MATH 4576 RINGS AND FIELDS — QUIZ 3

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(1) List all irreducible monic polynomials of degree 2 over Z_3.

Irreducible polynomials of degree 2 are just those having no roots. List all monic polynomial of degree 2 with nonzero conststnt term (that is such that 0 is not a root) and try:

- $x^2 + 1$ has no roots.
- $x^2 + 2$ has a root 1.
- $x^2 + x + 1$ has a root 1.
- $x^2 + x + 2$ has no roots.
- $x^2 + 2x + x$ has a root 2.
- $x^2 + 2x + 2$ has no roots.

Therefore, monic irreducible polynomials of degree 2 over Z_3 are:

$$x^2 + 1,\ x^2 + x + 2,\ x^2 + 2x + 2.$$

(2) The polynomial $f(x) = 2x^5 - 5x^3 + 8x^2 + 2x - 4$ has a complex root $1 - i$. Decompose $f(x)$ into a product of irreducible polynomials in $\mathbb{Q}[x]$, in $\mathbb{R}[x]$ and in $\mathbb{C}[x]$.

If $f(x)$ has a root $1 - i$ and its coefiicients are real numbers, it also has the conjugate root $1 + i$. Hence, it is divisible by the product $(x - 1 + i)(x - 1 - i) = x^2 - 2x + 2$. We get

$$f(x) = (x^2 - 2x + 2)(2x^3 + 4x^2 - x - 2).$$

The first factor is irreducible in $\mathbb{R}[x]$, since it has no real roots. Check whether the second one is irreducible in $\mathbb{Q}[x]$, that is has no roots (since it is of degree 3). Possible rational roots are $\pm 1, \pm 2, \pm \frac{1}{2}$ (the numerator must divide the constant term -2, the denominator must divide the leading coefficient 2). One easily checks that -2 is a root, and we get

$$2x^3 + 4x^2 - x - 2 = (x + 2)(2x^2 - 1).$$

The second factor here has no rational roots, so is irreducible in $\mathbb{Q}[x]$. Hence, the decomposition in $\mathbb{Q}[x]$ is

$$f(x) = (x^2 - 2x + 2)(x + 2)(2x^2 - 1).$$

In $\mathbb{R}[x]$, $2x^2 - 1 = (\sqrt{2}x - 1)(\sqrt{2}x + 1)$, so the decomposition in $\mathbb{R}[x]$ is

$$f(x) = (x^2 - 2x + 2)(x + 2)(\sqrt{2}x - 1)(\sqrt{2}x + 1).$$

Finally the decomposition in $\mathbb{C}[x]$ is

$$f(x) = (x - 1 + i)(x - 1 - i)(x + 2)(\sqrt{2}x - 1)(\sqrt{2}x + 1).$$

(3) Find a polynomial $f(x) \in \mathbb{R}[x]$ having roots $-1, 2, i$.

Since $f(x) \in \mathbb{R}[x]$ and has a root i, it also has a root $-i$. Therefore, we have the following possibility:

$$f(x) = (x+1)(x-2)(x-i)(x+i) = x^4 - x^3 - x^2 - x - 2.$$

(All other polynomials from $\mathbb{R}[x]$ having these roots are divisible by $f(x)$.)

(4) Prove that the following poloynomials are irreducible in $\mathbb{Q}[x]$:

(a) $3x^5 + 21x^3 - 98x^2 + 35x - 21$.

All coefficients of this polynomial, except the leading one, are divisible by 7 and the constant term is not divisible by $7^2 = 49$. Therefore this polynomial is irreducible by the Eisenstein criterion.

(b) $x^4 + 8x^3 - 6x^2 + 6x + 8$. Is it possible to apply the Eisenstein criterion to this polynomial?

The only prime number dividing all coefficients, except the leading one, is 2, but $2^2 = 4$ divides the constant term. Therefore, we cannot apply the Eisenstrein criterion.

Modulo 2 this polynomial is x^4, reducible. Try 3. Modulo 3 we get $\bar{f}(x) = x^4 + 2x^3 + 2$. One easily verifies that it has no roots in Z_3. Therefore, if $\bar{f}(x)$ is reducible, it has a qudratic factor. We have found all monic irreducible quadratic polynomial over Z_3 in the problem 1. Now we check that $\bar{f}(x)$ is divisible neither by $x^2 + 1$, nor by $x^2 + x + 2$, nor by $x^2 + 2x + 2$. Therefore $\bar{f}(x)$ is irreducible, hence $f(x)$ is irreducible as well.