t^2, z, \bar{t^5} \rangle, \\
R_0 = \langle 1, t, t^2, \bar{t^5} \rangle.
\]

Now one can easily write down all generating subspaces $V$ from each $W = M/nM$ (up to automorphisms of $M$). Consider the case $M = A'$ (the most complicated). First, as $V$ is generating, it has to contain at least two elements of the form: $a_1 = 1 + \lambda_1 z + \lambda_2 \bar{t^5}$ and $a_2 = t^2 + \mu_1 z + \mu_2 \bar{t^5}$. Dividing by $a_1$ (which is the image of an invertible element from $A'$), we may suppose that $a_1 = 1$. Suppose that $V = \langle a_1, a_2 \rangle$. If $\mu_1 \neq 0$, one can replace $V$ by $(1 - (\mu_1 + \mu_2)(2\mu_1)^{-1})V$, thus obtaining a subspace of the same form but with $\mu_2 = 0$. Hence, we get two 1-parametre families of $S$-ideals:

\[
\begin{align*}
F_1(\mu) &= S(1, t^2 + \mu z), \\
F_2(\mu) &= S(1, t^2 + \mu t^5).
\end{align*}
\]

Suppose now that $V$ contains another element, which can be chosen in the form $b = \eta_1 z + \eta_2 \bar{t^6}$. If $\eta_1 \neq 0$, we may suppose that $\mu_1 = 0$ and then, multiplying by $1 - \eta_2 / \eta_1 z$, also $\eta_2 = 0$, i.e. $b = z$. But then, multiplying by $1 - \mu_2 \bar{t^5}$, we get $\mu_2 = 0$, which gives only one more ideal:

\[
I_1 = S(1, t^2, z).
\]

If $\eta_1 = 0$, we may suppose $\mu_2 = 0$, obtaining 1-parametre family:

\[
F_3(\mu) = S(1, t^2 + \mu z, \bar{t^5}).
\]

Of course, we have to add to those families also the ideal $A'$ itself (corresponding to $V = W$). But we can remark that $d(A') \leq 4$, hence $\dim A'/mA' \leq 4$ and $mA' = nA'$. Thus, any generating subspace in $A'/mA'$ (with respect to $S$) must
coincide with $A'/mA'$ itself. Therefore, we need not consider the case $SI = A'$ when calculating the $R$-ideals. The same argument is valid, of course, in each case, when we have such $S$-ideal $L$ that $nL = mL$ (it is always the case, when $\dim(L/nS) = d_0$): we may exclude them while calculating the $R$-ideals. In particular, here we may exclude all $S'$-ideals.

Quite analogous observations give us the following list of $S$-ideals (which are not $S'$-ideals): $S' = S'$:

$$
F_4(\mu) = S\langle 1, z + \mu z^3 \rangle,
F_5(\mu) = S\langle 1, z^2 + \mu z^3 \rangle,
S = S\langle 1 \rangle,
I_2 = S\langle 1, z^3 \rangle,
I_3 = S\langle 1, z, z^2 \rangle,
I_4 = S\langle 1, z, z^3 \rangle,
I_5 = S\langle 1, z^2, z^3 \rangle,
$$

$S' = A$:

$$
F_6(\mu) = S\langle 1, z + \mu \ell^5 \rangle, \quad \mu \neq 0,
F_7(\mu) = S\langle 1, z + \mu \ell^5, \ell^6 \rangle, \quad \mu \neq 0,
F_8(\mu) = S\langle 1, \ell^5 + \mu \ell^6 \rangle,
F_9(\mu) = S\langle 1, \ell^5 + \mu \ell^6 \rangle,
I_6 = S\langle 1, \ell^5, \ell^6 \rangle.
$$

$S' = A^*$:

$$
F_{10}(\mu) = S\langle 1, t + \mu z \rangle,
F_{11}(\mu) = S\langle 1, t + \mu z, \ell^6 \rangle,
I_7 = S\langle 1, t, z \rangle.
$$

$S' = R_0$:

$$
I_8 = S\langle 1, t, t^2 \rangle.
$$

Now we can pass to $R$-ideals $I$. Put $M = SI$. It would be one of the ideals $F_{1-11}(\mu), I_{1-8}$ or $S$. One can easily check, that in the following cases $mM = nM$, so we need not to consider them:
\[ F_i(\mu) \quad \text{for } i \neq 4, 8, 10, \]
\[ F_i(\mu) \quad \text{for } i = 4, 10 \text{ and } \mu \neq 0, \]
\[ I_i \quad \text{for } i \neq 2. \]

In the case \( M = F_4(0) \) we have: \( M/\mathfrak{m}M = \langle 1, z, t^{13} \rangle \). Hence, the only proper generating subspace is \( \langle 1, z \rangle \), which gives one new ideal:
\[ I_9 = R\langle 1, z \rangle. \]

Analogously, in the case \( M = F_{10}(0) \) we obtain also one new ideal:
\[ I_{10} = R\langle 1, t \rangle. \]

In the case \( M = F_8 \) we have \( M/\mathfrak{m}M = \langle 1, t^5 + \mu t^6, t^{10} \rangle \). Hence, there is again only one proper generating subspace, namely, \( \langle 1, t^5 + \mu t^6 \rangle \) and we obtain a new 1-parametre family:
\[ F_{12}(\mu) = R\langle 1, u(\mu) \rangle, \quad \text{where } u(\mu) = t^5 + \mu t^6. \]

In the case \( M = S \) we have: \( M/\mathfrak{m}M = \langle 1, t^{10}, t^{13} \rangle \). As there is no element with valuation 3 in \( S \), we get here a new family parametrized by the projective line:
\[ F_{13}(\lambda_0 : \lambda_1) = R\langle 1, \lambda_0 t^{10} + \lambda_1 t^{13} \rangle. \]

At last, the case \( M = I_2 \) also gives a new 1-parametre family:
\[ F_{14}(\mu) = R\langle 1, z^3 + \mu t^{10}, t^{13} \rangle. \]

Thus, we have described all \( R \)-ideals and proved that \( \text{par}(1, R) = 1 \). Quite similar (mainly, easier) calculations show that \( \text{par}(1, R) = 1 \) all other singularities of type \( \mathcal{W} \).

**Remark:** In the list of Schappert [10] the ideals \( F_2(\mu) \) (which are indeed overings of \( R \)) and \( I_3 \) are missed.

### 3.2. Ideals of singularities of type \( Z \)

Now consider the singularities of type \( Z \). Here we suppose that \( s = 2 \) and \( R \) contains elements \( x, y \) with \( \nu(x) = (1, 8), \nu(y) = (\infty, 3) \) (SUS of type \( Z_{19} \)). One
can check that in this case \( R \supset (t_1^2, t_2^{17})R_0 \) and, moreover, its minimal overring contains \((t_1^2, t_2^{14})\). Hence, we may suppose that

\[
R = \langle 1, x, y, y^2, y^3, xy, y^4 \rangle + (t_1^2, t_2^{14})R_0.
\]

It is convenient to take for \( t_1 \) the first component of \( x \) and choose \( t_2 \) in such way that \( y^3 = (0, t_2)x \). Now

\[
S = \langle 1, x, y, y^2, y^3 \rangle + (t_1, t_2^{11})R_0
\]

and

\[
S' = D_1 \oplus S_2,
\]

where \( D_1 = k[[t_1]] \) and \( S_2 = k[[t_2, t_2^0]] \) (the simple plane curve singularity of type \( E_8 \)). Here is the list of all \( S_2 \)-ideals (cf. [6],[8]) given by their generators over \( S_2 \) and over \( S \):

<table>
<thead>
<tr>
<th>Ideal</th>
<th>( S_2 )-generators</th>
<th>( S )-generators</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S_2 )</td>
<td>1</td>
<td>1, ( t_2^5, t_2^{10} )</td>
</tr>
<tr>
<td>A</td>
<td>1, ( t_2^2 )</td>
<td>1, ( t_2^5, t_2^{10} )</td>
</tr>
<tr>
<td>( A^* )</td>
<td>1, ( t_2^2 )</td>
<td>1, ( t_2^2, t_2^4 )</td>
</tr>
<tr>
<td>B</td>
<td>1, ( t_2^4 )</td>
<td>1, ( t_2^2, t_2^4 )</td>
</tr>
<tr>
<td>( B^* )</td>
<td>1, ( t_2 )</td>
<td>1, ( t_2, t_2^2 )</td>
</tr>
<tr>
<td>( B' )</td>
<td>1, ( t_2^2, t_2^4 )</td>
<td>1, ( t_2^2, t_2^4 )</td>
</tr>
<tr>
<td>( D_2 )</td>
<td>1, ( t_2, t_2^2 )</td>
<td>1, ( t_2, t_2^2 )</td>
</tr>
</tbody>
</table>

Any (full) \( S' \)-ideal is of the form \( D_1 \oplus N \), where \( N \) is an \( S_2 \)-ideal. Consider first the \( S \)-ideals \( I \) such that \( S'I = D_1 \oplus A \). Then \( nI = n + \langle (0, t_2^{10}) \rangle \),

\[
S'I/nI = W = \langle (1, 0), (0, 1), (0, t_2^5), (0, t_2^2) \rangle
\]

and any generating subspace \( V \subseteq W \) contains an element of the form \((1, a)\) and also elements of the form \((\alpha_1, 1+\mu_1 t_2^2), (\alpha_2, t_2^2+\mu_2 t_2^5)\). Multiplying by \((1, 1-\mu_1 t_2^2)\), we may suppose that \( V \) contains \((\alpha_1, 1)\). Then, if \( \dim V = 2 \), there are only two possibilities:

\[
V = \langle (1, 1), (0, \mu t_2^5 + t_2^2) \rangle \quad \text{and} \quad V = \langle (0, 1), (1, \mu t_2^5 + t_2^2) \rangle.
\]

15
If \( \dim V = 3 \), we can add also an element of the form \((\beta, t_2^\circ)\) or \((1, 0)\) It gives three more possibilities:

\[
V = \langle (1, 1), (0, t_2^\circ), (0, t_2^\circ) \rangle,
\]
\[
V = \langle (0, 1), (\lambda_0, t_2^\circ), (\lambda_1, t_2^\circ) \rangle,
\]
\[
V = \langle (1, 0), (0, 1), (\mu t_2^\circ + t_2^\circ) \rangle,
\]

where \((\lambda_0 : \lambda_1)\) is a point of the projective line. Hence, here is the list of the corresponding \(S\)-ideals (which are not \(S'\)-ideals):

\[
F_1(\mu) = S\langle (1, 1), (0, \mu t_2^\circ + t_2^\circ) \rangle,
\]
\[
F_2(\mu) = S\langle (0, 1), (1, \mu t_2^\circ + t_2^\circ) \rangle,
\]
\[
F_3(\lambda_0 : \lambda_1) = S\langle (0, 1), (\lambda_0, t_2^\circ), (\lambda_1, t_2^\circ) \rangle,
\]
\[
F_4(\mu) = S\langle (1, 0), (0, 1), (\mu t_2^\circ + t_2^\circ) \rangle,
\]
\[
I_1 = S\langle (1, 1), (0, t_2^\circ), (0, t_2^\circ) \rangle.
\]

Remark that the ideals \(F_4(\mu)\) are decomposable (as modules), while all other ideals of this list are indecomposable.

Analogous calculations give the following list of all \(S\)-ideals \(I\), which are not \(S'\)-ideals:

\[
S'I = S':
\]
\[
F_5(\lambda_0 : \lambda_1) = S\langle (\lambda_0, 1), (\lambda_1, t_2^\circ) \rangle,
\]
\[
S = S\langle (1, 1) \rangle,
\]
\[
I_2 = S\langle (1, 1), (0, t_2^{10}) \rangle,
\]
\[
I_3 = S\langle (1, 1), (0, t_2^\circ), (0, t_2^{10}) \rangle,
\]
\[
I_4 = S\langle (0, 1), (1, t_2^\circ), (0, t_2^{10}) \rangle,
\]
\[
I_5 = S\langle (0, 1), (1, t_2^{10}) \rangle,
\]
\[
I_6 = S\langle (0, 1), (0, t_2^\circ), (1, t_2^{10}) \rangle,
\]
\[
I_7 = S\langle (1, 0), (0, 1) \rangle,
\]
\[
I_8 = S\langle (1, 0), (0, 1), (0, t_2^\circ) \rangle,
\]
\[
I_9 = S\langle (1, 0), (0, 1), (0, t_2^{10}) \rangle,
\]

\[
S'I = D_1 \oplus A^*, \ nI = n + \langle (0, t_2^\circ), (0, t_2^{10}) \rangle:
\]

16
\[
\begin{align*}
F_6(\lambda_0 : \lambda_1) &= S\langle (\lambda_0, 1), (\lambda_1, t_2^0) \rangle, \\
F_7(\lambda_0 : \lambda_1) &= S\langle (\lambda_0, 1), (\lambda_1, t_2^0), (0, t_2^0) \rangle, \\
I_{10} &= S\langle (0, 1), (0, t_2^0), (1, t_2^0) \rangle, \\
I_{11} &= S\langle (1, 0), (0, 1), (0, t_2^0) \rangle.
\end{align*}
\]

\[S' I = D_1 \oplus B, \quad n I = n + \langle (0, t_2^0), (0, t_2^0) \rangle: \]

\[
\begin{align*}
F_8(\mu) &= S\langle (1, 1), (0, t_2^0 + \mu t_2^0) \rangle, \\
F_9(\mu) &= S\langle (0, 1), (1, t_2^0 + \mu t_2^0) \rangle, \\
F_{10}(\lambda_0 : \lambda_1) &= S\langle (0, 1), (\lambda_0, t_2^0), (\lambda_1, t_2^0) \rangle, \\
F_{11}(\mu) &= S\langle (1, 0), (0, 1), (0, t_2^0 + \mu t_2^0) \rangle, \\
I_{12} &= S\langle (1, 1), (0, t_2^0), (0, t_2^0) \rangle.
\end{align*}
\]

\[S' I = D_1 \oplus B^*, \quad n I = n + \langle (0, t_2^0), (0, t_2^0), (0, t_2^0) \rangle: \]

\[
\begin{align*}
F_{12}(\lambda_0 : \lambda_1) &= S\langle (\lambda_0, 1), (\lambda_1, t_2) \rangle, \\
F_{13}(\lambda_0 : \lambda_1) &= S\langle (\lambda_0, 1), (\lambda_1, t_2), (0, t_2^0) \rangle, \\
I_{13} &= S\langle (0, 1), (0, t_2), (1, t_2^0) \rangle, \\
I_{14} &= S\langle (1, 0), (0, 1), (0, t_2) \rangle.
\end{align*}
\]

\[S' I = D_1 \oplus B', \quad n I = n + \langle (0, t_2^0), (0, t_2^0), (0, t_2^0) \rangle: \]

\[
\begin{align*}
I_{15} &= S\langle (1, 1), (0, t_2^0), (0, t_2^0) \rangle \\
I_{16} &= S\langle (0, 1), (1, t_2^0), (0, t_2^0) \rangle \\
I_{17} &= S\langle (0, 1), (0, t_2^0), (1, t_2^0) \rangle.
\end{align*}
\]

\[S' I = R_0: \]

\[
I_{18} = S\langle (1, 1), (0, t_2), (0, t_2^0) \rangle.
\]

17
\[ I_{19} = \mathcal{S}(0, 1), (1, t_2), (0, t_2^2) \] \[ I_{20} = \mathcal{S}(0, 1), (0, t_2), (1, t_2^2) \].

Here \( \mu \) denotes an element of \( k \) and \( (\lambda_0 : \lambda_1) \) a point of the projective line over \( k \).

Now pass to the calculation of \( R \)-ideals, which are not \( S \)-ideals. Again one can verify that for the following \( S \)-ideals \( M \) we have \( mM = nM \), so we do not need to consider the case, when \( SI = M \):

\[
\begin{align*}
F_i(\mu) & \quad \text{for } i \neq 1, 8, \\
F_i(\lambda_0 : \lambda_1) & \quad \text{for all } i, \\
F_8(\mu) & \quad \text{for } \mu \neq 0, \\
I_i & \quad \text{for } i \neq 2, 7.
\end{align*}
\]

For \( M = F_1(\mu), \ M/mM = \langle (1, 1), (0, \mu t_2^5 + t_2^5), (0, t_2^6) \rangle \). If \( \mu \neq 0 \), the multiplication by itself maps the second of these elements to the third one, which is congruent modulo \( mM \) to \( \mu^{-1} t_2^0 \). Hence, here we obtain only the following new 1-parametre family of ideals:

\[ F_{14}(\mu) = R\langle (1, 1), (0, \mu t_2^5 + t_2^5) \rangle , \mu \neq 0. \]

If \( \mu = 0 \), we obtain the following new family:

\[ F_{15}(\mu) = R\langle (1, 1), (t_2^5 + \mu t_2^5) \rangle. \]

In the case \( M = F_8(0), M/mM = \langle (1, 1), (0, t_2^4 + \mu t_2^5), (0, t_2^6) \rangle \). Hence, we also obtain only one new family of ideals:

\[ F_{16}(\mu) = R\langle (1, 1), (t_2^4 + \mu t_2^5) \rangle. \]

The remaining cases: \( S \) and \( I_i \ (i = 2, 7) \), give the ring \( R \) itself and the following ideals:

\[
\begin{align*}
F_{17}(\lambda_0 : \lambda_1) & = R\langle (1, 1), (\lambda_0 t_1, \lambda_1 t_2^{13}) \rangle , \\
F_{18}(\mu) & = R\langle (1, 1), (\mu t_1, t_2^{10}) \rangle , \\
I_{21} & = R\langle (1, 0), (0, 1) \rangle,
\end{align*}
\]

18
\[ I_{22} = R\langle (1, t_2^{13}), (0, 1) \rangle. \]

Therefore, we have proved that in this case also \( \text{par} (1, R) = 1 \). Quite similar calculations prove the same for all other singularities of type \( Z \).

\textbf{Remark:} Here also the family \( F_{13} \) fails in the Schappert’s list of ideals.

### 3.3. Ideals of singularities of type \( E \)

Now consider the case of singularities \( E \). In this case \( d_0 = 3 \), so it follows from [3] that each \( R \)-ideal is isomorphic either to an overring of \( R \) or to its dual module. Therefore, we only need to find all overrings. But if \( I \) is an overring of \( R \) then \( SI \) is also an overring of \( S \). Hence, at each stage of our inductive process we may restrict ourselves by considering only overrings. To be complete, we always mark, which of these overrings are Gorenstein (i.e. self-dual).

Here we suppose that \( s = 3 \) and \( R \) contains elements \( x, y \) with \( v(x) = (1, 1, 1) \) and \( v(y) = (\infty, k, 3) \) with \( k > 3 \) (SUS of type \( E_{3,p} \)). Of course, we suppose here that \( x = t = (t_1, t_2, t_3) \). Again, passing to the minimal overring, we may suppose that \( R \) contains \( (t_1^{k+1}, t_2^{k+2}, t_3^k)R_0 \) and is generated by \( x \) and \( y \) modulo this ideal. Then \( S \) contains \( (t_1^{k+1}, t_2^{k+2}, t_3^k)R_0 \) and is generated modulo this ideal by \( x \) and \( z \), where \( y = xz \) and \( v(z) = (\infty, k - 1, 2) \) (it is a SUS of type \( E_{1,p} \)). The ring \( S' \) contains \( (t_1^k, t_2^k, t_3^2)R_0 \) and is generated modulo this ideal by \( x \) and \( z' \), where \( v(z') = (\infty, k - 2, 1) \) and \( z = xz' \). Hence, \( S'' \) has only finitely many ideals up to isomorphism (it is a simple plane curve singularity of type \( D \)), cf. [6],[8]. Namely, here is the list of the overrings of \( S'' \) (except \( S'' \) itself):

\[
\begin{align*}
A_i &= S''\langle 1, (0, t_2^i, 0) \rangle \quad (1 \leq i \leq k - 2), \\
B_i &= S''\langle 1, (0, 0, 1), (0, t_2^i, 0) \rangle \quad (0 \leq i \leq k - 2), \\
B_{01} &= S''\langle 1, (0, 0, 0) \rangle, \\
B_{02} &= S''\langle 1, (0, 1, 0) \rangle, \\
R_0 &= S''\langle 1, (0, 0, 0), (0, 1, 0), (0, 0, 1) \rangle.
\end{align*}
\]

Among them, only \( A_i \) are non-Gorenstein. Moreover, \( Mn'' = Mn' \) for \( M = R_0 \) or \( M = B_i, i < k - 2 \), where \( n'' = \text{rad} S'' \), so we do not need further to consider these overrings. Remark also that in the case of \( B_{k-2} \) the generator \( (0, t_2^i, 0) \) above is superfluous.
As $S$ is Gorenstein, its overrings, except $S$ itself, are those of $S'$. Here is the list of factorial algebras $M/n'M$ for the overrings $M$ of $S'$:

$S''/n' = \langle 1, z', (0, 0, t_2^2) \rangle$,  
$A_i/n'A_i = \langle 1, (0, 0, t_3), (0, t_i^2, 0) \rangle$,  
$B_{k-2}/n'B_{k-2} = \langle 1, (0, 0, 1), (0, t_{k-2}^2, 0) \rangle$,  
$B_{01}/n'B_{01} = \langle 1, (1, 0, 0), (0, t_2, 0) \rangle$,  
$B_{02}/n'B_{02} = \langle 1, (0, 1, 0), (0, t_2, 0) \rangle$.

It is easy now to find all proper subalgebras of these algebras and, hence, the overrings of $S$ (except $S$ itself), which are not overrings of $S''$:

$S' = S\langle 1, (0, 0, t_2^2) \rangle$,  
$A_{k-1} = S\langle 1, (0, 0, t_2^2) \rangle$,  
$F_i(\mu) = S\langle 1, (0, t_i^2, \mu t_3) \rangle$ \,(1 \leq i \leq k - 2),  
$B_{k-1} = S\langle 1, (0, 0, 1) \rangle$,  
$B'_{01} = S\langle 1, (1, 0, 0) \rangle$,  
$B'_{02} = S\langle 1, (0, 1, 0) \rangle$.

Among them only $S'$, $A_{k-1}$ and $F_i(0)$ are non-Gorenstein. Remark also that $Mm = M'n'$ for all overrings of $S''$, so we do not need to consider them further. To find all overrings of $R$, which are not overrings of $S$, calculate the factorial algebras $M/mM$ for the rings of the preceding list and $S$:

$S/m = \langle 1, z, (0, 0, t_3^2) \rangle$,  
$S'/mS' = \langle 1, z, (0, 0, t_3^2) \rangle$,  
$A_{k-1}/mA_{k-1} = \langle 1, (0, 0, t_3^2), (0, 0, t_{k-1}^2) \rangle$,  
$F_i(\mu)/mF_i(\mu) = \langle 1, (0, t_i^2, \mu t_3), (0, 0, t_3^2) \rangle$,  
$B_{k-1}/mB_{k-1} = \langle 1, (0, 0, 1), (0, t_{k-1}^2) \rangle$,  
$B'_{01}/mB'_{01} = \langle 1, (1, 0, 0), (0, t_2^2, 0) \rangle$,  
$B'_{02}/mB'_{02} = \langle 1, (0, 1, 0), (t_i^2, 0, 0) \rangle$.

It gives us the following list of the overrings of $R$, which are not overrings of $S$ (except $R$ itself):
\[ R' = R \langle 1, (0, 0, \tau_1^i) \rangle, \]
\[ F'_{k-1}(\mu) = R \langle 1, \mu z + (0, 0, \tau_1^i) \rangle, \]
\[ F_{k-1}(\mu) = R \langle 1, (0, \mu \tau_1^{k-1}, \tau_2^i) \rangle \quad (\mu \neq 1), \]
\[ F_i'(\mu) = R \langle 1, (0, \tau_1^i, \tau_2^i) \rangle \quad (1 \leq i \leq k - 2); \]
\[ B_k = R \langle 1, (0, 0, 1) \rangle, \]
\[ B_{01}^\prime = R \langle 1, (1, 0, 0) \rangle, \]
\[ B_{02}^\prime = R \langle 1, (0, 1, 0) \rangle. \]

Here \( F_{k-1}(\mu), B_k, B_{01}^\prime \) and \( B_{02}^\prime \) are Gorenstein.
Thus, we have proved that \( \text{par} (1, R) = 1 \). Analogous calculations show the same for all other SUS of type \( E \), which accomplishes the proof of Theorem 2.1.

References


