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Derived categories of nodal algebras ☆

Igor Burban [a,*] and Yuriy Drozd [b]

[a] Universität Kaiserslautern, Fachbereich Mathematik, Postfach 3049, 67653, Kaiserslautern, Germany
[b] Department of Mechanics and Mathematics, Kyiv Taras Shevchenko University, 01033 Kyiv, Ukraine

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Abstract

In this article we classify indecomposable objects of the derived categories of finitely-generated modules over certain infinite-dimensional algebras. The considered class of algebras (which we call nodal algebras) contains such well-known algebras as the complete ring of a double nodal point $\mathbf{k}[[x, y]]/(xy)$ and the completed path algebra of the Gelfand quiver. As a corollary we obtain a description of the derived category of Harish-Chandra modules over $SL_2(\mathbb{R})$. We also give an algorithm, which allows to construct projective resolutions of indecomposable complexes. In the appendix we prove the Krull–Schmidt theorem for homotopy categories.

Keywords: Derived categories; Tame matrix problems; Harish-Chandra modules; Krull–Schmidt theorem

1. Introduction

Let A be a *pure noetherian complete algebra*, i.e., an associative $\mathbf{k}$-algebra such that:

(1) Its center C is a complete local noetherian $\mathbf{k}$-algebra.
(2) A is finitely generated C-module without minimal submodules.

Denote by r the radical of A. It was shown in [14] that A is tame if and only if it satisfies the following conditions:

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* Corresponding author.
E-mail addresses: burban@mathematik.uni-kl.de (I. Burban), yuriy@drozd.org (Yu. Drozd).

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(1) The algebra $\tilde{A} = \mathrm{End}_A(r)$ is hereditary.
(2) $\mathrm{rad}(\tilde{A}) = r$.
(3) For any simple left A-module U the length of the left A-module $\tilde{A} \otimes_A U$ is at most 2 and for any simple right A-module V the length of the right A-module $V \otimes_A \tilde{A}$ is at most 2.

We call algebras satisfying these conditions *nodal* algebras.

Our description of the derived category of a nodal algebra shows that it is tame at least in "pragmatic" sense, i.e., one can obtain a list of its indecomposable objects as a union of one-dimensional families and some discrete set of objects staying apart. Unfortunately, the definition of derived tameness proposed in [21] can be only applied to finite-dimensional algebras of finite global dimension and nodal algebras usually satisfy neither of these conditions.

The methods developed in this article can be also applied to finite-dimensional gentle and skew-gentle algebras considered in [3,4,20,27,28], as well as to some other algebras [9] and to some derived categories of coherent sheaves [10]. An advantage of these methods is that they also work in cases, when an algebra has infinite homological dimension and describe the derived category of bounded from the *right* complexes. The developed technique allows to write down projective resolutions of indecomposable complexes.

For the sake of simplicity we suppose that the field $\mathbf{k}$ is algebraically closed. Let us rewrite the definition of nodal algebras in a more transparent form. Let U be a simple A-module, $P \longrightarrow U$ its projective covering. Then we have an exact sequence

$$0 \longrightarrow rP \longrightarrow P \longrightarrow U \longrightarrow 0.$$

Apply the functor $\tilde{A} \otimes_A$ to this sequence. We get

$$\tilde{A} \otimes_A rP \longrightarrow \tilde{A} \otimes_A P \longrightarrow \tilde{A} \otimes_A U \longrightarrow 0.$$

But $r = \mathrm{rad}(\tilde{A})$, hence $\mathrm{Im}(\tilde{A} \otimes_A rP \longrightarrow \tilde{A} \otimes_A P) = r \otimes_A P = \mathrm{rad}(\tilde{A} \otimes_A P)$. So we have an exact sequence

$$0 \longrightarrow \mathrm{rad}\big(\tilde{A} \otimes_A P\big) \longrightarrow \tilde{A} \otimes_A P \longrightarrow \tilde{A} \otimes_A U \longrightarrow 0.$$

Therefore $\tilde{A} \otimes_A U$ is a direct sum of simple $\tilde{A}$-modules. Let $U_1, U_2, \ldots, U_m$ be the set of all non-isomorphic simple A-modules, $V_1, V_2, \ldots, V_n$ the set of all non-isomorphic simple $\tilde{A}$-modules. Consider the graph Γ with vertices U_i, V_j, $i = 1, \ldots, m$, $j = 1, \ldots, n$. There is an arrow from U_i to V_j if and only if V_j is a direct summand of $\tilde{A} \otimes_A U_i$. Then, as it was shown in [14], the last condition in the criteria of tameness is equivalent to the following condition: all connected components of Γ are of the form:

(1) $V' \longleftarrow U \longrightarrow V''$.
(2) $U' \longrightarrow V \longleftarrow U''$.
(3) $U \longrightarrow V$.

Let us consider some examples.

Example 1.1. Let $A = \mathbf{k}[[x, y]]/(xy)$, $\mathfrak{m} = (x, y)$ be its maximal ideal. Then $\tilde{A} = \mathrm{End}_A(\mathfrak{m}) = \mathbf{k}[[x]] \times \mathbf{k}[[y]]$. Let U be the unique simple A-module, V_1, V_2 be simple $\tilde{A}$-modules. Then the graph Γ has the form

$$V_1 \longleftarrow U \longrightarrow V_2.$$

Example 1.2. Let

$$A = \left\{ \begin{pmatrix} f_{11} & tf_{12} \\ f_{21} & f_{22} \end{pmatrix} \middle| f_{ij} \in \mathbf{k}[[t]],\ 1 \leqslant i, j \leqslant 2;\ f_{11}(0) = f_{22}(0) \right\} \subseteq \mathrm{Mat}_2\bigl(\mathbf{k}[[t]]\bigr).$$

As one can easily observe, A is just the algebra $\mathbf{k}\langle\langle x, y\rangle\rangle/(x^2, y^2)$ ($\mathbf{k}\langle\langle x, y\rangle\rangle$ is the algebra of formal power series in two non-commutative variables). The endomorphism algebra of its radical is just

$$\left\{ \begin{pmatrix} f_{11} & tf_{12} \\ f_{21} & f_{22} \end{pmatrix} \middle| f_{ij} \in \mathbf{k}[[t]],\ 1 \leqslant i, j \leqslant 2 \right\} \subseteq \mathrm{Mat}_2\bigl(\mathbf{k}[[t]]\bigr).$$

It is easy to see that it is just the completed path algebra of the quiver

$$\begin{tikzcd} \mathbf{1} \bullet \arrow[r, bend right, "\mathbf{y}"'] & \bullet \mathbf{2} \arrow[l, bend right, "\mathbf{x}"'] \end{tikzcd}$$

Indeed, an isomorphism is given by

$$x \mapsto \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}, \qquad y \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The graph Γ again has the form

$$V_1 \longleftarrow U \longrightarrow V_2.$$

Here and further on we consider the natural completion of path algebras, namely, the J-adic one, where J is the ideal generated by all arrows.

Example 1.3. Let A be the completed path algebra of the Gelfand quiver

$$\begin{tikzcd} \cdot \arrow[r, bend right, "\alpha_-"'] & \cdot \arrow[l, bend right, "\alpha_+"'] \arrow[r, bend left, "\beta_+"] & \cdot \arrow[l, bend left, "\beta_-"] \end{tikzcd} \qquad \alpha_-\alpha_+ = \beta_-\beta_+$$

As one can easily see,

$$A \cong \left\{ \begin{pmatrix} f_{11} & tf_{12} & tf_{13} \\ f_{21} & f_{22} & tf_{23} \\ f_{31} & tf_{32} & f_{33} \end{pmatrix} \,\middle|\, f_{ij} \in \mathbf{k}[[t]],\ 1 \leqslant i, j \leqslant 3 \right\} \subseteq \mathrm{Mat}_3\big(\mathbf{k}[[t]]\big).$$

An isomorphism is given by

$$\alpha_+ \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \beta_+ \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\alpha_- \mapsto \begin{pmatrix} 0 & t & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \beta_- \mapsto \begin{pmatrix} 0 & 0 & t \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

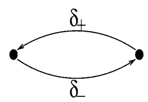
The endomorphism algebra $\tilde{A} = \mathrm{End}(\mathrm{rad}(A))$ is

$$\tilde{A} \cong \left\{ \begin{pmatrix} f_{11} & tf_{12} & tf_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{pmatrix} \,\middle|\, f_{ij} \in \mathbf{k}[[t]],\ 1 \leqslant i, j \leqslant 3 \right\} \subseteq \mathrm{Mat}_3\big(\mathbf{k}[[t]]\big).$$

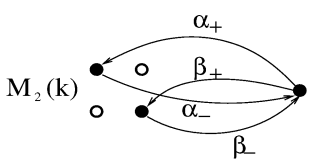
and is Morita equivalent to

$$\left\{ \begin{pmatrix} f_{11} & tf_{12} \\ f_{21} & f_{22} \end{pmatrix} \,\middle|\, f_{ij} \in \mathbf{k}[[t]],\ 1 \leqslant i, j \leqslant 2 \right\} \subseteq \mathrm{Mat}_2\big(\mathbf{k}[[t]]\big),$$

which is the completed path algebra of the quiver



Note that $\tilde{A}$ is isomorphic to the completed path algebra of the following (non-basic) quiver



$$\beta_+ \alpha_+ = \beta_- \alpha_-$$

Let U_1, U_2, U_3 be simple A-modules, V, W simple $\tilde{A}$-modules. Our graph Γ has the form

$$U_1 \longrightarrow V \longleftarrow U_2, \qquad U_3 \longrightarrow W.$$

2. The main construction

Let A be a semi-perfect associative **k**-algebra (not necessarily finite-dimensional), $A \subset \tilde{A}$ be an embedding such that $r = \mathrm{rad}(A) = \mathrm{rad}(\tilde{A})$. Let $I \subset A$ be a two-sided $\tilde{A}$-ideal containing r. It means that $r \subseteq I = I\tilde{A} = \tilde{A}I$, thus A/I and $\tilde{A}/I$ are semi-simple algebras.

Let $\tilde{A} \otimes_A$ be the derived functor of the tensor product. We want to describe the fibers of the map

$$\mathrm{Ob}\big(D^-(A\text{-mod})\big) \longrightarrow \mathrm{Ob}\big(D^-\big(\tilde{A}\text{-mod}\big)\big).$$

Remark 2.1. A-mod denotes the category of *finitely-generated* A-modules. We always consider objects of derived categories as complexes of projective modules.

Definition 2.2. Consider the following category of triples of complexes TC_A

(1) Objects are triples $(\widetilde{\mathcal{P}}_\bullet, \mathcal{M}_\bullet, i)$, where
$\widetilde{\mathcal{P}}_\bullet \in D^-(\tilde{A}\text{-mod})$,
$\mathcal{M}_\bullet \in D^-(A/I\text{-mod})$,
$i : \mathcal{M}_\bullet \longrightarrow \tilde{A}/I \otimes_{\tilde{A}} \widetilde{\mathcal{P}}_\bullet$ a morphism in $D^-(A/I\text{-mod})$, such that $\tilde{i} : \tilde{A}/I \otimes_A \mathcal{M}_\bullet \longrightarrow \tilde{A}/I \otimes_{\tilde{A}} \widetilde{\mathcal{P}}_\bullet$ is an isomorphism in $D^-(\tilde{A}/I\text{-mod})$.

(2) Morphisms $(\widetilde{\mathcal{P}}_{\bullet_1}, \mathcal{M}_{\bullet 1}, i_1) \longrightarrow (\widetilde{\mathcal{P}}_{\bullet_2}, \mathcal{M}_{\bullet 2}, i_2)$ are pairs (Φ, φ),

$$\widetilde{\mathcal{P}}_{\bullet_1} \xrightarrow{\Phi} \widetilde{\mathcal{P}}_{\bullet_2}, \qquad \mathcal{M}_{\bullet_1} \xrightarrow{\varphi} \mathcal{M}_{\bullet_2},$$

such that

$$\begin{tikzcd}
\tilde{A}/I \otimes_{\tilde{A}} \widetilde{\mathcal{P}}_{\bullet_1} \arrow[d, "\Phi\otimes\mathrm{id}"'] & \mathcal{M}_{\bullet_1} \arrow[l, "i_1"'] \arrow[d, "\varphi"] \\
\tilde{A}/I \otimes_{\tilde{A}} \widetilde{\mathcal{P}}_{\bullet_2} & \mathcal{M}_{\bullet_2} \arrow[l, "i_2"]
\end{tikzcd}$$

is commutative.

Remark 2.3. If an algebra A has infinite homological dimension, then we are forced to deal with the derived category of right bounded complexes (in order to define the left derived functor of the tensor product). In case A has finite homological dimension we can suppose that all complexes above are *bounded* from both sides.

Theorem 2.4. *The functor*

$$D^-(A\text{-mod}) \xrightarrow{\mathbf{F}} \mathrm{TC}_A,$$

$\mathcal{P}_\bullet \longrightarrow (\tilde{A} \otimes_A \mathcal{P}_\bullet, A/I \otimes_A \mathcal{P}_\bullet, i : A/I \otimes_A \mathcal{P}_\bullet \longrightarrow \tilde{A}/I \otimes_A \mathcal{P}_\bullet)$ *has the following properties*:

(1) $\mathbf{F}$ *is dense* (*i.e., every triple* $(\widetilde{\mathcal{P}}_\bullet, \mathcal{M}_\bullet, i)$ *is isomorphic to some* $\mathbf{F}(\mathcal{P}_\bullet)$).
(2) $\mathbf{F}(\mathcal{P}_\bullet) \cong \mathbf{F}(\mathcal{Q}_\bullet) \Longleftrightarrow \mathcal{P}_\bullet \cong \mathcal{Q}_\bullet$.
(3) $\mathbf{F}(\mathcal{P}_\bullet)$ *is indecomposable if and only if so is* $\mathcal{P}_\bullet$ (*note that this property is an easy formal consequence of the previous two properties*).
(4) $\mathbf{F}$ *is full.*

Remark 2.5. $\mathbf{F}$ is not faithful. So it is not an equivalence of categories. A functor $\mathbf{F}$ satisfying the properties (1)–(4) is called *detecting functor* (see [2]).

Proof. The main point to be clarified is: having a triple $\mathcal{T} = (\widetilde{\mathcal{P}}_\bullet, \mathcal{M}_\bullet, i)$ how can we reconstruct $\mathcal{P}_\bullet$? The exact sequence

$$0 \longrightarrow I\widetilde{\mathcal{P}}_\bullet \longrightarrow \widetilde{\mathcal{P}}_\bullet \longrightarrow \tilde{A}/I \otimes_{\tilde{A}} \widetilde{\mathcal{P}}_\bullet \longrightarrow 0$$

of complexes in A-mod gives a distinguished triangle

$$I\widetilde{\mathcal{P}}_\bullet \longrightarrow \widetilde{\mathcal{P}}_\bullet \longrightarrow \tilde{A}/I \otimes_{\tilde{A}} \widetilde{\mathcal{P}}_\bullet \longrightarrow I\widetilde{\mathcal{P}}_\bullet[-1]$$

in $D^-(A\text{-mod})$. The properties of triangulated categories imply that there is a morphism of triangles

$$\begin{tikzcd}
I\widetilde{\mathcal{P}}_\bullet \arrow[r] & \widetilde{\mathcal{P}}_\bullet \arrow[r] & \tilde{A}/I \otimes_{\tilde{A}} \widetilde{\mathcal{P}}_\bullet \arrow[r] & I\widetilde{\mathcal{P}}_\bullet[-1] \\
I\widetilde{\mathcal{P}}_\bullet \arrow[r] \arrow[u, "\mathrm{id}"] & \mathcal{P}_\bullet \arrow[r] \arrow[u, "\Phi"'] & \mathcal{M}_\bullet \arrow[r] \arrow[u, "i"'] & I\widetilde{\mathcal{P}}_\bullet[-1], \arrow[u, "\mathrm{id}"]
\end{tikzcd}$$

where $\mathcal{P}_\bullet = \mathrm{cone}(\mathcal{M}_\bullet \longrightarrow I\widetilde{\mathcal{P}}_\bullet[-1])[1]$. Set $\mathbf{G}(\mathcal{T}) = \mathcal{P}_\bullet$. Taking a cone is not a functorial operation. It gives an intuitive explanation why the functor $\mathbf{F}$ is not an equivalence. The properties of triangulated categories immediately imply that the constructed map (not a functor!)

$$\mathbf{G} : \mathrm{Ob}(\mathrm{TC}_A) \longrightarrow \mathrm{Ob}\big(D^-(A\text{-mod})\big)$$

sends isomorphic objects into isomorphic ones and $\mathbf{GF}(\mathcal{P}_\bullet) \cong \mathcal{P}_\bullet$. Now we have to show that $\mathbf{FG}(\widetilde{\mathcal{P}}_\bullet, \mathcal{M}_\bullet, i) \cong (\widetilde{\mathcal{P}}_\bullet, \mathcal{M}_\bullet, i)$. $\square$

Lemma 2.6. *In the above notations, let $\widetilde{P}$ be a projective $\tilde{A}$-module, M be an A/I-module, $i : M \longrightarrow \widetilde{P}/I\widetilde{P}$ be an A/I-module monomorphism such that the induced map $\tilde{i} : \tilde{A}/I \otimes_{A/I} M \longrightarrow \widetilde{P}/I\widetilde{P}$ is an isomorphism. Consider the pull-back diagram*

$$\begin{tikzcd} 0 \arrow[r] & I\widetilde{P} \arrow[r] \arrow[d, "\mathrm{id}"] & P \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, "i"] & 0 \\ 0 \arrow[r] & I\widetilde{P} \arrow[r] & \widetilde{P} \arrow[r, "\pi"] & \widetilde{P}/I\widetilde{P} \arrow[r] & 0. \end{tikzcd}$$

Then P is a projective A-module and $\tilde{A} \otimes_A P \longrightarrow \widetilde{P}$ is an isomorphism.

Consider the image $\bar{I}$ of the ideal I in $\tilde{A}/r$. Since $\tilde{A}/r$ is semi-simple, we can find an ideal $\bar{J}$ in $\tilde{A}/r$ such that $\bar{I} + \bar{J} = \tilde{A}/r$, $\bar{I} \cap \bar{J} = 0$. By the Chinese remainder theorem we have $\tilde{A}/r = \tilde{A}/I \times \tilde{A}/J$.

Let $\overline{\widetilde{P}} = \widetilde{P}/r\widetilde{P}$. Then $\overline{\widetilde{P}} = \overline{\widetilde{P}}_1 \oplus \overline{\widetilde{P}}_2$, where $\overline{\widetilde{P}}_1$ is an $\tilde{A}/I$-module and $\overline{\widetilde{P}}_2$ an $\tilde{A}/J$-module. But then $\widetilde{P}$ also decomposes into a direct sum: $\widetilde{P} = \widetilde{P}_1 \oplus \widetilde{P}_2$, where $\overline{\widetilde{P}}_i = \widetilde{P}_i/r\widetilde{P}_i$, $i = 1, 2$ (we use the fact that there is a bijection between projective and semi-simple modules: $\widetilde{P} \longleftrightarrow \widetilde{P}/r\widetilde{P}$).

Then we have:

$$I\widetilde{P}_1 = r\widetilde{P}_1, \qquad I\widetilde{P}_2 = \widetilde{P}_2.$$

Indeed, $\widetilde{P}_1/r\widetilde{P}_1$ is an A/I-module, so $I\widetilde{P}_1 \subseteq r\widetilde{P}_1$. But $r \subseteq I$, hence $r\widetilde{P}_1 \subseteq I\widetilde{P}_1$. So, $I\overline{\widetilde{P}}_1 = r\overline{\widetilde{P}}_1$. Analogously, $J\widetilde{P}_2 \subseteq r\widetilde{P}_2$. But $I + J = \tilde{A}$, so

$$\widetilde{P}_2 = I\widetilde{P}_2 + J\widetilde{P}_2 \subseteq I\widetilde{P}_2 + r\widetilde{P}_2 \subseteq \widetilde{P}_2.$$

Hence, by Nakayama's lemma $I\widetilde{P}_2 = \widetilde{P}_2$.

Our diagram has now the form:

$$\begin{tikzcd} 0 \arrow[r] & \widetilde{P}_2 \oplus r\widetilde{P}_1 \arrow[r] \arrow[d, "\mathrm{id}"] & P \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, "i"] & 0 \\ 0 \arrow[r] & \widetilde{P}_2 \oplus r\widetilde{P}_1 \arrow[r] & \widetilde{P}_1 \oplus \widetilde{P}_2 \arrow[r, "\pi"] & \widetilde{P}_1/r\widetilde{P}_1 \arrow[r] & 0. \end{tikzcd}$$

Since $P \longrightarrow \widetilde{P}_1 \oplus \widetilde{P}_2$ is a monomorphism, $\widetilde{P}_2$ is a direct summand of P. Moreover, $\widetilde{P}_2$ is a projective A-module. Indeed, let $\widetilde{Q}$ be any projective $\tilde{A}$-module satisfying $I\widetilde{Q} = \widetilde{Q}$. Without loss of generality suppose that $\widetilde{Q}$ is a direct summand of $\tilde{A}$. Then

$$\widetilde{Q} = I\widetilde{Q} \subseteq I\tilde{A} \subseteq A \subseteq \tilde{A}.$$

But if the embedding $\widetilde{Q} \longrightarrow \tilde{A}$ splits, then $\widetilde{Q} \longrightarrow A$ splits too. Hence, $\widetilde{Q}$ is a projective A-module.

Note that the canonical map $\tilde{A} \otimes_A \widetilde{P}_2 \longrightarrow \widetilde{P}_2$ is an isomorphism. Indeed,

$$\tilde{A} \otimes_A \widetilde{P}_2 = \tilde{A} \otimes_A I\widetilde{P}_2 = \tilde{A}I \otimes_A \widetilde{P}_2 = I \otimes_A \widetilde{P}_2.$$

But $\widetilde{P}_2$ is a flat A-module, hence

$$I \otimes_A \widetilde{P}_2 = I\widetilde{P}_2 = \widetilde{P}_2.$$

So we get

$$\begin{array}{ccccccccc} 0 & \longrightarrow & r\widetilde{P}_1 & \longrightarrow & P_1 & \longrightarrow & M & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle \mathrm{id}} & & \big\downarrow & & \big\downarrow{\scriptstyle i} & & \\ 0 & \longrightarrow & r\widetilde{P}_1 & \longrightarrow & \widetilde{P}_1 & \xrightarrow{\pi} & \widetilde{P}_1/r\widetilde{P}_1 & \longrightarrow & 0. \end{array}$$

We know that $\tilde{i}: \tilde{A}/I \otimes_{A/I} M \longrightarrow \widetilde{P}_1/I\widetilde{P}_1$ is an isomorphism. But then $\tilde{A}/r \otimes_{A/r} M \longrightarrow \widetilde{P}_1/r\widetilde{P}_1$ is an isomorphism, too. Indeed $IM = 0$, since M is a submodule of $\widetilde{P}_1/I\widetilde{P}_1$. But $I + J = A$, hence $JM = M$ and $\tilde{A}/J \otimes_{A/r} M = 0$. Therefore

$$\tilde{A}/I \otimes_{A/I} M = \tilde{A}/I \otimes_{A/r} M \cong \bigl(\tilde{A}/I \oplus \tilde{A}/J\bigr) \otimes_{A/r} M \cong \tilde{A}/r \otimes_{A/r} M.$$

Now we have to show that P_1 is projective and $\tilde{A} \otimes_A P \longrightarrow \widetilde{P}_1$ is an isomorphism. Let $P(M)$ be a projective covering of M.

$$\begin{array}{ccccccccc} & & & & & & P(M) & & \\ & & & & & \swarrow{\scriptstyle \psi} & \big\downarrow & & \\ 0 & \longrightarrow & r\widetilde{P}_1 & \longrightarrow & P_1 & \longrightarrow & M & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle \mathrm{id}} & & \big\downarrow & & \big\downarrow{\scriptstyle i} & & \\ 0 & \longrightarrow & r\widetilde{P}_1 & \longrightarrow & \widetilde{P}_1 & \longrightarrow & \widetilde{P}_1/r\widetilde{P}_1 & \longrightarrow & 0. \end{array}$$

Apply the functor $A/r \otimes_A$ to the first row of this diagram. We get: $\overline{\psi}: P(M)/rP(M) \longrightarrow P/rP$ is an isomorphism. Hence by Nakayama's lemma ψ is an epimorphism. Consider the composition map $P(M) \longrightarrow \widetilde{P}_1$. The induced map $\tilde{A} \otimes_A P(M) \longrightarrow \widetilde{P}_1$ is an isomorphism modulo r. Since both modules are projective, it is indeed an isomorphism. We get: $P(M) \longrightarrow \tilde{A} \otimes P(M) \longrightarrow \widetilde{P}_1$ is a monomorphism. But then $\psi: P(M) \longrightarrow P_1$ is a monomorphism too. So it is an isomorphism. And we have shown also that $\tilde{A} \otimes_A P \longrightarrow \widetilde{P}_1$ is an isomorphism.

We finish now the proof of the theorem. Let $(\widetilde{\mathcal{P}}_\bullet, \mathcal{M}_\bullet, i)$ be a triple. Without loss of generality, suppose that $\widetilde{\mathcal{P}}_\bullet$ is a minimal complex and $\mathcal{M}_\bullet$ a complex with zero differentials. Then $\widetilde{\mathcal{P}}_\bullet/r\widetilde{\mathcal{P}}_\bullet$ is a complex with zero differentials too and the map $i: \mathcal{M}_\bullet \longrightarrow \widetilde{\mathcal{P}}_\bullet/I\widetilde{\mathcal{P}}_\bullet$ has the property that $\tilde{i}: \tilde{A}/I \otimes_{A/I} \mathcal{M}_\bullet \longrightarrow \widetilde{\mathcal{P}}_\bullet/I\widetilde{\mathcal{P}}_\bullet$ is an isomorphism of *complexes*. Consider now the pull-back diagram in the abelian category of complexes of A-modules.

$$
\begin{tikzcd}
0 \arrow[r] & I\widetilde{\mathcal{P}}_\bullet \arrow[r] \arrow[d, "\mathrm{id}"] & \mathcal{P}_\bullet \arrow[r, "\Psi"] \arrow[d, "\Phi"] & \mathcal{M}_\bullet \arrow[r] \arrow[d, "i"] & 0 \\
0 \arrow[r] & I\widetilde{\mathcal{P}}_\bullet \arrow[r] & \widetilde{\mathcal{P}}_\bullet \arrow[r, "\pi"] & \widetilde{\mathcal{P}}_\bullet/I\widetilde{\mathcal{P}}_\bullet \arrow[r] & 0.
\end{tikzcd}
$$

From Lemma 2.6 follows that:

(1) $\mathcal{P}_\bullet$ is a complex of projective A-modules;
(2) $(\mathrm{id}\otimes\Phi, \mathrm{id}\otimes\Psi) : (\tilde{A}\otimes_A \mathcal{P}_\bullet, A/I\otimes_A \mathcal{P}_\bullet, A/I\otimes_A \mathcal{P}_\bullet \longrightarrow \tilde{A}/I\otimes_A \mathcal{P}_\bullet) \longrightarrow (\widetilde{\mathcal{P}}_\bullet, \mathcal{M}_\bullet, i)$ is an isomorphism in the category of triples.

It remains to show that **F** is full.

Let $(\Phi, \varphi) : (\widetilde{\mathcal{P}}_{\bullet 1}, \mathcal{M}_{\bullet 1}, i_1) \longrightarrow (\widetilde{\mathcal{P}}_{\bullet 2}, \mathcal{M}_{\bullet 2}, i_2)$ be a morphism in TC_A, where $\mathcal{M}_{\bullet 1}$ and $\mathcal{M}_{\bullet 2}$ are complexes with zero differentials. Since we are dealing with complexes of projective objects, Φ and φ can be represented by morphisms of complexes. Let us moreover suppose $\widetilde{\mathcal{P}}_{\bullet 1}$ and $\widetilde{\mathcal{P}}_{\bullet 2}$ to be minimal. Then $\tilde{A}/I \otimes_{\tilde{A}} \widetilde{\mathcal{P}}_{\bullet 1}$, $i = 1, 2$, are complexes with zero differentials, too.

$$
\begin{tikzcd}
\mathcal{M}_{\bullet 1} \arrow[r, "\varphi"] \arrow[d, "i_1"] & \mathcal{M}_{\bullet 2} \arrow[d, "i_2"] \\
\widetilde{\mathcal{P}}_{\bullet 1}/I\widetilde{\mathcal{P}}_{\bullet 1} \arrow[r, "\overline{\Phi}"] & \widetilde{\mathcal{P}}_{\bullet 2}/I\widetilde{\mathcal{P}}_{\bullet 2}
\end{tikzcd}
$$

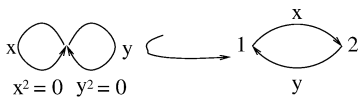
is commutative *in the category of complexes*. The properties of pull-back imply the existence of a morphism of complexes $\mathcal{P}_{\bullet 1} \longrightarrow \mathcal{P}_{\bullet 2}$ such that

$$
\begin{tikzcd}
 & \mathcal{P}_{\bullet 2} \arrow[rr] \arrow[dd] & & \mathcal{M}_{\bullet 2} \arrow[dd] \\
\mathcal{P}_{\bullet 1} \arrow[ur] \arrow[rr] \arrow[dd] & & \mathcal{M}_{\bullet 1} \arrow[ur] \arrow[dd] & \\
 & \widetilde{\mathcal{P}}_{\bullet 2} \arrow[rr] & & \widetilde{\mathcal{P}}_{\bullet 2}/I\widetilde{\mathcal{P}}_{\bullet 2} \\
\widetilde{\mathcal{P}}_{\bullet 1} \arrow[ur] \arrow[rr] & & \widetilde{\mathcal{P}}_{\bullet 1}/I\widetilde{\mathcal{P}}_{\bullet 1} \arrow[ur] &
\end{tikzcd}
$$

is commutative. Hence it gives a lift of a morphism (Φ, φ) we are looking for. So the functor **F** is full, which accomplishes the proof of the theorem.

3. The case of $D^{-}(\mathbf{k}\langle\langle x, y\rangle\rangle/(x^2, y^2))$

Consider the embedding of completed path algebras $A \longrightarrow \tilde{A}$:



Take $I = (x, y)$, then $A/I = \mathbf{k}$, $\tilde{A}/I = \mathbf{k} \times \mathbf{k}$ and $A/I \longrightarrow \tilde{A}/I$ is just the diagonal map.

As we have seen in the previous section, a complex $\mathcal{P}_\bullet$ of the derived category $D^{-}(A\text{-mod})$ is defined by some triple $(\widetilde{\mathcal{P}}_\bullet, \mathcal{M}_\bullet, i)$. Since A/I-mod can be identified with the category of $\mathbf{k}$-vector spaces, the map $i : \mathcal{M}_\bullet \longrightarrow \widetilde{\mathcal{P}}_\bullet/I\widetilde{\mathcal{P}}_\bullet$ is given by a collection of linear maps

$$H_k(i) : H_k(\mathcal{M}_\bullet) \longrightarrow H_k\big(\widetilde{\mathcal{P}}_\bullet/I\widetilde{\mathcal{P}}_\bullet\big).$$

The map $H_k(i)$ is a $\mathbf{k}$-linear map of a $\mathbf{k}$-module into a $\mathbf{k} \times \mathbf{k}$-module. Hence it is given by two matrices $H_k(i|1)$ and $H_k(i|2)$. From the non-degeneracy condition of the category of triples it follows that both of these matrices are invertible.

The algebra $\tilde{A}$ has homological dimension 1. Moreover, it is an order. By a theorem of Dold (see [13]), an indecomposable complex from $D^{-}(\tilde{A}\text{-mod})$ is isomorphic to

$$\cdots \longrightarrow 0 \longrightarrow \underbrace{M}_{i} \longrightarrow 0 \longrightarrow \cdots,$$

where M is an indecomposable $\tilde{A}$-module. But $\tilde{A}$ is a hereditary order over $\mathbf{k}[[t]]$. Hence (see [15]) an indecomposable finitely-generated $\tilde{A}$-module is isomorphic in the derived category to a shift of P_i, $i = 1, 2$, or to

$$P_i \xrightarrow{\varphi} P_j \quad (i, j = 1, 2),$$

where $P_i = \tilde{A}e_i$ (e_i is the idempotent corresponding to the point i of the graph), φ is a morphism given by a path going from j to i and $\operatorname{Im}(\varphi) \subseteq rP_j$, thus

$$\tilde{A}/I \otimes_{\tilde{A}} \big(P_i \xrightarrow{\varphi} P_j\big) = \mathbf{k}_i \xrightarrow{0} \mathbf{k}_j.$$

Let

$$\widetilde{\mathcal{P}}_\bullet = \bigoplus \widetilde{\mathcal{P}}_{\bullet i}^{n_i}$$

be a decomposition of $\widetilde{\mathcal{P}}_\bullet$ into a direct sum of indecomposables. This decomposition implies a division of matrices $H_k(i|1)$ and $H_k(i|2)$ into horizontal stripes.

The next question is: which transformations can we perform with the matrices $H_k(i|1)$ and $H_k(i|2)$?

We can do simultaneously any elementary transformation of columns of $H_k(i|1)$ and $H_k(i|2)$ (they correspond to the automorphisms of $\mathcal{M}_\bullet$). From the definition of the category of triples it follows that row transformations are induced by morphisms in $D^-(\tilde{A}\text{-mod})$.

Let us now describe the morphisms between indecomposable complexes from $D^-(\tilde{A}\text{-mod})$, which are non-zero after applying $\tilde{A}/I$. Due to [15] they are just

$$
\begin{tikzcd}
P_2 \arrow[r, "\cdot x"] & P_1 & & P_1 \arrow[r, "\cdot y"] \arrow[d] & P_2 \arrow[d, "\cdot x"] \\
P_1 \arrow[u, "\cdot y"] \arrow[r, "\cdot yx"] & P_1 \arrow[u] & & P_1 \arrow[r, "\cdot yx"] \arrow[d] & P_1 \arrow[d, "\cdot y"] \\
P_2 \arrow[u, "\cdot x"] \arrow[r, "\cdot xyx"] & P_1 \arrow[u] & & P_1 \arrow[r, "\cdot yxy"] \arrow[d] & P_2 \arrow[d, "\cdot x"] \\
P_1 \arrow[u, "\cdot y"] \arrow[r, "\cdot yxyx"] & P_1 \arrow[u] & & P_1 \arrow[r, "\cdot yxyx"] \arrow[d] & P_1 \arrow[d] \\
\cdots \arrow[u] & \cdots \arrow[u] & & \cdots & \cdots
\end{tikzcd}
$$

$$
\begin{tikzcd}
P_1 \arrow[r, "\cdot y"] & P_2 & & P_2 \arrow[r, "\cdot x"] \arrow[d] & P_1 \arrow[d, "\cdot y"] \\
P_2 \arrow[u, "\cdot x"] \arrow[r, "\cdot xy"] & P_2 \arrow[u] & & P_2 \arrow[r, "\cdot xy"] \arrow[d] & P_2 \arrow[d, "\cdot x"] \\
P_1 \arrow[u, "\cdot y"] \arrow[r, "\cdot yxy"] & P_2 \arrow[u] & & P_2 \arrow[r, "\cdot xyx"] \arrow[d] & P_1 \arrow[d, "\cdot y"] \\
P_2 \arrow[u, "\cdot x"] \arrow[r, "\cdot xyxy"] & P_2 \arrow[u] & & P_2 \arrow[r, "\cdot xyxy"] \arrow[d] & P_2 \arrow[d] \\
\cdots \arrow[u] & \cdots \arrow[u] & & \cdots & \cdots
\end{tikzcd}
$$

$$
\begin{tikzcd}
0 \arrow[r] \arrow[d] & P_i \arrow[d] \\
P_j \arrow[r, "\varphi"] & P_i
\end{tikzcd}
$$

$$\begin{tikzcd} P_j \arrow[r, "\varphi"] \arrow[d] & P_i \arrow[d] \\ P_j \arrow[r] & 0 \end{tikzcd}$$

Moreover, we always have a morphism

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & P_i \arrow[r] \arrow[d, "\lambda\cdot\mathrm{id}"] & P_j \arrow[d] \\ P_k \arrow[r] & P_i \arrow[r] & 0 \end{tikzcd}$$

Now note that we have the following cases:

(1) A morphism

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & P_i \arrow[r] \arrow[d, "\lambda\cdot\mathrm{id}"] & P_j \arrow[d] \\ P_k \arrow[r] & P_i \arrow[r] & 0 \end{tikzcd}$$

induces

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & \mathbf{k}_i \arrow[r, "0"] \arrow[d, "\lambda"] & \mathbf{k}_j \arrow[d] \\ \mathbf{k}_k \arrow[r, "0"] & \mathbf{k}_i \arrow[r] & 0 \end{tikzcd}$$

(2) A morphism

$$\begin{tikzcd} P_j \arrow[r] \arrow[d, "\varphi"] & P_i \arrow[d, "\lambda\cdot\mathrm{id}"] \\ P_k \arrow[r] & P_i \end{tikzcd}$$

where $\varphi \in \mathrm{rad}(\tilde{A})$, induces

$$\begin{tikzcd} \mathbf{k}_j \arrow[r] \arrow[d, "0"] & \mathbf{k}_j \arrow[d, "\lambda"] \\ \mathbf{k}_k \arrow[r, "0"] & \mathbf{k}_i \end{tikzcd}$$

(3) Analogously we have that

$$\begin{tikzcd} P_i \arrow[r] \arrow[d, "\lambda\cdot\mathrm{id}"] & P_j \arrow[d, "\varphi"] \\ P_i \arrow[r] & P_k \end{tikzcd}$$

where $\varphi \in \operatorname{rad}(\tilde{A})$, induces

$$\begin{tikzcd} \mathbf{k}_i \arrow[r, "0"] \arrow[d, "\lambda"] & \mathbf{k}_j \arrow[d, "0"] \\ \mathbf{k}_i \arrow[r, "0"] & \mathbf{k}_k \end{tikzcd}$$

(4) In the same way

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & P_i \arrow[d, "\lambda\cdot\mathrm{id}"] \\ P_j \arrow[r] & P_i \end{tikzcd}$$

induces

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & \mathbf{k}_i \arrow[d, "\lambda"] \\ \mathbf{k}_j \arrow[r, "0"] & \mathbf{k}_i \end{tikzcd}$$

and

$$\begin{tikzcd} P_j \arrow[r, "\varphi"] \arrow[d, "\lambda\cdot\mathrm{id}"] & P_i \arrow[d] \\ P_j \arrow[r] & 0 \end{tikzcd}$$

induces

$$\begin{tikzcd} \mathbf{k}_j \arrow[r, "0"] \arrow[d, "\lambda"] & \mathbf{k}_i \arrow[d] \\ \mathbf{k}_j \arrow[r] & 0 \end{tikzcd}$$

(5) And finally an endomorphism

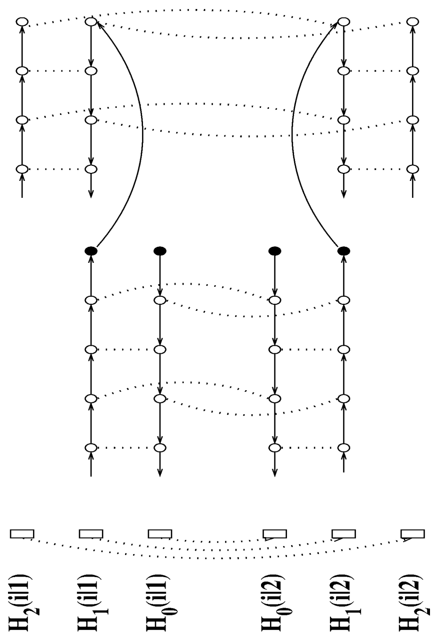
$$\begin{tikzcd} P_j \arrow[r, "\varphi"] \arrow[d, "\lambda\cdot\mathrm{id}"] & P_i \arrow[d, "\lambda\cdot\mathrm{id}"] \\ P_j \arrow[r, "\varphi"] & P_i \end{tikzcd}$$

induces

$$\begin{tikzcd} \mathbf{k}_i \arrow[r, "0"] \arrow[d, "\lambda"] & \mathbf{k}_j \arrow[d, "\lambda"] \\ \mathbf{k}_i \arrow[r, "0"] & \mathbf{k}_j \end{tikzcd}$$

and the same for P_i, $i = 1, 2$.

From what has been said we observe that the matrix problem describing the derived category $D^-(A\text{-mod})$ is given by the following partially ordered set (bunch of chains, see [5] or Appendix B [16]).



In this picture we assume that complexes are shifted in such a way that all $H_k(i) = 0$ for $i < 0$. Small circles correspond to the horizontal stripes, small rectangles correspond to the vertical stripes, dotted lines between circles show the related stripes (i.e., those which come from the same object of the derived category), vertical arrows describe the possible transformations between different horizontal stripes:

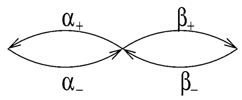
Explicitly saying, we can do the following transformations with our matrices $H_\bullet(i)$:

(1) We can do any simultaneous elementary transformations of the columns of the matrices $H_k(i|1)$ and $H_k(i|2)$, $k \in \mathbb{Z}$.
(2) We can do any simultaneous transformations of rows inside conjugated blocks.
(3) We can add a scalar multiple of any row from a block with lower weight to any row of a block of a higher weight (inside the big matrix, of course). These transformations can be proceeded independently inside $H_k(i|1)$ and $H_k(i|2)$, $k \in \mathbb{Z}$.

This matrix problem belongs to the well-known representations of bunches of chains (see [5,7,26] and Appendix). From here we conclude that there are three types of indecomposable objects: bands, finite strings (both correspond to complexes of finite projective dimension) and infinite strings (which correspond to complexes of infinite projective dimension). In Section 6 we shall explain, how the combinatoric of band and string representations can be used to write down explicit projective resolutions of complexes.

4. Gelfand quiver

In this section we shall see that our technique allows us to describe the derived category of representations of the completed path algebra of the quiver



$\alpha_-\alpha_+ = \beta_-\beta_+$

The classification of indecomposable representations of this quiver can be reduced to representations of bunches of *semi-chains*, see [5]. It is not surprising that the description of the derived category is reduced to the problem of the same type. Consider the embedding given in Example 1.3. In this case we have: $A/I = \mathbf{k} \times \mathbf{k}$, $\tilde{A}/I = M_2(\mathbf{k})$ and $A/I \longrightarrow \tilde{A}/I$ the diagonal mapping. Now we have to answer the following:

Question 4.1. Let M be a $\mathbf{k} \times \mathbf{k}$-module, M' be a $M_2(\mathbf{k})$-module, $\varphi : M \longrightarrow M'$ a map of $\mathbf{k} \times \mathbf{k}$-modules ($M'$ is supplied with $\mathbf{k} \times \mathbf{k}$-module structure using the diagonal embedding). The map of $\mathbf{k} \times \mathbf{k}$-modules is given by two matrices $\varphi(1)$ and $\varphi(2)$. Which conditions should satisfy $\varphi(1)$ and $\varphi(2)$ in order $\tilde{i} : M_2(\mathbf{k}) \otimes_{\mathbf{k}\times\mathbf{k}} M \longrightarrow M'$ to be an isomorphism?

Let $M = \langle v_1, v_2, \dots, v_m; w_1, w_2, \dots, w_n \rangle = \mathbf{k}(1)^m \oplus \mathbf{k}(2)^n$. There is only one indecomposable $M_2(\mathbf{k})$-module: $\mathbf{k}^2$. So,

$$M' = \langle u_1', u_1''; u_2', u_2''; \dots; u_N', u_N'' \rangle = (\mathbf{k}^2)^N,$$

where the action of matrix units are:

$$e_{11}u_i' = u_i', \qquad e_{21}u_i' = u_i'', \qquad e_{12}u_i' = 0, \qquad e_{22}u_i' = 0,$$

and, analogously,

$$e_{11}u_i'' = 0, \qquad e_{21}u_i'' = 0, \qquad e_{12}u_i'' = u_i', \qquad e_{22}u_i'' = u_i''.$$

Let

$$\varphi(v_i) = \sum_{j=1}^{N} \alpha_{ji} u_j' + \sum_{j=1}^{N} \alpha_{ji}' u_j''.$$

Since φ is a $\mathbf{k} \times \mathbf{k}$-module homomorphism,

$$0 = \varphi(e_{22}v_i) = e_{22}\varphi(v_i).$$

So all $\alpha_{ji}' = 0$. Analogously,

$$\varphi(w_i) = \sum_{j=1}^{N} \beta_{ji} u_j''.$$

On the other hand, any $M_2(\mathbf{k})$ homomorphism $\psi : (\mathbf{k}^2)^n \longrightarrow (\mathbf{k}^2)^m$ is given by an $m \times n$ matrix (α_{ij}) with the entries from $\mathbf{k}$ (see [17, Theorem 1.7.5]). Namely, if $(e_1', e_1'', e_2', e_2'', \ldots, e_n', e_n'')$ and $(f_1', f_1'', f_2', f_2'', \ldots, f_m', f_m'')$ are canonical bases of $(\mathbf{k}^2)^n$ and $(\mathbf{k}^2)^m$ then

$$\psi\bigl(e_j'\bigr) = \sum_{j=1}^{m} \alpha_{ij} f_i', \qquad \psi\bigl(e_j''\bigr) = \sum_{j=1}^{m} \alpha_{ij} f_i''.$$

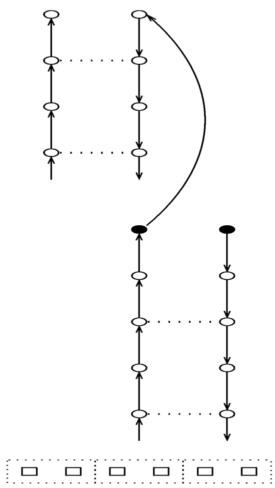
Consider now a $M_2(\mathbf{k})$-module $M_2(\mathbf{k}) \otimes_{\mathbf{k}\times\mathbf{k}} M$. It is generated by

$$e_{11} \otimes v_1,\ e_{21} \otimes v_1;\ e_{11} \otimes v_2,\ e_{21} \otimes v_2;\ \ldots;\ e_{11} \otimes v_m,\ e_{21} \otimes v_m;$$

$$e_{12} \otimes w_1,\ e_{22} \otimes w_1;\ e_{12} \otimes w_2,\ e_{22} \otimes w_2;\ \ldots;\ e_{12} \otimes w_n,\ e_{22} \otimes w_n.$$

Since $\tilde{\varphi}(e \otimes v) = e\varphi(v)$, it is easy to see that φ is given by $N \times (n+m)$-matrix $(\varphi(1) \mid \varphi(2))$. So, $\tilde{\varphi}$ is an isomorphism if $(\varphi(1) \mid \varphi(2))$ is square and invertible.

Now let us return to the Gelfand quiver. The morphisms in $D^-(\tilde{A}\text{-mod})$ were discussed in the previous subsection. We are able to write Bondarenko's partially ordered set:



This picture shows the division of matrices $H_k(i)$, $k \in \mathbb{Z}$, into horizontal and vertical stripes. Each of these matrices is divided into two vertical blocks $H_k(i|1)$ and $H_k(i|2)$ (which correspond to the fact that we have an embedding $\mathbf{k} \times \mathbf{k} \longrightarrow M_2(\mathbf{k})$) and horizontal blocks that correspond to indecomposables of $D^-(\tilde{A}\text{-mod})$. In the same way as in the previous section we have an ordering on the horizontal stripes.

We can perform the following transformations with matrices $H_\bullet(i)$:

(1) We can do independently elementary transformations of columns of $H_k(i|1)$ and $H_k(i|2)$.
(2) We can do any simultaneous transformations of rows inside conjugated blocks.
(3) We can add a scalar multiple of any row from a block with lower weight to any row of a block of a higher weight.

This problem belongs to the class of representations of bunches of *semi-chains*. The description of indecomposable objects was obtained in [5,11] and later elaborated in [12]. Since we get in this case infinitely many matrices, certain modifications should be done, see [7] and Appendix. Namely, there are the following types of indecomposable objects: bands, bispecial strings, finite and infinite special strings, finite and infinite strings. We shall give more details in the over-next section.

Remark 4.2. In fact we have shown (see [22]) that the derived category of the Harish-Chandra modules over $SL_2(\mathbb{R})$ is tame.

5. Matrix problem for a general nodal algebra

Let A be a nodal algebra, which is supposed to be basic, T its center $\tilde{A} = \mathrm{End}_A(\mathrm{rad}(A))$. Recall that we have 3 types of simple A-modules (see [14]):

(1) Such simple left A-modules U that $l_A(\tilde{A} \otimes_A U) = 1$.
(2) $l_A(\tilde{A} \otimes_A U) = 2$, $l_{\tilde{A}}(\tilde{A} \otimes_A U) = 2$.
(3) $l_A(\tilde{A} \otimes_A U) = 2$, $l_{\tilde{A}}(\tilde{A} \otimes_A U) = 1$.

It follows from the definition of a nodal algebra that A and $\tilde{A}$ have the common radical: $r = \mathrm{rad}(A) = \mathrm{rad}(\tilde{A})$. Hence we have an embedding of semi-simple algebras $A/r \longrightarrow \tilde{A}/r$. Since A is basic and $\mathbf{k}$ algebraically closed, A/r is isomorphic to a product of several copies of $\mathbf{k}$.

The conditions (1)–(3) above imply that each simple component of $\tilde{A}/r$ is isomorphic either to $\mathbf{k}$ or to $M_2(\mathbf{k})$ and the induced map $A/r \longrightarrow \tilde{A}/r$ acts as follows:

(1) A simple component of A/r is mapped isomorphically onto a simple component of $\tilde{A}/r$.
(2) A simple component of A/r is embedded diagonally into a product of two simple components of $\tilde{A}/r$, both isomorphic to $\mathbf{k}$.
(3) A product of two simple components of A/r is mapped isomorphically onto the diagonal subalgebra of a simple component of $\tilde{A}/r$ isomorphic to $M_2(\mathbf{k})$.

Let I be an ideal in $\tilde{A}$ generated by the radical and idempotents of the first type. Then I is an ideal in A, too. Moreover, the factor-algebras A/I and $\tilde{A}/I$ are semi-simple in this case. So, the conditions of the main theorem are fulfilled.

Let $\tilde{A} = \prod_{n=1}^{N} \tilde{A}_n$, where all $\tilde{A}_n$ are hereditary orders, $C(\tilde{A}_n)$ be the basic algebra corresponding to $\tilde{A}_n$. Since it is a hereditary order over $\mathbf{k}[[t]]$ (by Noether normalization there is a finite ring extension $\mathbf{k}[[t]] \longrightarrow T$), it is isomorphic to the completed path algebra of some cycle of length d_n (it follows from the classification of hereditary orders over a complete discrete valuation ring, see [8,23] or [18]). Let us introduce some numbering of the vertices of the cycles $C(\tilde{A}_n)$. For the sake of convenience we number the vertices of $C(\tilde{A}_n)$ by elements $[1], [2], \ldots, [d_n]$ of $\mathbb{Z}/d_n\mathbb{Z}$. So each simple $\tilde{A}$-module U correspond to a pair (n, ν), where $n \in 1, \ldots, N$, $\nu \in \mathbb{Z}/d_n\mathbb{Z}$. Namely, n denotes the number of the component $\tilde{A}_n$ that acts non-trivially on U, ν is the number of the vertex from the cycle $C(\tilde{A}_n)$ corresponding to U.

In order to consider the category of triples TC_A we have to consider morphisms in the derived category $D^-(\tilde{A}\text{-mod})$. From what we have seen above it follows that it is enough to consider morphisms in $D^-(C(\tilde{A}_n)\text{-mod})$, $n \in 1, \ldots, N$.

Let C be a cycle of length m. Then the category of finitely generated left C-modules is hereditary. Hence any indecomposable object of $D^-(C\text{-mod})$ is isomorphic to $0 \longrightarrow M \longrightarrow 0$, where M is an indecomposable C-module. Moreover, either M is projective or it has a resolution $P \xrightarrow{\varphi} Q$, where P and Q are indecomposable projective C-modules, φ a morphism, given by some path on the quiver C [15]; denote $l(\varphi) = \mathrm{length}(\mathrm{coker}\,\varphi)$.

The morphisms of A-modules (which are non-zero modulo the radical) are of the following form (see [15]):

$$\begin{tikzcd}
P_{\nu+1} \arrow[r, "\varphi_1"] & P_\nu & P_\nu \arrow[r, "\psi_1"] \arrow[d] & P_{\nu-1} \arrow[d, "c_{\nu-1}"] \\
P_{\nu+2} \arrow[r, "\varphi_2"] \arrow[u, "c_{\nu+1}"] & P_\nu \arrow[u] & P_\nu \arrow[r, "\psi_2"] \arrow[d] & P_{\nu-2} \arrow[d, "c_{\nu-2}"] \\
P_{\nu+3} \arrow[r, "\varphi_3"] \arrow[u, "c_{\nu+2}"] & P_\nu \arrow[u] & P_\nu \arrow[r, "\psi_3"] \arrow[d] & P_{\nu-3} \arrow[d, "c_{\nu-3}"] \\
\cdots \arrow[u, "c_{\nu+3}"] & \cdots \arrow[u] & \cdots & \cdots
\end{tikzcd}$$

where $c_{\nu+i} : P_{\nu+i+1} \longrightarrow P_{\nu+i}$ is the map given by an arrow going from the vertex $\nu + i$ to $\nu + i + 1$. There are also morphisms in the derived category, which correspond to Ext^1-groups:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & Q \arrow[d] \\
0 \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & 0 \arrow[d] \\
R \arrow[r] & P \arrow[r] & 0
\end{tikzcd}$$

Let us now construct the partially ordered set, which describes the matrix problem corresponding to the category of triples TC_A for a given nodal algebra A.

Let $C(\tilde{A}_n)$ be a basic algebra (which is a cycle) corresponding to $\tilde{A}_n$. Consider a complex

$$\bigl(P_{\nu+l(\varphi)} \xrightarrow{\varphi} P_\nu\bigr)[f],$$

where $\nu \in \mathbb{Z}/d_n\mathbb{Z}$, φ a morphism of projective modules given by the path of the length $l(\varphi)$, $f \in \mathbb{Z}$ the shift of the complexes.

Denote $\mathbf{J}(C(\tilde{A}_n))$ the set of simple $\tilde{A}_n$-modules, which correspond to direct summands of $\tilde{A} \otimes_A U$, where U is a simple A-module of second or third type.

Let ν, $\nu + l(\varphi) \in \mathbf{J}(C(\tilde{A}_n))$. Then we associate to this complex two symbols $\alpha(n, \nu, l(\varphi), f)$ and $\beta(n, \nu + l(\varphi), l(\varphi), f + 1)$. In case only ν (respectively $\nu + l(\varphi)$ or neither of both) belongs to $\mathbf{J}(C(\tilde{A}_n))$ we associate with it only $\alpha(n, \nu, l(\varphi), f)$

(respectively $\beta(n, \nu + l(\varphi), l(\varphi), f + 1)$ or nothing). In the same way a symbol $\rho(n, \nu, f)$ corresponds to the object

$$(0 \longrightarrow P_\nu \longrightarrow 0)[f].$$

We are ready now to introduce our partially ordered set.

Definition 5.1. We introduce a Bondarenko's partially ordered set together with equivalence relation in several steps.

(1) Let $1 \leqslant n \leqslant N$, $\nu \in \mathbb{Z}/d_n\mathbb{Z}$, $\nu \in \mathbf{J}(C(\tilde{A}_n))$.

$$\mathbf{E}_\nu^{(f)}(n) = \bigl\{\alpha(n, \nu, i, f), \beta(n, \nu, i, f) \mid i \geqslant 1\bigr\} \cup \bigl\{\rho(n, \nu, f)\bigr\}, \quad f \in \mathbb{Z}.$$

(2) $$\mathbf{E}_\nu(n) = \bigcup_{f \in \mathbb{Z}} \mathbf{E}_\nu^{(f)}(n), \qquad \mathbf{E}(n) = \bigcup_{\nu \in \mathbf{J}(\tilde{A}_n)} \mathbf{E}_\nu(n), \qquad \mathbf{E} = \bigcup_{n=1}^{N} \mathbf{E}(n).$$

(3) We can introduce a partial order an equivalence relation on $\mathbf{E}$.
   (a) First of all

$$\alpha(n, \nu, i_1, f) \geqslant \alpha(n, \nu, i_2, f), \qquad \beta(n, \nu, i_1, f) \leqslant \beta(n, \nu, i_2, f)$$

   for $i_1 \geqslant i_2$.
   (b) Furthermore,

$$\alpha(n, \nu, i, f) \geqslant \rho(n, \nu, f) \geqslant \beta(n, \nu, j, f)$$

   for all $i, j \geqslant 1$, $f \in \mathbb{Z}$.
   (c) If $i \in \mathbb{N}$ and $\nu \in \mathbf{J}(C(\tilde{A}_n))$ are such that $i + \nu \in \mathbf{J}(C(\tilde{A}_n))$, then

$$\alpha(n, \nu, i, f) \sim \beta(n, \nu + i, i, f + 1), \quad f \in \mathbb{Z}.$$

(4) Let $1 \leqslant n \leqslant N$, $\nu \in \mathbb{Z}/d_n\mathbb{Z}$, $\nu \in \mathbf{J}(C(\tilde{A}_n))$. The set $\mathbf{F}(n, \nu, f)$, $f \in \mathbb{Z}$ consists either from one or two elements.
   (a) If U is a simple module of a second type, (n, ν), (m, μ) corresponding simple $\tilde{A}$-modules, then the sets $\mathbf{F}(n, \nu, f) = \{g(n, \nu, f)\}$ and $\mathbf{F}(m, \mu, f) = \{g(m, \mu, f)\}$ are the sets consisting from one element. Moreover $g(n, \nu, f) \sim g(m, \mu, f)$.
   (b) In case U is an A-module of the third type, (n, ν) corresponds to $B \otimes_A U$, then

$$\mathbf{F}(n, \nu, f) = \bigl\{g'(n, \nu, f), g''(n, \nu, f)\bigr\}.$$

   It is however convenient to assume that $\mathbf{F}(n, \nu, f) = \{g(n, \nu, f)\}$ and $g(n, \nu, f) \sim g(n, \nu, f)$.

Let us point out that we are interested in only in *bounded from the right representations* of the constructed bunch of semi-chains (since we want to describe the derived category of bounded from the right complexes). Moreover, the non-degeneracy condition from the definition of the category of triples implies certain non-degeneracy restrictions on our matrices. However they concern only the discrete series of representations, for continuous series they are automatically satisfied (see [5]).

To sum everything up we formulate the main result of this article:

Theorem 5.2. *Let A be a nodal algebra. The description of indecomposable objects of $D^-(A\text{-mod})$ can be reduced to the description of indecomposable representations of a bunch of semi-chains, described in the previous definition. In particular, there are* 3 *types of indecomposable objects in $D^-(A\text{-mod})$:*

(1) *Bands $\mathcal{B}(w, m, \lambda)$.*
(2) *Strings (which can be usual, special and bispecial).*
(3) *Complexes P_i and $P_i \xrightarrow{\varphi} P_j$, where i and j correspond to simple A-modules of the first type.*

In particular, a nodal algebra is derived-tame in "pragmatic sense".

Remark 5.3. For the ring $A = \mathbf{k}[[x, y]]/(xy)$ it was shown in [10], how to describe complexes, corresponding to objects of A-mod with respect to the canonical inclusion

$$A\text{-mod} \longrightarrow D^-(A\text{-mod}).$$

6. Description of indecomposable complexes via gluing diagrams

In this section we want to show, how the combinatoric of bands and strings can be applied to write down explicit projective resolutions of indecomposable complexes. We shall consider two "typical examples": the case of $A = \mathbf{k}\langle\langle x, y\rangle\rangle/(x^2, y^2)$ and the case of the completed path algebra of the Gelfand quiver.

6.1. The case of $D^-(\mathbf{k}\langle\langle x, y\rangle\rangle/(x^2, y^2)\text{-mod})$

Let $A = \mathbf{k}\langle\langle x, y\rangle\rangle/(x^2, y^2)$, r its radical, $\tilde{A} = \mathrm{End}_A(r)$. As we have seen in previous sections, the description of indecomposable objects of the derived category $D^-(A\text{-mod})$ can be reduced to a matrix problem of type "representations of bunches of chains." There are two types of indecomposable complexes in this case: bands $\mathcal{B}(w, m, \lambda)$ and strings $\mathcal{U}(w)$.

Let us rewrite the corresponding partially ordered sets in this special case. We have a family of sets

$$\mathbf{F}(k) = \{g(1, k), g(2, k)\}, \quad k \in \mathbb{Z},\ g(1, k) \sim g(2, k),$$

which correspond to columns of the matrices $H_k(i|1)$ and $H_k(i|2)$; and two families of sets $\mathbf{E}_1(k) = \{\alpha(1,i,k), \beta(1,i,k) \mid i \geqslant 1\} \cup \{\rho(1,k)\}$ and $\mathbf{E}_2(k) = \{\alpha(2,i,k), \beta(2,i,k) \mid i \geqslant 1\} \cup \{\rho(2,k)\}$, which label horizontal blocks of matrices $H_k(i|1)$ and $H_k(i|2)$, $k \in \mathbb{Z}$.

The symbols $\{\rho(1,k)\}$ and $\{\rho(2,k)\}$ correspond to the kth shift of projective $\tilde{A}$-modules P_1 and P_2. The element $\beta(1+i,i,k+1)$ is conjugated to $\alpha(1,i,k)$ and $\beta(2+i,i,k+1)$ is conjugated to $\alpha(2,i,k)$, where $1+i$ and $2+i$ have to be taken modulo 2.

Let w be some word containing a subword $\beta(1+i,i,k) \sim \alpha(1,i,k)$. If i is even, then it comes from the complex $(P_1 \xrightarrow{\varphi} P_1)[k]$, where $\operatorname{coker}(\varphi)$ is an indecomposable $\tilde{A}$-module of the length i. In what follows we shall say that φ has length $\operatorname{coker}(\varphi)$. If i is odd, then this subword corresponds to $(P_2 \xrightarrow{\varphi} P_1)[k]$.

As we shall see, an indecomposable complex from the derived category $D^-(A\text{-mod})$ can be viewed as a gluing of complexes

$$P_1, \quad P_2, \quad P_1 \xrightarrow{\varphi} P_1, \quad P_2 \xrightarrow{\psi} P_2, \quad P_1 \xrightarrow{\phi} P_2, \quad P_2 \xrightarrow{\theta} P_1.$$

Suppose we have a subword

$$\beta(1+i,i,k+1) \sim \alpha(1,i,k) - g(1,k) \sim g(2,k) - \alpha(2,j,k) \sim \beta(2+j,j,k+1).$$

It can be interpreted as a gluing of complexes

$$\begin{tikzcd} P_{i+1} \arrow[r, "\varphi_1"] & P_1 \arrow[d, dotted, no head] \\ P_{j+2} \arrow[r, "\varphi_2"] & P_2 \end{tikzcd}$$

shown by the dotted line. Here the indices $i+1$ and $j+2$ must be taken modulo 2, φ_1 and φ_2 have the length i and j respectively.

The subword

$$\beta(1+i,i,k+1) \sim \alpha(1,i,k) - g(1,k) \sim g(2,k) - \beta(2+j,j,k) \sim \alpha(2,j,k-1),$$

corresponds to the gluing of the type

$$\begin{tikzcd} P_{i+1} \arrow[r, "\varphi_1"] & P_1 \arrow[d, dotted, no head] & \\ & P_{j+2} \arrow[r, "\varphi_2"] & P_2 \end{tikzcd}$$

and so on.

It is convenient to describe gluing of the complexes by a gluing diagram.

Example 6.1. Consider the band data $\mathcal{B}(w, m, \lambda)$, where $w = \alpha(2,2,0) \sim \beta(2,2,1) - g(2,1) \sim g(1,1) - \alpha(1,3,1) \sim \beta(2,3,2) - g(2,2) \sim g(1,2) - \beta(1,2,2) \sim \alpha(1,2,1) - g(1,1) \sim g(2,1) - \alpha(2,1,1) \sim \beta(1,1,0) - g(1,0) \sim g(2,0)$.

Then it corresponds to the following gluing diagram:

$$\begin{tikzcd}
P_2^m \arrow[r, "xyxI_m"] \arrow[ddd, dotted, no head] & P_1^m \arrow[d, dotted, no head] & \\
& P_2^m \arrow[r, "xyI_m"] & P_2^m \arrow[d, dotted, no head, "J_m(\lambda)"] \\
& P_2^m \arrow[r, "xI_m"] \arrow[d, dotted, no head] & P_1^m \\
P_1^m \arrow[r, "yxI_m"'] & P_1^m &
\end{tikzcd}$$

This gluing diagram gives the complex

$$\begin{tikzcd}
& A^m \arrow[dr, "xyI_m"] & \\
A^m \arrow[ur, "xyxI_m"] \arrow[dr, "xyI_m"'] & & A^m \\
& A^m \arrow[ur, "xJ_m(\lambda)"'] &
\end{tikzcd}$$

or, the same,

$$A^m \xrightarrow{\begin{pmatrix} xyxI_m \\ yxI_m \end{pmatrix}} A^{2m} \xrightarrow{(xyI_m(\lambda)\ \ xJ_m(\lambda))} A^m.$$

Example 6.2. Consider the string data $\mathcal{U}(w)$, where $w = \cdots - g(2,1) \sim g(1,1) - \beta(1,1,1) \sim \alpha(2,1,0) - g(2,0) \sim g(1,0) - \alpha(1,1,0) \sim \beta(2,1,1) - g(2,1) \sim g(1,1) - \alpha(1,1,1) \sim \beta(2,1,2) - \cdots$.

The gluing diagram is

$$\begin{tikzcd}
P_2 \arrow[r, "x"] \arrow[d, dash, dotted] & P_1 \arrow[d, dash, dotted] & P_2 \arrow[r, "x"] \arrow[d, dash, dotted] & P_1 \arrow[d, dash, dotted] & \\
{} & P_2 \arrow[r, "x"] & P_1 & P_2 \arrow[r, "x"] & P_1 \arrow[d, dash, dotted] \\
{} \arrow[d, dash, dotted] & P_1 \arrow[r, "y"] \arrow[d, dash, dotted] & P_2 \arrow[d, dash, dotted] & P_1 \arrow[r, "y"] \arrow[d, dash, dotted] & P_2 \\
P_1 \arrow[r, "x"] & P_2 & P_1 \arrow[r, "x"] & P_2 &
\end{tikzcd}$$

This string complex is the minimal resolution of the simple module **k**

$$\cdots \longrightarrow A^2 \xrightarrow{\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}} A^2 \xrightarrow{\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}} A.$$

There are also finite strings.

Example 6.3. Consider the string $\mathcal{U}(w)$ given by

$$w = \rho(1,1) - g(1,1) \sim g(2,1) - \beta(2,3,1) \sim \alpha(1,3,0) - g(1,0) \sim g(2,0) - \rho(2,0).$$

$$\begin{tikzcd}
P_1 \arrow[d, dash, dotted] & \\
P_2 \arrow[r, "yxy"] & P_1 \arrow[d, dash, dotted] \\
 & P_2
\end{tikzcd}$$

It corresponds to the complex

$$A \xrightarrow{yxy} A.$$

6.2. The case of the Gelfand quiver

Let A be the completed path algebra of the Gelfand quiver, r its radical and $\tilde{A} = \mathrm{End}_A(r)$. Let P_1, P_2, P_3, be indecomposable projective A-modules, P, Q indecomposable projective $\tilde{A}$-modules and suppose that $\tilde{A} \otimes_A P_1 = \tilde{A} \otimes_A P_2 = P$ and $\tilde{A} \otimes_A P_3 = Q$.

We have seen that the description of indecomposable objects of the derived category $D^-(A\text{-mod})$ can be reduced to a matrix problem of the type "representations of bunches of semi-chains."

The combinatorics of indecomposable objects is similar to the case of bunches of chains. Continuous series of representations are still bands $\mathcal{B}(w, m, \lambda)$, but the structure of discrete series is much more complicated. There are bispecial strings $\mathcal{U}(w, m, \delta_1, \delta_2)$, finite and infinite special strings $\mathcal{U}(w, \delta)$ and finite and infinite usual strings $\mathcal{U}(w)$. In this case there are also complexes (certain discrete series) which do not come from the matrix problem.

Let us rewrite the partially ordered set in this case. We have a family of sets $\mathbf{F}(k) = \{g(k)\}$ with equivalence relation $g(k) \sim g(k)$. The set

$$\mathbf{E}(k) = \{\alpha(i,k), \beta(i,k) \mid i \in \mathbb{N}\} \cup \{\rho(k)\}$$

is a chain with the total order

$$\beta(j_2,k) \geqslant \beta(j_1,k) \geqslant \rho(k) \geqslant \alpha(i_1,k) \geqslant \alpha(i_2,k)$$

for all natural numbers $i_1 \geqslant i_2$ and $j_1 \geqslant j_2$ and $k \in \mathbb{Z}$.

If $i = 2l$ is even then we have conjugate points $\alpha(2l,k)$ and $\beta(2l,k+1)$, and the subword $\alpha(2l,k) \sim \beta(2l,k+1)$ corresponds to the complex $(P \xrightarrow{\varphi} P)[k]$, where φ is the unique path from b to itself of the length $2l$. If $i = 2l+1$ is even, then elements $\alpha(2l+1,k)$ and $\beta(2l+1,k)$ correspond to complexes

$$(P \xrightarrow{\varphi} Q)[k] \quad \text{and} \quad (Q \xrightarrow{\varphi} P)[k-1]$$

respectively, where φ has the length $2l+1$.

As in the case of dihedral algebra, the combinatorics of bands and strings can be simplified.

$$\beta(2i,k+1) \sim \alpha(2i,k) - g(k) \sim g(k) - \alpha(2j,k) \sim \beta(2j,k+1)$$

codes the gluing

$$\begin{tikzcd} P \arrow[r, "2i"] & P \arrow[d, dotted, no head] \\ P \arrow[r, "2j"] & P \end{tikzcd}$$

etc. There is an algorithm which associates to a band or string data the corresponding complex of projective modules. A complex of projective A-modules is obtained as a gluing of the complexes of $\tilde{A}$-modules $P \xrightarrow{\varphi} P$, $Q \xrightarrow{\phi} P$, $P \xrightarrow{\psi} Q$ and P. In order to keep the notation simpler we shall write instead of the map φ only its length $l(\varphi)$ (which defines φ uniquely).

Example 6.4. Consider the band data: $(w, 1, \lambda)$, where

$$\begin{aligned} w = \alpha(2,0) &\sim \beta(2,1) - g(1) \sim g(1) - \alpha(6,1) \\ &\sim \beta(6,2) - g(2) \sim g(2) - \beta(4,2) \sim \alpha(4,1) - g(1) \\ &\sim g(1) - \beta(4,1) \sim \alpha(4,0) - g(0) \sim g(0). \end{aligned}$$

It gives the following gluing diagram

$$\begin{tikzcd}
 & P \arrow[r, "2"] \arrow[d, dotted] & P \\
P \arrow[r, "4"] & P & \\
P \arrow[r, "6"] \arrow[u, dotted] & P & \\
 & P \arrow[r, "4"] \arrow[u, dotted] & P \arrow[uuu, dotted, "\lambda"']
\end{tikzcd}$$

Dotted lines here are directed: the direction of the arrow shows that there is a map of complexes which induces a non-zero map modulo the radical in the corresponding component of the complex.

Now we introduce the rule "of moving of an arrow":

(1) Any time we have the situation

$$\begin{tikzcd}
\bullet \arrow[r, "a"] \arrow[d, dotted] & \bullet \\
\bullet &
\end{tikzcd}$$

we move the arrow (preserving its sign):

$$\begin{tikzcd}
\bullet \arrow[r, "a"] \arrow[d, dotted] & \bullet \\
\bullet \arrow[ur, "a"] &
\end{tikzcd}$$

(2) Any time we have the situation

$$\begin{tikzcd}
 & \bullet \arrow[d, dotted] \\
\bullet \arrow[r, "b"] & \bullet
\end{tikzcd}$$

we can move the arrow taking the opposite sign:

$$\begin{tikzcd}
& \bullet \arrow[d, dotted] \\
\bullet \arrow[r, "b"] \arrow[ur, "-b"] & \bullet
\end{tikzcd}$$

(3) If we have the situation

$$\begin{tikzcd}
\bullet \arrow[r, "a"] \arrow[d, dotted, "\lambda"] & \bullet \\
\bullet &
\end{tikzcd}$$

we move the arrow (preserving its sign):

$$\begin{tikzcd}
\bullet \arrow[r, "a"] \arrow[d, dotted, "\lambda"] & \bullet \\
\bullet \arrow[ur, "\lambda a"] &
\end{tikzcd}$$

Applying this rule to the band data above we get the following picture

$$\begin{tikzcd}
& P \arrow[r, "2"] \arrow[d, dotted] \arrow[dddr, "\lambda 2"] & P \\
P \arrow[r, "6"] \arrow[ur, "6"] \arrow[dr, "4"] \arrow[ddr, "-4"] & P \arrow[ur, "-2"] \arrow[ddr, "-\lambda 2"] & \\
P \arrow[r, "4"] \arrow[u, dotted] \arrow[dr, "-4"] & P \arrow[dr, "4"] & \\
& P \arrow[r, "4"] \arrow[u, dotted] & P \arrow[uuu, dotted, "\lambda"]
\end{tikzcd}$$

Now we can insert instead of P one of the symbols P_1 or P_2 following the rule that every dotted line has to connect symbols with different subscripts.

$$\begin{tikzcd}
 & P_1 \arrow[r, "2"] \arrow[ddr, "\lambda 2"] & P_1 \\
P_2 \arrow[r, "6"] \arrow[ur, "6"] \arrow[dr, "4"] \arrow[ddr, "-4"'] & P_2 \arrow[ur, "-2"] \arrow[dr, "-\lambda 2"] & \\
P_1 \arrow[r, "4"] \arrow[dr, "-4"'] & P_1 \arrow[dr, "4"] & \\
 & P_2 \arrow[r, "4"] & P_2
\end{tikzcd}$$

It corresponds to the complex $\mathcal{P}_\bullet$

$$P_2 \oplus P_1 \xrightarrow{\begin{pmatrix} \varphi_6 & 0 \\ \varphi_6 & 0 \\ \varphi_4 & \varphi_4 \\ -\varphi_4 & -\varphi_4 \end{pmatrix}} P_1 \oplus P_2 \oplus P_1 \oplus P_2 \xrightarrow{\begin{pmatrix} \varphi_2 & -\varphi_2 & 0 & 0 \\ \lambda\varphi_2 & -\lambda\varphi_2 & \varphi_4 & \varphi_4 \end{pmatrix}} P_1 \oplus P_2,$$

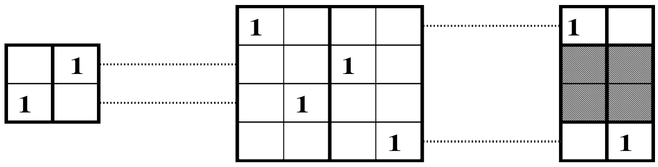
where φ_{2k} always denotes the map $\varphi_{2k} : P_i \longrightarrow P_j$ of the length $2k$. Let us compute the triple $(\widetilde{\mathcal{P}}_\bullet, \mathcal{M}_\bullet, i)$. Observe that after applying $\tilde{A} \otimes_A$ to $P_i \xrightarrow{\varphi} P_j$ $(i, j \in \{1, 2\})$ we get $P \xrightarrow{\varphi} P$. It holds:

$$\widetilde{\mathcal{P}}_\bullet := \tilde{A} \otimes_A \mathcal{P}_\bullet = \left(P \oplus P \xrightarrow{\begin{pmatrix} \varphi_6 & 0 \\ \varphi_6 & 0 \\ \varphi_4 & \varphi_4 \\ -\varphi_4 & -\varphi_4 \end{pmatrix}} P \oplus P \oplus P \oplus P \xrightarrow{\begin{pmatrix} \varphi_2 & -\varphi_2 & 0 & 0 \\ \lambda\varphi_2 & -\lambda\varphi_2 & \varphi_4 & \varphi_4 \end{pmatrix}} P \oplus P \right).$$

Denote $M = M_2(\mathbf{k})$. The map $i : A/I \otimes_A \mathcal{P}_\bullet \longrightarrow \tilde{A}/I \otimes \widetilde{\mathcal{P}}_\bullet$ is

$$\begin{tikzcd}
\mathbf{k}_2 \oplus \mathbf{k}_1 \arrow[r, "0"] \arrow[d, "i_2"] & \mathbf{k}_1 \oplus \mathbf{k}_2 \oplus \mathbf{k}_1 \oplus \mathbf{k}_2 \arrow[r, "0"] \arrow[d, "i_1"] & \mathbf{k}_1 \oplus \mathbf{k}_2 \arrow[d, "i_0"] \\
M \oplus M \arrow[r, "0"] & M \oplus M \oplus M \oplus M \arrow[r, "0"] & M \oplus M
\end{tikzcd}$$

where i_0, i_1 and i_2 are given by matrices

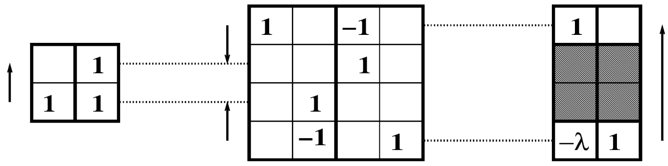


$\mathbf{i_2}$ $\mathbf{i_1}$ $\mathbf{i_0}$

We have the following chain equivalence:

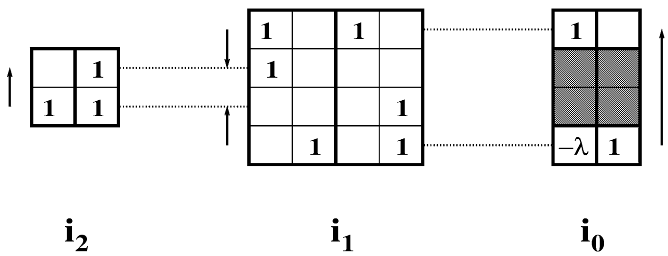
$$\begin{tikzcd}[column sep=huge, row sep=huge]
P \oplus P \arrow[r, "{\begin{pmatrix} \varphi_6 & 0 \\ \varphi_6 & 0 \\ \varphi_4 & \varphi_4 \\ -\varphi_4 & \varphi_4 \end{pmatrix}}"] \arrow[d, "{\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}}"'] & P \oplus P \oplus P \oplus P \arrow[r, "{\begin{pmatrix} \varphi_2 & -\varphi_2 & 0 & 0 \\ \lambda\varphi_2 & -\lambda\varphi_2 & \varphi_4 & \varphi_4 \end{pmatrix}}"] \arrow[d, "{\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}}"] & P \oplus P \arrow[d, "{\begin{pmatrix} 1 & 0 \\ -\lambda & 1 \end{pmatrix}}"] \\
P \oplus P \arrow[r, "{\begin{pmatrix} 0 & 0 \\ \varphi_6 & 0 \\ 0 & \varphi_4 \\ 0 & 0 \end{pmatrix}}"] & P \oplus P \oplus P \oplus P \arrow[r, "{\begin{pmatrix} \varphi_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \varphi_4 \end{pmatrix}}"] & P \oplus P
\end{tikzcd}$$

This map transforms the matrices i_0, i_1 and i_2 to the form

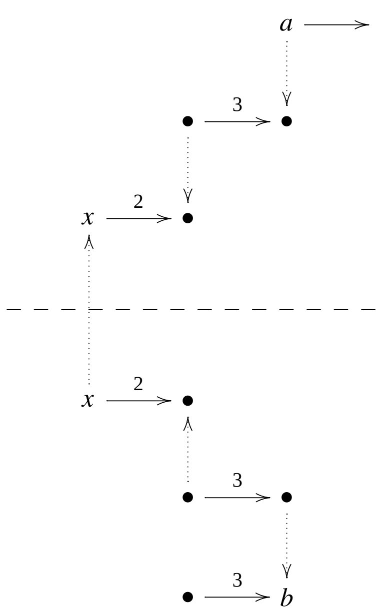


$\mathbf{i_2}$ $\mathbf{i_1}$ $\mathbf{i_0}$

Doing the allowed transformations of rows and columns we transform them into the canonical form (see [5]).

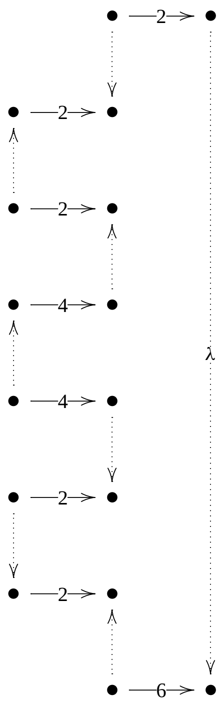


Suppose that a dotted line joins two points with equal weights. How to choose the direction of this line? We can do it by means of the following rule. Let us suppose that a gluing diagram has a subpart



We have to find first pair of points (a, b) which are non-symmetric with respect to the axe of symmetry. In our case it holds: $a < b$. The arrow looks in the direction of the smaller point (see [5]). In case when there are many dotted arrows joining points with equal weights, we have to consider for each pair its own axe of symmetry.

Example 6.5. Consider the following gluing diagram (band):



If a word w is symmetric, then we set directions of both dotted arrows intersecting the symmetry axe simultaneously clockwise or anticlockwise.

Let us now consider the case of discrete series. The first type of them are bispecial strings $\mathcal{U}(w, m, \delta_1, \delta_2)$. They are given by some word w, by a natural number m and by two symbols $\delta_1, \delta_2 \in \{-, +\}$. Consider the following example:

Example 6.6. Let $\delta_1 = +$, $\delta_2 = -$, $m = 5$ and $w = (+)g(1) - \alpha(4, 1) \sim \beta(4, 2) - g(2) \sim g(2) - \beta(6, 2) \sim \alpha(6, 1) - g(1) \sim g(1) - \beta(2, 1) \sim \alpha(2, 0) - g(0)(-)$. Then we get the following gluing diagram

$$\begin{tikzcd}
5P \arrow[r, "4"] \arrow[d, dotted] & 5P(+) & \\
5P \arrow[r, "6"] & 5P & \\
& 5P \arrow[u, dotted] \arrow[r, "2"] & 5P(-)
\end{tikzcd}$$

It correspond to the complex:

$$\begin{tikzcd}
5P_1 \arrow[r, "\varphi_4 I_5^{r+}"] & 3P_1 \oplus 2P_2 & \\
5P_2 \arrow[ur, "\varphi_4 I_5^{r+}"] \arrow[r, "\varphi_6 I_5"] \arrow[dr, "-\varphi_6 I_5"'] & 5P_2 \arrow[dr, "\varphi_2 J_5^{r-}"] & \\
 & 5P_1 \arrow[r, "\varphi_2 J_5^{r-}"] & 2P_1 \oplus 3P_2
\end{tikzcd}$$

where I_5^{r+} and J_5^{r-} are the following matrices:

$$I_5^{r+} = \left(\begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right), \qquad J_5^{r-} = \left(\begin{array}{ccccc} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ \hline 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

The matrix J_5^{r-} is obtained by the following rule: we take the 5×5 matrix

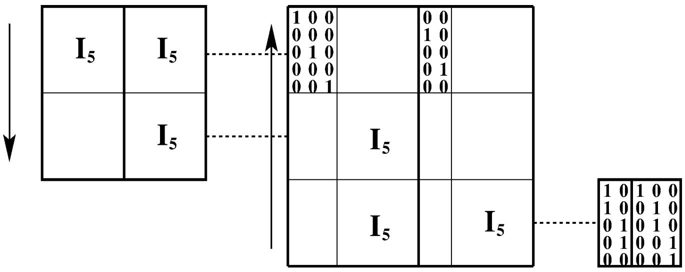
$$J_5 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and permute rows putting rows with even and odd numbers in separate horizontal blocks. The superscript "–" means that the block with even rows comes first. The same rule applied to the identity matrix I_5 gives I_5^{r+}.

The triple corresponding to this complex is isomorphic to $(\widetilde{\mathcal{P}}_\bullet, \mathcal{M}_\bullet, i)$ where

$$\widetilde{\mathcal{P}}_\bullet = \left(5P \oplus 5P \xrightarrow{\begin{pmatrix} I_5 & 0 \\ 0 & I_5 \\ 0 & 0 \end{pmatrix}} 5P \oplus 5P \oplus 5P \xrightarrow{(\,0\;\;0\;\;I_5\,)} 5P \right),$$

$\mathcal{M}_\bullet = (\mathbf{k}_1^5 \oplus \mathbf{k}_2^5 \xrightarrow{0} \mathbf{k}_1^3 \oplus \mathbf{k}_2^2 \oplus \mathbf{k}_1^5 \oplus \mathbf{k}_2^5 \xrightarrow{0} \mathbf{k}_2^5 \oplus \mathbf{k}_1^5)$ and i_0, i_1, i_2 are given by matrices



In the last example both special ends were sinks. In the case when one of the special end is source we have to modify our rule a little bit.

Example 6.7. Consider the following bispecial string: $m = 4$, $\delta_1 = +$, $\delta_2 = +$, $w = (+)g(2) - \beta(2,2) \sim \alpha(2,1) - g(1) \sim g(1) - \beta(2,1) \sim \alpha(2,0) - g(0)(+)$.

$$\begin{tikzcd}
(+)P \arrow[r, "2"] & P \\
& P \arrow[u, dotted] \arrow[r, "4"] & P(+)
\end{tikzcd}$$

It corresponds to the complex

$$\begin{tikzcd}
2P_1 \oplus 2P_2 \arrow[r, "\varphi_2 I_4^{c+}"] \arrow[dr, "-\varphi_2 I_4^{c+}"'] & 4P_1 \arrow[dr, "\varphi_4 J_4^{r+}"] & \\
& 4P_2 \arrow[r, "\varphi_4 J_4^{r+}"] & 2P_1 \oplus 2P_2
\end{tikzcd}$$

where

$$I_4^{c+} = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \quad \text{and} \quad J_4^{r+} = \left(\begin{array}{cccc} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ \hline 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right).$$

The matrix I_4^{c+} can be computed by the following rule: we take the matrix

$$I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and group odd and even columns into separate blocks.

The last remark concern the case when a bispecial string is given by a “short” diagram.

Example 6.8. Let $m = 5$, $\delta_1 = +$, $\delta_2 = -$ and w is just

$$(+)P \xrightarrow{\varphi_2} P(-).$$

Then the corresponding complex is

$$3P_1 \oplus 2P_2 \xrightarrow{\varphi_2 C} 2P_1 \oplus 3P_2$$

where $C = I_5^{c+} \cdot (J_5^{r-})^{-1}$.

There are also special and usual strings, which can be finite and infinite.

Example 6.9. Let $w = \beta(1,1) - g(1) \sim g(1) - \alpha(1,1)$. Then the string $\mathcal{U}(w)$ corresponds to the gluing diagram

$$\begin{tikzcd} & P \arrow[r, "1"] \arrow[d, dotted] & Q \\ Q \arrow[r, "1"] & p & \end{tikzcd}$$

It defines the complex

$$\begin{tikzcd} & P_1 \arrow[r, "\varphi_1"] & P_3 \\ P_3 \arrow[r, "\varphi_1"] \arrow[ur, "-\varphi_1"] & P_2 \arrow[ur, "-\varphi_1"] & \end{tikzcd}$$

One can recognize in this complex a projective resolution of the simple A-module U_1.

Example 6.10. Let $\delta = +$, $w = (+)g(0) - \alpha(2,0) \sim \beta(2,1) - g(1) \sim g(1) - \alpha(4,1) \sim \beta(4,2) - g(2) \sim g(2) - \alpha(2,2) \sim \beta(2,3) - g(3) \sim g(3) - \cdots$. The infinite special string $\mathcal{U}(w,\delta)$ is given by the gluing diagram

$$\begin{tikzcd} \cdots & P \arrow[r, "2"] \arrow[d, dotted, no head] & P \arrow[d, dotted, no head] & P \arrow[r, "2"] \arrow[d, dotted, no head] & P(+) \\ \cdots P \arrow[r, "4"] & P & P \arrow[r, "4"] & P & \end{tikzcd}$$

It corresponds to the complex

$$\begin{tikzcd}
\cdots & & P_2 \arrow[r, "\varphi_2"] \arrow[dr, "\varphi_2"'] & P_1 \arrow[dr, "\varphi_4"] & P_2 \arrow[r, "\varphi_2"] & P_1 \\
& \cdots P_2 \arrow[r, "\varphi_4"] \arrow[ur, "-\varphi_4"] & P_1 \arrow[ur, "-\varphi_2"'] & P_2 \arrow[r, "\varphi_4"] \arrow[ur, "-\varphi_4"] & P_1 \arrow[ur, "\varphi_2"] &
\end{tikzcd}$$

This complex belongs to $D^{-}(A\text{-mod})$ and does not belong to $D^{b}(A\text{-mod})$.

Example 6.11. Let $w = \rho(1,0) - g(0) \sim g(0) - \alpha(1,0)$. The the string complex $\mathcal{U}(w)$ is given by the gluing diagram

$$\begin{tikzcd}
& P_1 \\
P_3 \arrow[r, "\varphi_1"] \arrow[ur, "-\varphi_1"] & P_2
\end{tikzcd}$$

This complex is isomorphic to a projective resolution of a module which is finitely generated but not finite-dimensional.

There are finally complexes which are not coming from the matrix problem. They are just complexes of the form $P_3 \xrightarrow{\varphi} P_3$, which come from triples $(Q \xrightarrow{\varphi} Q, 0, 0)$.

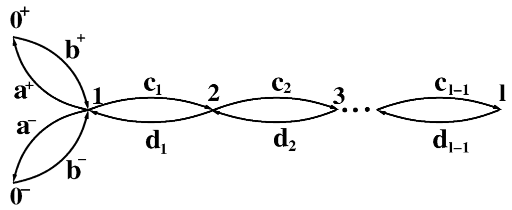
The description of complexes for a general nodal algebra can be obtained by combining the combinatorics of complexes of the derived category of the dihedral algebra and of the Gelfand quiver.

7. Derived categories and Harish-Chandra modules

In [24] it was proven that there are only two cases of compact Lie groups, for which the category of Harish-Chandra modules is tame: $SL_2(\mathbb{R})$ and $SO(1,n)$. As a corollary of the theorem we obtain that the derived category of Harish-Chandra modules is also tame in both of these cases. We have already seen it for $SL_2(\mathbb{R})$.

Let $SO_0(1,n)$ be the connected component of $SO(1,n)$.

(1) Let $n = 2l$. Then the category of Harish-Chandra modules over $SO_0(1, n)$ at a singular point is equivalent to the category of finite-dimensional representations of the completed path algebra of the following quiver (see [24]):



where we have the relations:

$$\gamma = b^+a^+ = b^-a^-,$$

and γ is nilpotent,

$$a^\pm d_1 = 0, \qquad c_1 b^\pm = 0,$$

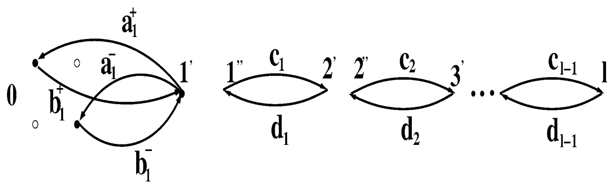
$$c_{i+1}c_i = 0, \quad d_i d_{i+1} = 0, \quad i = 1, \ldots, l-2.$$

Moreover, all

$$\vartheta_i = d_i c_i, \quad i = 1, \ldots, l-1,$$

are nilpotent.

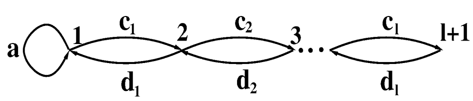
This algebra can be embedded into



(this algebra is the endomorphism algebra of the radical of A. However, it is not so important).

The simple A-module, corresponding to the vertex $l+1$ is of the first type. Those, which correspond to $1, 2, \ldots, l-1$ are of the second type, $0^\pm$ are of the third type.

(2) If $n = 2l + 1$. Then the category of Harish-Chandra modules over $SO_0(1, n)$ is described by the completed path algebra of the following quiver:



with relations:

$$ad_1 = 0, \qquad c_1 a = 0, \qquad d_1 a = 0, \qquad ac_1 = 0,$$

a is nilpotent,

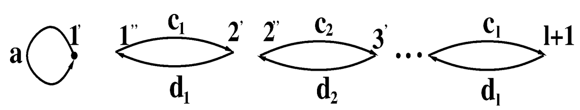
$$c_{i+1}c_i = 0, \quad d_i d_{i+1} = 0, \quad i = 1, \dots, l-1,$$

and all

$$\vartheta_i = d_i c_i, \quad i = 1, \dots, l,$$

are nilpotent.

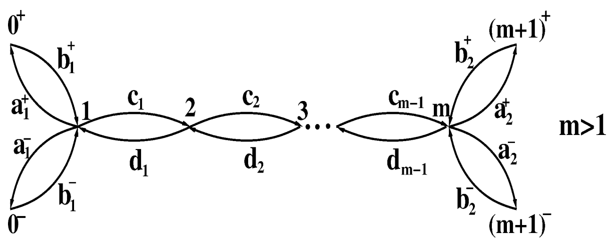
It can be embedded into



The simple module, corresponding to the vertex l is of the first type, all other simple modules are of the second type.

Let us consider two more examples (see [5] for a description of indecomposable modules over these algebras).

Example 7.1. Consider the completed path algebra of the following quiver:



where we have the relations

$$b_i^+ a_i^+ = b_i^- a_i^-, \quad i = 1, 2,$$

$$a_i^{\sigma_1} b_i^{\sigma_2} = 0, \quad i = 1, 2, \quad \sigma_1, \sigma_2 \in \{-, +\},$$

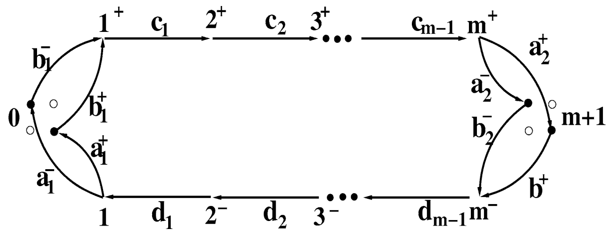
$$c_i d_i = 0, \quad d_i c_i = 0, \quad 1 \leqslant i \leqslant m-1,$$

and finally

$$\gamma = b_2^+ a_2^+ c_{m-1} c_{m-2} \cdots c_1 b_1^+ a_1^+ d_1 d_2 \cdots d_{m-1}$$

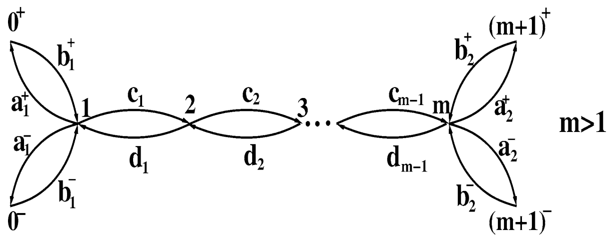
is nilpotent.

As one easily observes, we can embed this algebra into



The simple A-modules, corresponding to vertices $1, 2, \dots, m$ are of the second type, $0^{\pm}$, $(m+1)^{\pm}$ are of the third type.

Example 7.2. Consider the completed path algebra of quiver:



where we have the relations:

$$\gamma_i = b_i^+ a_i^+ = b_i^- a_i^-, \quad i = 1, 2,$$

and γ_i, $i = 1, 2$, are nilpotent,

$$a_1^{\pm} d_1 = 0, \qquad c_1 b_1^{\pm} = 0, \qquad a_2^{\pm} c_{m-1} = 0, \qquad d_{m-1} b_2^{\pm} = 0,$$

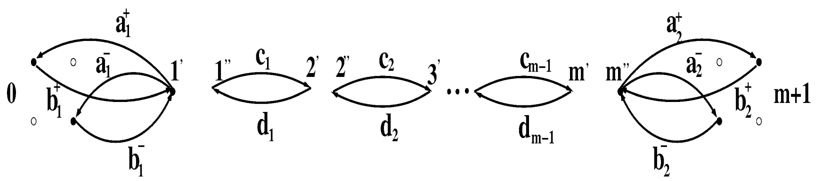
$$c_{i+1} c_i = 0, \quad d_i d_{i+1} = 0, \quad i = 1, \dots, m-2.$$

Moreover, all

$$\vartheta_i = d_i c_i, \quad i = 1, \dots, m-1,$$

are nilpotent.

We can embed this algebra into:



The simple A-modules, corresponding to vertices $1, 2, \dots, m$ are of the second type, $0^{\pm}$, $(m+1)^{\pm}$ are of the third type.

Remark 7.3. It can be checked that all algebras from this section are nodal and every embedding is embedding into the endomorphism algebra of the radical.

Acknowledgments

The authors thank the referee for a very careful reading of the paper and numerous remarks. The first author thanks V.M. Bondarenko, C.-M. Ringel and W. Sörgel for helpful discussions of the results of this article.

Appendix A. Krull–Schmidt theorem for homotopy categories

Let $\mathfrak{C}$ be an additive category. We denote by $C(\mathfrak{C})$ the category of chain complexes with entries from $\mathfrak{C}$ and by $K(\mathfrak{C})$ the factorcategory of $C(\mathfrak{C})$ modulo homotopy. If $\mathfrak{C}$ is abelian, we denote by $D(\mathfrak{C})$ the *derived category* of $\mathfrak{C}$, that is the category of quotients of $K(\mathfrak{C})$ with respect to the set of morphisms inducing isomorphism of homologies. We fix a commutative ring $\mathbf{S}$ and consider $\mathbf{S}$-*categories*, namely, suppose that all sets $\mathfrak{C}(A, B)$ are modules over $\mathbf{S}$ and the multiplication of morphisms is $\mathbf{S}$-bilinear.

Definition A.1. An additive category $\mathfrak{C}$ is called:

- *local* if every object $A \in \mathfrak{C}$ decomposes into a finite direct sum of objects with local endomorphism rings;
- ω-*local* if every object $A \in \mathfrak{C}$ decomposes into a finite of countable direct sum of objects with local endomorphism rings;
- *fully additive* if any idempotent morphism in $\mathfrak{C}$ splits, that is defines a decomposition into a direct sum;
- *locally finite* (over $\mathbf{S}$) if it is fully additive and all morphism spaces $\mathfrak{C}(A, B)$ are finitely generated $\mathbf{S}$-modules. Especially if $\mathbf{S}$ is a field, a locally finite category is called *locally finite-dimensional*.

Evidently, every locally finite category is local; moreover, an endomorphism algebra $\mathfrak{C}(A, A)$ in a locally finite category is a *finite* $\mathbf{S}$-*algebra*, i.e., such that the underlying $\mathbf{S}$-module is finitely generated. It is known that any local (or ω-local) category is fully additive; moreover, a decomposition into a direct sum of objects with local endomorphism rings is always unique; in other words, any local (or ω-local) category is a Krull–Schmidt one, cf. [1, Theorem 3.6].

For a local category $\mathfrak{C}$ denote by $\operatorname{rad}\mathfrak{C}$ its *radical*, that is the set of all morphisms $f : A \to B$, where $A, B \in \operatorname{Ob}\mathfrak{C}$, such that no component of the matrix presentation of f with respect to some (hence any) decomposition of A and B into a direct sum of indecomposable objects is invertible. Note that if $f \notin \operatorname{rad}\mathfrak{C}$, there is a morphism $g : B \to A$

such that $fgf = f$ and $gfg = g$. Hence both gf and fg are nonzero idempotents, which define decompositions $A \cong A_1 \oplus A_2$ and $B \cong B_1 \oplus B_2$ such that the matrix presentation of f with respect to these decompositions is diagonal: $\left(\begin{smallmatrix} f_1 & 0 \\ 0 & f_2 \end{smallmatrix}\right)$, and f_1 is invertible. Obviously, if $\mathfrak{C}$ is locally finite-dimensional, then $\operatorname{rad}\mathfrak{C}(A, B)$ coincide with the set of all morphisms $f : A \to B$ such that gf (or fg) is nilpotent for any morphism $g : B \to A$.

Proposition A.2. *Suppose that* $\mathbf{S}$ *is a complete local noetherian ring with maximal ideal* $\mathfrak{m}$. *If* $\mathfrak{C}$ *is a locally finite category over* $\mathbf{S}$, *the categories* $C(\mathfrak{C})$ *and* $K(\mathfrak{C})$ *are* ω*-local* (*in particular, Krull–Schmidt*). *Moreover, a morphism* $f_\bullet : A_\bullet \to B_\bullet$ *from one of these categories belongs to the radical if and only if all components* $f_n g_n$ (*or* $g_n f_n$) *are nilpotent modulo* $\mathfrak{m}$ *for any morphism* $g_\bullet : B_\bullet \to A_\bullet$.

Proof. Denote by $\mathbf{k} = \mathbf{S}/\mathfrak{m}$ the residue field of $\mathbf{S}$. We use the following simple statement from linear algebra. $\square$

Lemma A.3. *Let* Λ *be a finite-dimensional* $\mathbf{k}$*-algebra and* a *be and element from* Λ. *There is a polynomial* $f(x) \in \mathbf{k}[x]$ *such that* $f(a)$ *is an idempotent and* $f(e) = e$ *for any idempotent* e *from any* $\mathbf{k}$*-algebra. Moreover,* $f(a)$ *is nilpotent* (*or invertible*) *if and only if so is* a.

Proof. Suppose that a polynomial $f(x)$ satisfies the condition $f(0) = 0$, $f(1) = 1$. Then $f(e) = e$ for any idempotent e from any finite-dimensional algebra.

We can embed Λ in an endomorphism algebra of some finite-dimensional vector space V, so we suppose that $\Lambda = \operatorname{End} V$. Decompose $V = V_0 \oplus V_1$ so that the restriction $a|_{V_0}$ is nilpotent and $a|_{V_1}$ is invertible. Replacing a by a^k for some k, one can suppose that $a|_{V_0} = 0$. Indeed, if we have found a polynomial $f(x)$ such that $f(a^k)$ is idempotent, then $f^k(a) = f(a^k)$ hence $f^k(x)$ is the polynomial for a^k we are looking for. In particular, if $a^k = 0$, then we can take $f(x) = x^k$. Set $b = a|_{V_1}$ Since b is invertible, there is a polynomial $g(x)$ such that $g(b) = 1$ and $g(0) = 0$. If 1 is an eigenvalue of b, then $g(1) = 1$, whence $g(e) = e$ for every idempotent e. If 1 is not an eigenvalue of b, then $(xh(x), x^2 - x) = x$, where $h(x)$ is the minimal polynomial of b, hence there is a polynomial $f(x)$ such that $f(x) \equiv g(x) \pmod{xh(x)}$ and $f(x) \equiv x \pmod{x^2 - x}$. Therefore, $f(b) = 1$ and $f(e) = e$ for every idempotent e, which accomplishes the proof of the lemma. $\square$

Recall also a known result, which can be easily deduced, for instance, from [25, Section III.8].

Lemma A.4. *There are polynomials* $G_n(x) \in \mathbb{Z}[x]$ *with* $G_n(0) = 0$ *and such that for every ring* Λ, *any ideal* $I \subseteq \Lambda$ *and any element* $a \in \Lambda$ *such that* $a^2 \equiv a \bmod I$, $G_n(a)^2 \equiv G_n(a) \bmod I^{n+1}$ *and* $a \equiv G_n(a) \bmod I$.

(For instance, $G_1(x) = 3x^2 - 2x^3$.)

Corollary A.5. *Let Λ be a finite algebra over a local noetherian ring* $\mathbf{S}$ *with maximal ideal* $\mathfrak{m}$ *and* $a \in \Lambda$. *For every* $n \in \mathbb{N}$ *there is a polynomial* $g(x) \in \mathbf{S}[x]$ *such that*

- $g(a)^2 \equiv g(a) \bmod \mathfrak{m}^{n+1}$;
- $g(e) \equiv e \bmod \mathfrak{m}^n$ *for every element* e *of an arbitrary finite* $\mathbf{S}$*-algebra such that* $e^2 \equiv e \bmod \mathfrak{m}^n$;
- $g(a) \equiv 1 \bmod \mathfrak{m}$ *if and only if* a *is invertible*;
- $g(a) \equiv 0 \bmod \mathfrak{m}$ *if and only if* a *is nilpotent modulo* $\mathfrak{m}$.

Proof. Find a polynomial $f(x)$ over $\mathbf{k} = \mathbf{S}/\mathfrak{m}$ such that $f(\bar{a})$ is idempotent in $\Lambda/\mathfrak{m}\Lambda$, where $\bar{a} = a + \mathfrak{m}\Lambda \in \Lambda/\mathfrak{m}\Lambda$ and $f(\bar{e}) = \bar{e}$ for any idempotent $\bar{e}$ of any $\mathbf{k}$-algebra; especially $f(0) = 0$. Lift $f(x)$ to a polynomial $F(x) \in \mathbf{S}[x]$ such that $F(0) = 0$. Then $F(a)$ is idempotent modulo $\mathfrak{m}$ and if $e^2 \equiv e \bmod \mathfrak{m}^n$, then $F(e) \equiv e \bmod \mathfrak{m}$ (by the construction of $F(x)$) and $eF(e) \equiv F(e) \bmod \mathfrak{m}^n$ (it is true for any polynomial $F(x)$ satisfying $F(0) = 0$). Set $g(x) = G_n(F(x))$. Then $g(a)$ is idempotent modulo $\mathfrak{m}^{n+1}$, just as $g(e)$ for every e that is idempotent modulo $\mathfrak{m}$. If, moreover, $e^2 \equiv e \bmod \mathfrak{m}^n$, then $g(e) \equiv e \bmod \mathfrak{m}$ and $eg(e) \equiv g(e) \bmod \mathfrak{m}^n$. Let $g(e) = e + r$; then $r = g(e) - e$ and $er = re = r$. Therefore it holds $(e + r)^2 \equiv e + 2r + r^2 \equiv e + r \bmod \mathfrak{m}^n$, wherefrom $r \equiv -r^2 \bmod \mathfrak{m}^n$. But then $r \equiv -r^2 \equiv -r^4 \equiv \cdots \bmod \mathfrak{m}^n$, so $r \equiv 0 \bmod \mathfrak{m}^n$.

Let now $a_\bullet$ be an endomorphism of a complex $A_\bullet$ from $C(\mathfrak{C})$. Consider the sets $I_n \subset \mathbb{Z}$ defined as follows: $I_0 = \{0\}$, $I_{2k} = \{l \in \mathbb{Z} \mid -k \leqslant l \leqslant k\}$ and $I_{2k-1} = \{l \in \mathbb{Z} \mid -k < l \leqslant k\}$. Obviously, $\bigcup_n I_n = \mathbb{Z}$, $I_n \subset I_{n+1}$ and $I_{n+1} \setminus I_n$ consists of a unique element l_n. Using Corollary A.5, we can construct a sequence of endomorphisms $a_\bullet^{(n)}$ such that

- $(a_i^{(n)})^2 \equiv a_i^{(n)} \bmod \mathfrak{m}^n$;
- $a_i^{(n+1)} \equiv a_i^{(n)} \bmod \mathfrak{m}^n$;
- $a_i^{(n)}$ is invertible or nilpotent modulo $\mathfrak{m}$ if and only if so is a_i.

Then one easily sees that setting $u_i = \lim_{n\to\infty} a_i^{(n)}$, we get an idempotent endomorphism $u_\bullet$ of $A_\bullet$, such that $u_i \equiv 0 \bmod \mathfrak{m}$ ($u_i \equiv 1 \bmod \mathfrak{m}$) if and only if a_i is nilpotent modulo $\mathfrak{m}$ (respectively a_i is invertible).

Especially, if either one of a_l is neither nilpotent nor invertible modulo $\mathfrak{m}$ or one of a_l is nilpotent modulo $\mathfrak{m}$ while another one is invertible, then $u_\bullet$ is neither zero nor identity. Hence the complex $A_\bullet$ decomposes. Thus $A_\bullet$ is indecomposable if and only if, for any endomorphism $a_\bullet$ of $A_\bullet$, either $a_\bullet$ is invertible or all components a_n are nilpotent modulo $\mathfrak{m}$. Since all algebras $\operatorname{End} A_n/\mathfrak{m} \operatorname{End} A_n$ are finite-dimensional, neither product $\alpha\beta$, where $\alpha, \beta \in \operatorname{End} A_n$ and one of them is nilpotent modulo $\mathfrak{m}$, can be invertible. Therefore, the set of endomorphisms $a_\bullet$ of an indecomposable complex $A_\bullet$ such that all components a_n are nilpotent modulo $\mathfrak{m}$ form an ideal R of $\operatorname{End} A_\bullet$ and $\operatorname{End} A_\bullet/R$ is a skew field. Hence $R = \operatorname{rad}(\operatorname{End} A_\bullet)$ and $\operatorname{End} A_\bullet$ is local.

Now we want to show that any complex from $C(\mathfrak{C})$ has an indecomposable direct summand. Consider an arbitrary complex $A_\bullet$ and suppose that $A_0 \neq 0$. For any idempotent endomorphism $e_\bullet$ of $A_\bullet$ at least one of the complexes $e(A_\bullet)$ or $(1 - e)(A_\bullet)$ has a nonzero component at the zero place. On the set of all endomorphisms of $A_\bullet$ we can introduce a partial ordering by writing $e_\bullet \geqslant e'_\bullet$ if and only if $e'_\bullet = e_\bullet e'_\bullet e_\bullet$ and both e_0 and e'_0 are non-zero. Let $e_\bullet \geqslant e'_\bullet \geqslant e''_\bullet \geqslant \cdots$ be a chain of idempotent endomorphisms of $A_\bullet$. As all endomorphism algebras $\operatorname{End} A_l$ are finitely generated $\mathbf{S}$-modules, the sequences $e_l, e'_l, e''_l, \ldots \in \operatorname{End} A_l$ stabilize for all l, so this chain has a lower bound (formed by the limit values of components). By Zorn's lemma, there is a minimal non-zero idempotent of $A_\bullet$, which defines an indecomposable direct summand.

Again, since all $\operatorname{End} A_l$ are finitely generated, for every n there is a decomposition $A_\bullet = B_\bullet^{(n)} \oplus \bigoplus_{i=1}^{r_n} B_{in\bullet}$ where all $B_{in\bullet}$ are indecomposable and $B_l^{(n)} = 0$ for $l \in I_n$. Moreover, one may suppose that $r_n \leqslant r_m$ for $m > n$ and $B_{in\bullet} = B_{im\bullet}$, for $i \leqslant r_n$. Evidently, it implies that $A_\bullet = \bigoplus_{i=1}^{r} B_{i\bullet}$ where $r = \sup_n r_n$ and $B_{i\bullet} = B_{in\bullet}$ for $i \leqslant r_n$, which accomplishes the proof of the Proposition A.2 for $C(\mathfrak{C})$.

Note now that the endomorphism ring of each complex $B_{i\bullet}$ in the category $K(\mathfrak{C})$ is a factor-ring of its endomorphism ring in $C(\mathfrak{C})$. Hence it is either local or zero; in the latter case the image of $B_{i\bullet}$ in $K(\mathfrak{C})$ is a zero object. Therefore, the claim is also valid for $K(\mathfrak{C})$. □

Corollary A.6. *Let* $\mathbf{S}$ *be local, complete and noetherian, and* $\mathbf{A}$ *be an* $\mathbf{S}$*-algebra finitely generated as* $\mathbf{S}$*-module. Then the derived category* $D^-(\mathbf{A}\text{-mod})$, *where* $\mathbf{A}$-mod *denotes the category of finitely generated* $\mathbf{A}$*-modules, is* ω*-local, in particular, Krull–Schmidt category.*

Proof. Indeed, $D^-(\mathbf{A}\text{-mod})$ coincide with the category $K^-(\mathbf{A}\text{-pro})$, where $\mathbf{A}$-pro denotes the category of finitely generated projective $\mathbf{A}$-modules. □

Remark A.7. The conditions of Proposition A.2 are essential indeed, and Krull–Schmidt theorem can fail even for the category of bounded complexes $C^b(\mathfrak{C})$ over a local category $\mathfrak{C}$ as the following example shows.

Let $\mathbf{R}$ be the localization of the polynomial ring $\mathbf{k}[x, y]$ at the maximal ideal (x, y), $\mathfrak{C} = \mathbf{R}$-pro be the category of free $\mathbf{R}$-modules of finite rank. Obviously, it is local. The category $K^-(\mathfrak{C})$ is equivalent to the category $D^-(\mathbf{R}\text{-mod})$ and contains the category $\mathbf{R}$-mod as a full subcategory. Denote by $\mathbf{S}$ the factor-ring $\mathbf{R}/(x^2y - y^3 + x^4)$. It is a local domain, but its completion $\widehat{\mathbf{S}}$ is not a domain: its normalization decomposes as $\mathbf{S}_1 \times \mathbf{S}_2 \times \mathbf{S}_3$, where each $\mathbf{S}_i \cong \mathbf{k}[[x]]$. In particular, $\widehat{\mathbf{S}}$ has three torsion-free modules $\mathbf{L}_i$ such that each $\mathbf{L}_i$ has a composition series with the factors $\mathbf{S}_j, \mathbf{S}_k$, where $\{i, j, k\} = \{1, 2, 3\}$ (it is the projection of $\widehat{\mathbf{S}}$ onto $\mathbf{S}_j \times \mathbf{S}_k$). It implies that $\mathbf{S}$ has torsion-free indecomposable modules M_1, M_2, N_1, N_2, N_3 with the following completions:

$$\widehat{M}_1 = \mathbf{S}_1 \oplus \mathbf{S}_2 \oplus \mathbf{S}_3; \qquad \widehat{M}_2 = L_1 \oplus L_2 \oplus L_3;$$

$$\widehat{N}_i = \mathbf{S}_i \oplus L_i \quad (i = 1, 2, 3).$$

(cf. [29]). Then $M_1 \oplus M_2 \cong N_1 \oplus N_2 \oplus N_3$, hence the category $\mathbf{S}$-mod, thus also $\mathbf{R}$-mod and $D^b(\mathbf{R}\text{-mod})$ are not local.

Appendix B. Bunches of chains

We summarize the results of [5,6], changing both the definition and the presentation of the answer to equivalent ones, which seem more convenient for our purpose. Moreover we add a description of some infinite-dimensional representations that occur in dealing with derived categories, together with a sketch of a proof. As V.M. Bondarenko has informed us, he has submitted a paper containing more details on infinite case [7]. Note that Bondarenko calls "bunch of semichained sets" what we call "bunch of chains". The reason can be seen if one compares our definitions.

Definition B.1. A *bunch of chains* $\mathfrak{X}$ consists of:

- An index set I, which we suppose finite or countable.
- For each $i \in I$, two chains (linearly ordered sets) $\mathfrak{E}_i$ and $\mathfrak{F}_i$.
  We set $\mathfrak{E} = \bigcup_{i \in I} \mathfrak{E}_i$, $\mathfrak{F} = \bigcup_{i \in I} \mathfrak{F}_i$ and $|\mathfrak{X}| = \mathfrak{E} \cup \mathfrak{F}$.
- A symmetric relation $\sim$ (not an equivalence!) on $|\mathfrak{X}|$ such that for every x there is at most one y with $x \sim y$ (maybe $x = y$).

We define an equivalence relation $\approx$ on $|\mathfrak{X}|$ such that $x \approx y$ means either $x = y$ or $x \sim y$, and set $\widetilde{\mathfrak{X}} = |\mathfrak{X}|/\approx$. We write $x - y$ if there is an index $i \in I$ such that $x \in \mathfrak{E}_i$, $y \in \mathfrak{F}_i$ or vice versa. For each $x \in |\mathfrak{X}|$ such that $x \sim x$ we introduce two new elements x', x'' and set $\mathfrak{X}^* = (|\mathfrak{X}| \setminus \{x \mid x \sim x\}) \cup \{x', x'' \mid x \sim x\}$. We subdivide $\mathfrak{X}^*$ into $\mathfrak{E}^* = \bigcup_i \mathfrak{E}_i^*$ and $\mathfrak{F}^* = \bigcup_i \mathfrak{F}_i^*$, which are the images of $\mathfrak{E}_i$ and $\mathfrak{F}_i$; for instance x' and x'' are in $\mathfrak{E}_i^*$ if $x \in \mathfrak{E}_i$. We consider the ordering $<$ on $|\mathfrak{X}|$, which is just the union of orderings on all $\mathfrak{E}_i$ and $\mathfrak{F}_i$, and extend it, as well as the relation $-$, onto $\mathfrak{X}^*$ so that each "new" element x' or x'' inherits all relations that the element x has. For instance, $x' < y$ with $y \in |\mathfrak{X}|$ means that $x < y$; $x'' - z'$ means that $x - z$, etc. Note that the elements x', x'' are always non-comparable. On the other hand, we extend the equivalence $\approx$ to $\mathfrak{X}^*$ trivially (each new element x' or x'' is unique in its $\approx$-class), and set $\widetilde{\mathfrak{X}}^* = \mathfrak{X}^*/\approx$.

A bunch of chains $\mathfrak{X}$ gives rise to a bimodule problem. Namely, we fix a field $\mathbf{k}$ and define a $\mathbf{k}$-category $\mathbf{A} = \mathbf{A}(\mathfrak{X})$ and an $\mathbf{A}$-bimodule $\mathbf{U} = \mathbf{U}(\mathfrak{X})$ as follows:

- $\operatorname{Ob} \mathbf{A} = \widetilde{\mathfrak{X}}^*$.
- If a, b are two equivalence classes, a basis of the morphism space $\mathbf{A}(a, b)$ consists of elements p_{yx} with $x \in a$, $y \in b$, $x < y$ and, if $a = b$, the identity morphism 1_x.
- The multiplication is given by the rule: $p_{zy} p_{yx} = p_{zx}$ if $z < y < x$, while all other possible products are zeros.
- A basis of $\mathbf{U}(a, b)$ consists of elements u_{yx} with $y \in b \cap \mathfrak{E}^*$, $x \in a \cap \mathfrak{F}^*$, $x - y$.
- The action of $\mathbf{A}$ on $\mathbf{U}$ is given by the rule: $p_{zy} u_{yx} = u_{zx}$ if $y < z$; $u_{yx} p_{xt} = u_{yt}$ if $x < t$, while all other possible products are zeros.

The category of *representations of the bunch* $\mathfrak{X}$ over the field $\mathbf{k}$ is then defined as the category $\mathrm{El}(\mathbf{U})$ of the elements of this bimodule. In other words, a representation is a set M of block matrices

$$M_i = \begin{pmatrix} \dots\dots\dots\dots \\ \dots\ M_{xy}\ \dots \\ \dots\dots\dots\dots \end{pmatrix}, \quad i \in \mathbf{I},\ x \in \mathfrak{E}_i^*,\ y \in \mathfrak{F}_i^*,\ M_{xy} \in \mathrm{Mat}(n_x \times n_y, \mathbf{k})$$

such that $x \approx y$ implies $n_x = n_y$. Two representations are isomorphic if and only if they can be obtained from one another by a sequence of the following *elementary transformations*:

- elementary transformations of rows (columns) in each horizontal (vertical) stripe; it means that they are performed simultaneously in all matrices M_{xy} with fixed x (respectively y); moreover, if $x \approx z$, the transformations of the x-stripe must be the same as those of z-stripe (certainly, if one of them is horizontal and the other is vertical, "the same" means "contragradient");
- if $x < y$, then scalar multiples of rows (columns) of the x-stripe can be added to rows (columns) of the y-stripe.

One easily sees that this definition coincides with that of [5,6].

The description of indecomposable representations from [5,6] rests upon a combinatorics, which we expound in terms of *strings and bands* alike to their use in the representation theory.

Definition B.2. Let $\mathfrak{X} = \{I, \mathfrak{E}_i, \mathfrak{F}_i, \sim\}$ be a bunch of chains.

(1) An $\mathfrak{X}$-*word* is a sequence $w = x_1 r_1 x_2 r_2 x_3 \dots r_{m-1} x_m$, where $x_k \in |\mathfrak{X}|$ and $r_k \in \{\sim, -\}$, such that for all possible values of k

(a) $x_k r_k x_{k+1}$ in $|\mathfrak{X}|$.
(b) $r_k \neq r_{k+1}$.

We call m the *length* of the word w. Possibly $m = 1$, i.e., $w = x$ for some $x \in |\mathfrak{X}|$. The elements x_1 and x_m are called the *ends* of the word w.

(2) We call an $\mathfrak{X}$-word *full* if, whenever x_1 is not a unique element in its $\approx$-class, then $r_1 = \sim$, and whenever x_m is not a unique element in its $\approx$-class, then $r_{m-1} = \sim$.

(3) We denote by w^* the *inverse word* $x_m r_{m-1} x_{m-1} \dots r_1 x_1$ and call an $\mathfrak{X}$-word *symmetric* if $w = w^*$. We call w *quasisymmetric* if it can be presented in the form $v \sim v^* \sim v \sim v^* \sim \dots \sim v$ for a shorter word v.

(4) We call the end x_1 (x_m) of the word w *special* if $x_1 \sim x_1$ and $r_1 = -$ (respectively $x_m \sim x_m$ and $r_{m-1} = -$). We call the word w

(1) *usual* if neither of its ends is special;
(2) *special* if one of its ends is special;
(3) *bispecial* if both its ends are special.

Note that a special word is never symmetric, while a bispecial word is always full; a quasisymmetric word is always bispecial.

(5) If $r_1 = r_{m-1} = \sim$ and $x_m - x_1$ in $\mathfrak{X}$, we call the word w an $\mathfrak{X}$-*cycle*. Note that in this case m is always even. For a cycle w we set $r_m = -$ and $x_{qm+k} = x_k$, $r_{qm+k} = r_k$ for all integers q, k.

(6) We call an $\mathfrak{X}$-cycle $w = x_1 r_1 x_2 r_2 x_3 \dots r_{m-1} x_m$ *non-periodic* if the sequence $x_1 r_1 x_2 r_2 \dots x_m r_m$ cannot be written as a multiple $vv \dots v$ of a shorter sequence v.

(7) A *shift* of a cycle w is defined as the cycle $w^{[k]} = x_{k+1} r_{k+1} x_{k+2} \dots r_{k-1} x_k$ for some even integer $0 \leqslant k < m$. We call a non-periodic cycle w *symmetric* if $w^* = w^{[k]}$ for some k. (Note that $w^{[k]} = w^{[l]}$ with $k \neq l$ is impossible if w is non-periodic.)

(8) For a cycle w and an even integer $0 \leqslant k < m$ we define $\nu(k, w)$ as the number of even integers $0 \leqslant i \leqslant k$ such that both x_{i-1} and x_i belong either to $\mathfrak{E}$ or to $\mathfrak{F}$.

Definition B.3. (1) A *usual string* datum is a non-symmetric full usual word.

(2) A *special string* datum is a pair (w, δ), where w is a special full word and $\delta \in \{+, -\}$.

(3) A *bispecial string* datum is a quadruple $(w, m, \delta_1, \delta_2)$, where w is a bispecial word, which is neither symmetric nor quasisymmetric, $m \in \mathbb{N}$ and $\delta_i \in \{+, -\}$.

(4) A *band* datum is a pair (w, f), where w is a non-periodic cycle and $f \in \mathbf{k}[t]$ is a *primary polynomial* over the field $\mathbf{k}$, i.e., a degree of an irreducible polynomial with leading coefficient 1, such that $f(0) \neq 0$ and if w is symmetric also $f(1) \neq 0$. If the field $\mathbf{k}$ is algebraically closed and $f = (t - \lambda)^d$, we write (w, d, λ) instead of (w, f).

(5) The following string data are called *equivalent*:

(a) usual string data w and w^*;
(b) special string data (w, δ) and (w^*, δ);
(c) bispecial string data $(w, m, \delta_1, \delta_2)$ and $(w^*, m, \delta_2, \delta_1)$.

(6) Two band data are called *equivalent* if they can be obtained from one another by a sequence of the following transformations:

(a) replace (w, f) by $(w^{[k]}, f)$ if $\nu(k, w)$ is even;
(b) replace (w, f) by $(w^{[k]}, \alpha^{-1} t^d f(1/t))$, where $d = \deg f$ and $\alpha = f(0)$ if $\nu(k, w)$ is odd;
(c) replace (w, f) by (w^*, f).

Note that if $f(t) = (t - \lambda)^d$, then $\alpha^{-1} t^d f(1/t) = (t - \lambda^{-1})^d$.

Then the main result of the papers [5,6] (see also [11]) can be reformulated as follows.

Theorem B.4. *There is one-to-one correspondence between isomorphism classes of indecomposable representations of a bunch of chains and equivalence classes of string and band data.*

We call indecomposable representations corresponding to usual string data (special string data, bispecial string data, band data) *usual strings* (respectively, *special strings*, *bispecial strings*, *bands*).

Appendix C. Infinite chains

For our purpose we have to consider some infinite representations of a bunch of chains. We suppose now that $I = \mathbb{N}$ and for every index $i \in I$ the set

$$\bigl\{ j \in I \mid (\exists x \in \mathfrak{E}_i \cup \mathfrak{F}_i)\ (\exists y \in \mathfrak{E}_j \cup \mathfrak{F}_j)\ x \sim y \bigr\}$$

is finite. Namely, we define the category $\mathrm{El}^{\infty}(\mathbf{U})$ just in the same way as $\mathrm{El}(\mathbf{U})$, but allowing infinitely many elements of $\mathfrak{X}^*$ to occur in every representation. On the contrary, we always suppose that for every $i \in I$ the sum of all dimensions n_x with $x \in \mathfrak{E}_i^* \cup \mathfrak{F}_i^*$ is finite. The last condition looks indispensable, since even when one considers the simplest case $\#I = 1, \mathfrak{E} = \{x\}, \mathfrak{F} = \{y\}, x \sim y$ (which means square matrices under conjugation), the classification of representations of infinite dimension is a wild problem.

To deal with such infinite representations we first establish a general result concerning infinite matrices over bimodules.

Definition C.1. Let $\mathbf{A}$ be a locally finite-dimensional category, $\mathbf{B}$ be its full subcategory and $\mathbf{U}$ be an $\mathbf{A}$-bimodule. We say that $\mathbf{U}$ is *triangular with respect to* $\mathbf{B}$ if, for every indecomposable objects A, B, C, where $B, C \in \mathbf{B}$ and $A \notin \mathbf{B}$, $\mathbf{A}(C, A)\mathbf{U}(B, C) = \mathbf{U}(B, C)\mathbf{A}(A, B) = 0$.

The following lemma is obvious.

Lemma C.2. *Let* $\mathbf{U}$ *be triangular with respect to* $\mathbf{B}$. *For any object* $A \in \mathbf{A}$ *choose a decomposition* $A \cong A_1 \oplus A_2$, *where* $A_1 \in \mathbf{B}$ *and* A_2 *has no direct summands from* $\mathbf{B}$. *For a morphism* $a \in \mathbf{A}(A, A')$ *or an element* $u \in \mathbf{U}(A, A')$ *denote, respectively, by* a_1 *or* u_1 *its component from* $\mathbf{A}(A_1, A'_1)$ *or* $\mathbf{U}(A_1, A'_1)$. *If* $a \in \mathbf{A}(A, A')$ *is a morphism in* $\mathrm{El}(\mathbf{U})$ *from* $u \in \mathbf{U}(A, A)$ *to* $v \in \mathbf{U}(A', A')$ *(i.e.,* $au = va$*), then* a_1 *is a morphism from* u_1 *to* v_1. *Especially if* a *is an isomorphism* $u \to v$, *then* a_1 *is an isomorphism* $u_1 \to v_1$.

Lemma C.3. *Let* $\mathbf{A}$ *be a locally finite-dimensional category that is a union of a chain* $\mathbf{A}_1 \subseteq \mathbf{A}_2 \subseteq \mathbf{A}_3 \subseteq \cdots$ *of full subcategories. Suppose that* $\mathbf{U}$ *is an* $\mathbf{A}$*-bimodule that is triangular with respect to each* $\mathbf{A}_i$. *Denote by* $\mathbf{A}^{\infty}$ *the category of infinite direct sums* $A = \bigoplus_{i=1}^{\infty} A_i$, *where* A_i *is an object from* $\mathbf{A}_i$ *with no direct summands from* $\mathbf{A}_{i-1}$, *and by* $\mathbf{U}^{\infty}$ *the natural extension of* $\mathbf{U}$ *onto* $\mathbf{A}^{\infty}$. *For each element* u *from* $\mathbf{U}^{\infty}(A, A)$ *denote by* u_n *its restriction onto* $\bigoplus_{i=1}^{n} a_i$. *If* u, v *are two elements such that* $u_n \cong v_n$ *for all* n, *then* $u \cong v$.

Proof. First suppose the field $\mathbf{k}$ *uncountable*. Consider the sets of isomorphisms $\mathrm{Iso}(u_n, v_n)$ and the natural mappings $\pi_{mn} : \mathrm{Iso}(u_m, v_m) \to \mathrm{Iso}(u_n, v_n)$ $(m > n)$ arising from the triangularity condition. These sets can be considered as algebraic (even affine) varieties, then π_{mn} are open morphisms of these varieties. In particular, the images $\operatorname{Im}\pi_{m1}$ form a decreasing chain of non-empty open subsets in $\operatorname{Iso}(u_1, v_1)$. Hence their intersection is also non-empty (cf., for instance, [19]). Take an element a_1 from this intersection and set $X_n = \pi_{n1}^{-1}(a_1)$ $(n \geqslant 2)$. Again they are algebraic varieties and $X'_n = \pi_{n2}(X_n)$ are non-empty open subsets of X_2, thus there is an element $a_2 \in \bigcap_{n=2}^{\infty} X'_n$. Continuing this process, we get a sequence a_n of elements from $\operatorname{Iso}(u_n, v_n)$ such that $\pi_{mn}(a_m) = a_n$ for all $m > n$. This sequence defines an isomorphism $a : u \to v$.

If $\mathbf{k}$ is arbitrary, take its uncountable extension $\tilde{\mathbf{k}}$ and consider extensions of $\mathbf{A}$ and $\mathbf{U}$ to $\tilde{\mathbf{k}}$. It is easy to see that $u \cong v$ if and only if their extensions are isomorphic, which accomplishes the proof. □

Note that using Lemma A.3 one can obtain the following analogue of Proposition A.2 (with almost the same proof).

Proposition C.4. *We use the suppositions and notations of Lemma* C.3. *If* $A = \bigoplus_{i=1}^{\infty} A_i \in \mathbf{A}^{\infty}$ *and* $a \in \operatorname{End} A$, *denote by* a_i *the component of* a *belongings to* $\operatorname{End} A_i$. *The category* $\operatorname{El}(\mathbf{U}^{\infty})$ *is* ω*-local* (*in particular, Krull–Schmidt*). *Moreover, if* $u \in \mathbf{U}^{\infty}(A, A)$ *is an indecomposable element from* $\operatorname{El}(\mathbf{U}^{\infty})$ *and* $a \in \operatorname{End} u$, *then either all* a_i *are invertible or all of them are nilpotent.*

Now we define *infinite* $\mathfrak{X}$*-words* as sequences $w = \dots x_1 r_1 x_2 r_2 x_3 \dots r_{m-1} x_m \dots$, which are one-side or two-side infinite, subject to conditions (a) and (b) of Definition B.2(1) and such that for each i the set $\{k \mid x_k \in \mathfrak{E}_i \cup \mathfrak{F}_i\}$ is finite. We apply to such words all terminology from Definitions B.2(2)–(4) and B.3(1), (2), (5)(a)(b). Then we can extend Theorem B.4 to infinite representations.

Theorem C.5. *Isomorphism classes of indecomposable infinite representations of a bunch of chains are in one-to-one correspondence with equivalence classes of infinite string data. Moreover, every infinite representation uniquely decomposes into a direct sum of indecomposable ones.*

Sketch of the proof (more details will appear in [7]). Let $\mathfrak{X}_m$ be the bunch of chains with the index set $I_m = \{1, 2, \dots, m\}$, the same chains $\mathfrak{E}_i, \mathfrak{F}_i$ and the same relation $\sim$, $\mathbf{A}_m = \mathbf{A}(\mathfrak{X}_m)$. Then we are in the situation of Lemma C.3. We define representations corresponding to infinite string data just as it has been done in [5,6] for finite case. One can show that all of them are indecomposable and their endomorphism rings are local. So we only have to prove that there are no more indecomposable infinite representations.

For each representation $M \in \operatorname{El}^{\infty}(\mathbf{U})$ we denote by M_m the restriction of M onto $\mathfrak{X}_m$, given by all matrices M_{xy} with $x, y \in \mathfrak{X}_m^*$. Lemma C.3 implies that $M \cong N$ if and only if $M_m \cong N_m$ for every m. Suppose that M is infinite and indecomposable and consider an indecomposable direct summand L of a representation M_m. The reduction procedure and the explicit description of strings and bands from [6] immediately imply the following facts.

(1) L cannot be a band or a bispecial string.

(2) If L is a usual or a special string, there is an integer $m' > m$ and an indecomposable direct summand L' of $M_{m'}$ such that the word w from the string datum corresponding to L is a part of the word w' from the string datum corresponding to L'.
(3) If K is another indecomposable direct summand of M_m, the number $m' > m$ and the representation L' from (2) can be chosen common for L and K.

It implies the first statement of the theorem. The Krull–Schmidt property follows from Proposition C.4. $\square$

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