MATH 4576 RINGS AND FIELDS — QUIZ 8
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Let K be the splitting field of the polynomial x^3+3 over $\mathbb{Q}$.

(1) Prove that $K = \mathbb{Q}(\sqrt[3]{3}, \omega)$, where $\omega^3 = 1$ and $\omega \neq 1$.

Since $(-\sqrt[3]{3})^3 = (-\omega\sqrt[3]{3})^3 = (-\omega^2\sqrt[3]{3})^3 = -3$, these elements are the roots of x^3+3 and the field K is generated by these elements. Obviously, it contains both $\sqrt[3]{3}$ and ω, and in $\mathbb{Q}(\sqrt[3]{3}, \omega)$ the polynomial x^3+3 splits into linear factors. Therefore, $\mathbb{Q}(\sqrt[3]{3}, \omega)$ is the splitting field of this polynomial.

(2) Prove that $[K : \mathbb{Q}] = 6$.

We have the chain of extensions $\mathbb{Q} \subset \mathbb{Q}(\sqrt[3]{3}) \subset \mathbb{Q}(\sqrt[3]{3}, \omega) = K$, so $[K : \mathbb{Q}] = [\mathbb{Q}(\sqrt[3]{3}) : \mathbb{Q}][\mathbb{Q}(\sqrt[3]{3}, \omega) : \mathbb{Q}(\sqrt[3]{3})]$. Since $\sqrt[3]{3}$ is a root of the irreducible polynomial $x^2 - 3 \in \mathbb{Q}[x]$, $[\mathbb{Q}(\sqrt[3]{3}) : \mathbb{Q}] = 3$. Now ω is a root of the polynomial $x^2+x+1 \in \mathbb{Q}(\sqrt[3]{3})[x]$. This polynomial has no real roots, hence no roots in $\mathbb{Q}(\sqrt[3]{3})$, therefore is irreducible, thus $[\mathbb{Q}(\sqrt[3]{3}, \omega) : \mathbb{Q}(\sqrt[3]{3})] = 2$ and $[K : \mathbb{Q}] = 6$.

(3) Find all automorphims of K over $\mathbb{Q}$.

Since K is a splitting field, it is normal and separable ($\operatorname{char} \mathbb{Q} = 0$) over $\mathbb{Q}$. Therefore, its Galois group G contains $6 = [K : \mathbb{Q}]$ elements. If $\sigma \in G$, it moves $\sqrt[3]{3}$ to a root of the same polynomial $x^2 - 3$ and ω to a root of x^2+x+1, that is

$$\sigma(\sqrt[3]{3}) \in \left\{ \sqrt[3]{3}, \omega\sqrt[3]{3}, \omega^2\sqrt[3]{3} \right\},$$
$$\sigma(\omega) \in \left\{ \omega, \omega^2 \right\}$$

and all these possibilities must occur, since $\#(G) = 6$. Thus G consists of the following 6 automorphisms:

$$\begin{aligned}
\sigma_1 &= \iota \quad \text{(identity map)},\\
\sigma_2 &: \sqrt[3]{3} \mapsto \sqrt[3]{3},\ \omega \mapsto \omega^2,\\
\sigma_3 &: \sqrt[3]{3} \mapsto \omega\sqrt[3]{3},\ \omega \mapsto \omega,\\
\sigma_4 &: \sqrt[3]{3} \mapsto \omega\sqrt[3]{3},\ \omega \mapsto \omega^2,\\
\sigma_5 &: \sqrt[3]{3} \mapsto \omega^2\sqrt[3]{3},\ \omega \mapsto \omega,\\
\sigma_6 &: \sqrt[3]{3} \mapsto \omega^2\sqrt[3]{3},\ \omega \mapsto \omega^2.
\end{aligned}$$

(4) Find an element θ such that $K = \mathbb{Q}(\theta)$ and all elements $\sigma(\theta)$, where σ runs through $\operatorname{Gal}_{\mathbb{Q}} K$.

According to the proof of Theorem 10.18, we have to find $c \in \mathbb{Q}$ such that

$$c \neq \frac{\alpha - \sqrt[3]{3}}{\omega^2 - \omega}, \quad \text{where } \alpha \in \left\{ \sqrt[3]{3}, \omega\sqrt[3]{3}, \omega^2\sqrt[3]{3} \right\}.$$

It is easy to see that $c = 1$ fits. Therefore, for instance, $K = \mathbb{Q}(\theta)$, where $\theta = \sqrt[3]{3} - \omega$. Then

$$\begin{aligned}
\sigma_1(\theta) &= \theta, \\
\sigma_2(\theta) &= \sqrt[3]{3} - \omega^2, \\
\sigma_3(\theta) &= \omega\sqrt[3]{3} - \omega, \\
\sigma_2(\theta) &= \omega\sqrt[3]{3} - \omega^2, \\
\sigma_2(\theta) &= \omega^2\sqrt[3]{3} - \omega, \\
\sigma_2(\theta) &= \omega^2\sqrt[3]{3} - \omega^2.
\end{aligned}$$