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Matrix problems and stable homotopy types of polyhedra[a]

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Abstract: This is a survey of the results on stable homotopy types of polyhedra of small dimensions, mainly obtained by H.-J. Baues and the author [3, 5, 6]. The proofs are based on the technique of matrix problems (bimodule categories).

Keywords: polyhedra, homotopy type, matrix problems, tame and wild problems
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[a] Dedicated to C. M. Ringel.

1 Introduction

This paper is a survey of some recent results on stable homotopy types of polyhedra. The common feature of these results is that their proofs use the technique of the so called *matrix problems*, which was mainly elaborated within framework of representation theory. I think that this technique is essential in homotopy theory too, and perhaps even in much more general setting of triangulated categories. I hope that the considerations of Section 3 are persuasive enough. Certainly, I could not cover all such results in an expository work, thus I have restricted to the stable homotopy classification of polyhedra of small dimensions obtained in [3, 5, 6, 7]. I tried to present these results in a homogeneous way and also to replace references to rather sophisticated topological sources by simpler ones. The latter mainly concerns with some basic facts about homotopy groups of spheres, which can be found in [18] or [21]. I also used the book [20] as a standard source of references; maybe some readers will prefer [19] or [10]. Most of these references are

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collected in Section 1. For the matrix problems I have chosen the language of *bimodule categories* explained in Section 2, since it seems to be the simplest one as well as the most appropriate for applications.

Note that almost the same arguments that are used in Sections 5 and 6 can be applied to the classification of polyhedra with only 2 non-trivial homology groups [6], while the dual arguments were applied to the spaces with only 2 non-trivial homotopy groups in [4]. Rather similar are also calculations in [17] (see also the Appendix by Baues and Henn to [3]). I hope that any diligent reader of this survey will be able to comprehend the arguments of these papers too.

I am extremely indebted to H.-J. Baues, who was my co-author and my guide to the topological problems, and to C. M. Ringel, whose wonderful organising activity had made such a pleasant and fruitful collaboration possible. H.-J. Baues and I obtained most of our joint results during my visits to the Max-Plank-Institut für Mathematik, and I highly acknowledge its support.

2 Generalities on stable homotopy types

All considered spaces are supposed *pathwise connected* and *punctured*; we denote by $*_X$ (or by $*$ if there can be no ambiguity) the marked point of the space X. B^n and S^{n-1} denote respectively the n-dimensional *ball* $\{\, \mathbf{x} \in \mathbb{R}^n \mid ||\mathbf{x}|| \le 1 \,\}$ and the $(n-1)$-dimensional *sphere* $\{\, \mathbf{x} \in \mathbb{R}^n \mid ||\mathbf{x}|| = 1 \,\}$, both with the marked point $(1, 0, \dots, 0)$. As usually, we denote by $X \vee Y$ the *bouquet* (or one point union) of X and Y, i.e. the factor space $X \sqcup Y$ by the relation $*_X = *_Y$, and identify it with $*_X \times Y \cup X \times *_Y \subset X \times Y$; we denote by $X \wedge Y$ the factor space $X \times Y / X \vee Y$. In particular, we denote by $\Sigma X = S^1 \wedge X$ the *suspension* of X and by $\Sigma^n X = \underbrace{\Sigma \dots \Sigma}_{n \text{ times}} X$ its n-th suspension. The word "*polyhedron*" is used as a synonym of "*finite CW-complex*." One can also consider bouquets of several spaces $\bigvee_{i=1}^{s} X_i$; if all of them are copies of a fixed space X, we denote such a bouquet by sX.

We recall several facts on stable homotopy category of CW-complexes. We denote by $\mathrm{Hot}(X, Y)$ the set of homotopy classes of continuous maps $X \to Y$ and by CW the *homotopy category* of polyhedra, i.e. the category whose objects are polyhedra and morphisms are homotopy classes of continuous maps. The suspension functor defines a natural map $\mathrm{Hot}(X, Y) \to \mathrm{Hot}(\Sigma X, \Sigma Y)$. Moreover, the Whitehead theorem [20, Theorem 10.28 and Corollary 10.29] shows that the suspension functor *reflects isomorphisms* of simply connected polyhedra. It means that if $f \in \mathrm{Hot}(X, Y)$, where X and Y are simply connected, f is an isomorphism (i.e. a homotopy equivalence) if and only if so is Σf. We set $\mathrm{Hos}(X, Y) = \varinjlim_n \mathrm{Hot}(\Sigma^n X, \Sigma^n Y)$. If $\alpha \in \mathrm{Hot}(\Sigma^n X, \Sigma^n Y)$, $\beta \in \mathrm{Hot}(\Sigma^m Y, \Sigma^m Z)$, one can consider the class $\Sigma^n \beta \circ \Sigma^m \alpha \in \mathrm{Hot}(\Sigma^{m+n} X, \Sigma^{n+m} Z)$, whose stabilization is, by definition, the product $\beta\alpha$ of the classes of α and β in $\mathrm{Hos}(X, Z)$. Thus we obtain the *stable homotopy category* of polyhedra CWS. Actually, if we only deal with *finite CW-complexes*, we need not go too far, since the Freudenthal theorem [20, Theorem 6.26] implies the following fact.

Proposition 2.1. If X, Y are of dimensions at most d and $(n-1)$-connected, where $d < 2n-1$, then the map $\mathrm{Hot}(X, Y) \to \mathrm{Hot}(\Sigma X, \Sigma Y)$ is bijective. If $d = 2n-1$, this map is surjective. In particular, the map $\mathrm{Hot}(\Sigma^m X, \Sigma^m Y) \to \mathrm{Hos}(X, Y)$ is bijective if $m > d - 2n + 1$ and surjective if $m = d - 2n + 1$.

Here $(n-1)$-*connected* means, as usually, that $\pi_k(X)$, the k-th homotopy group of X, is trivial for $k \leq n-1$. Thus for *all* polyhedra of dimension at most d the map $\mathrm{Hot}(\Sigma^m X, \Sigma^m Y) \to \mathrm{Hos}(X, Y)$ is bijective if $m \geq d$ and surjective if $m = d-1$.

Note also that the natural functor $\mathsf{CW} \to \mathsf{CWS}$ reflects isomorphisms of simply connected polyhedra.

Since we are only interested in stable homotopy classification, we identify, in what follows, polyhedra and continuous maps with their images in CWS. We denote by CWF the full subcategory of CWS consisting of all spaces X with torsion free homology groups $\mathrm{H}_i(X) = \mathrm{H}_i(X, \mathbb{Z})$ for all i.

Recall that any suspension $\Sigma^n X$ is an *H-cogroup* [20, Chapter 2], commutative if $n \geq 2$, so the category CWS is an additive category. Moreover, one can deduce from the Adams' theorem [20, Theorem 9.21] that this category is actually *fully additive*, i.e. every idempotent $e \in \mathrm{Hos}(X, X)$ splits. In our case it means that there is a decomposition $\Sigma^m X \simeq Y \vee Z$ for some m, such that e comes from the map $\varepsilon : Y \vee Z \to Y \vee Z$ with $\varepsilon(y) = y$ for $y \in Y$ and $\varepsilon(z) = *_{Y \vee Z}$ for $z \in Z$. We call a polyhedron X *indecomposable* if $X \simeq Y \vee Z$ implies that either Y or Z are contractible (i.e. isomorphic in CW to the 1-point space). Actually, the category CWS is a *triangulated category* [16]. The suspension plays the role of shift, while the *triangles* are the *cone sequences* $X \xrightarrow{f} Y \to Cf \to \Sigma X$ (and isomorphic ones), where $Cf = CX \cup_f Y$ is the *cone of the map* f, i.e the factor space $CX \sqcup Y$ by the relation $(x, 0) \sim f(x)$; $CX = X \times I / X \times 1$ is the cone over the space X. Note that cone sequences coincide with *cofibration sequences* in the category CWS [20, Proposition 8.30]. Recall that a cofibration sequence is a such one

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{\Sigma f} \Sigma Y \tag{1}$$

that for every polyhedron P the induced sequences

$$\begin{gathered}\mathrm{Hos}(P, X) \xrightarrow{f_*} \mathrm{Hos}(P, Y) \xrightarrow{g_*} \mathrm{Hos}(P, Z) \xrightarrow{h_*} \mathrm{Hos}(P, \Sigma X) \xrightarrow{\Sigma f_*} \mathrm{Hos}(P, \Sigma Y), \\ \mathrm{Hos}(\Sigma Y, P) \xrightarrow{\Sigma f^*} \mathrm{Hos}(\Sigma X, P) \xrightarrow{h^*} \mathrm{Hos}(Z, P) \xrightarrow{g^*} \mathrm{Hos}(Y, P) \xrightarrow{f^*} \mathrm{Hos}(X, P)\end{gathered} \tag{2}$$

are exact. In particular, we have an exact sequence of *stable homotopy groups*

$$\pi_k^S(X) \xrightarrow{f_*} \pi_k^S(Y) \xrightarrow{g_*} \pi_k^S(Z) \xrightarrow{h_*} \pi_{k-1}^S(X) \xrightarrow{\Sigma f_*} \pi_{k-1}^S(Y), \tag{3}$$

where $\pi_k^S(X) = \varinjlim_m \pi_{k+m}(\Sigma^m X) = \mathrm{Hos}(S^k, X)$. Certainly, one can prolong the sequences (2) and (3) into infinite exact sequences just taking further suspensions.

Every CW-complex is obtained by *attaching cells*. Namely, if X^n is the n-th skeleton of X, then there is a bouquet of balls $B = mB^{n+1}$ and a map $f : mS^n \to X^n$ such that X^{n+1} is isomorphic to the cone of f, i.e. to the space $X^n \cup_f B$. It gives cofibration sequences like (1) and exact sequences like (2) and (3).

We denote by CW_n^k the full subcategory of CW formed by $(n-1)$-connected $(n+k)$-dimensional polyhedra and by CWF_n^k the full subcategory of CW_n^k formed by the polyhedra X with torsion free homology groups $\mathrm{H}_i(X)$ for all i. Proposition 2.1 together with the fact that every map of CW-complexes is homotopic to a cell map, also implies the following result.

Proposition 2.2. The suspension functor Σ induces equivalences $\mathsf{CW}_n^k \to \mathsf{CW}_{n+1}^k$ for all $n > k+1$. Moreover, if $n = k+1$, the suspension functor $\Sigma : \mathsf{CW}_n^k \to \mathsf{CW}_{n+1}^k$ is a *full representation equivalence*, i.e. it is full, dense and reflects isomorphisms. (*Dense* means that every object from CW_{n+1}^k is isomorphic (i.e. homotopy equivalent) to ΣX for some $X \in \mathsf{CW}_n^k$.)

Therefore, setting $\mathsf{CW}^k = \mathsf{CW}_{k+2}^k \simeq \mathsf{CW}_n^k$ for $n > k+1$, we can consider it as a full subcategory of CWS. The same is valid for $\mathsf{CWF}^k = \mathsf{CWF}_{k+2}^k$. Note also that CW_n^k naturally embeds into CW_{n+1}^k. It leads to the following notion [2].

Definition 2.3. An *atom* is an indecomposable polyhedron $X \in \mathsf{CW}_{k+1}^k$ not belonging to the image of CW_k^k. A *suspended atom* is a polyhedron $\Sigma^m X$, where X is an atom.

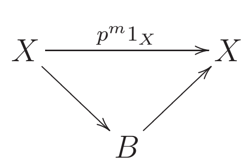
Then we have an obvious corollary.

Corollary 2.4. Every object from CW_n^k with $n \geq k+1$ is isomorphic (i.e. homotopy equivalent) to a bouquet $\bigvee_{i=1}^s X_i$, where X_i are suspended atoms. Moreover, any suspended atom is indecomposable (thus indecomposable objects are just suspended atoms).

Note that the decomposition in Corollary 2.4 is, in general, not unique [14]. That is why an important question is the structure of the *Grothendieck group* $K_0(\mathsf{CW}^k)$. By definition, it is the group generated by the isomorphism classes $[X]$ of polyhedra from CW^k subject to the relations $[X \vee Y] = [X] + [Y]$ for all possible X, Y. The following results of Freyd [14, 10] describe the structure of this group.

Definition 2.5. (1) Two polyhedra $X, Y \in \mathsf{CW}^k$ are said to be *congruent* if there is a polyhedron $Z \in \mathsf{CW}^k$ such that $X \vee Z \simeq Y \vee Z$ (in CW^k).

(2) A polyhedron $X \in \mathsf{CW}^k$ is said to be *p-primary* for some prime number p if there is a bouquet of spheres B such that the map $p^m 1_X : X \to X$ can be factored through B, i.e. there is a commutative diagram



Theorem 2.6 (Freyd). The group $K_0(\mathsf{CW}^k)$ (respectively $K_0(\mathsf{CWF}^k)$) is a free abelian group with a basis formed by the congruence classes of p-primary suspended atoms from CW^k (respectively from CWF^k) for all prime numbers $p \in \mathbb{N}$.

Therefore, if we know the "place" of every atom class $[X]$ in $K_0(\mathsf{CW}^k)$ or $K_0(\mathsf{CWF}^k)$, i.e. its presentation as a linear combination of classes of p-primary suspended atoms, we can deduce therefrom all decomposition rules for CW^k or CWF^k.

3 Bimodule categories

We also recall main notions concerning *bimodule categories* [11, 13]. Let $\mathbf{A}, \mathbf{B}$ be two fully additive categories. An $\mathbf{A}$-$\mathbf{B}$-*bimodule* is, by definition, a biadditive bifunctor $\mathsf{U} : \mathbf{A}^\circ \times \mathbf{B} \to \mathbf{Ab}$. As usual, given an element $u \in \mathsf{U}(A,B)$ and morphisms $\alpha \in \mathbf{A}(A',A)$, $\beta \in \mathbf{B}(B,B')$, we write $\beta u \alpha$ instead of $\mathsf{U}(\alpha,\beta)u$. Given such a functor, we define the *bimodule category* $\mathbf{El}(\mathsf{U})$ (or the category of *elements of the bimodule* U, or the category of *matrices over* U) as follows.

- The set of *objects* of $\mathbf{El}(\mathsf{U})$ is the disjoint union

$$\operatorname{ob}\mathbf{El}(\mathsf{U}) = \bigsqcup_{\substack{A \in \operatorname{ob}\mathbf{A} \\ B \in \operatorname{ob}\mathbf{B}}} \mathsf{U}(A,B).$$

- A *morphism* from $u \in \mathsf{U}(A,B)$ to $u' \in \mathsf{U}(A',B')$ is a pair (α,β) of morphisms $\alpha \in \mathbf{A}(A,A')$, $\beta \in \mathbf{B}(B,B')$ such that $u'\alpha = \beta u$ in $\mathsf{U}(A,B')$.
- The product $(\alpha',\beta')(\alpha,\beta)$ is defined as the pair $(\alpha'\alpha,\beta'\beta)$.

Obviously, $\mathbf{El}(\mathsf{U})$ is again a fully additive category.

Suppose that $\operatorname{ob}\mathbf{A} \supset \{ A_1, A_2, \dots, A_n \}$, $\operatorname{ob}\mathbf{B} \supset \{ B_1, B_2, \dots, B_m \}$ such that every object $A \in \operatorname{ob}\mathbf{A}$ $(B \in \operatorname{ob}\mathbf{B})$ decomposes as $A \simeq \bigoplus_{i=1}^n k_i A_i$ (respectively, $B \simeq \bigoplus_{i=1}^m l_i B_i$). Then $\mathbf{A}^\circ$ (respectively, $\mathbf{B}$) is equivalent to the category of finitely generated projective right (left) modules over the ring of matrices $(a_{ij})_{n\times n}$ with $a_{ij} \in \mathbf{A}(A_j,A_i)$ (respectively, $(b_{ij})_{m\times m}$ with $b_{ij} \in \mathbf{B}(B_j,B_i)$). We denote these rings respectively by $|\mathbf{A}|$ and $|\mathbf{B}|$. We also denote by $|\mathsf{U}|$ the $|\mathbf{A}|$-$|\mathbf{B}|$-bimodule consisting of matrices $(u_{ij})_{m\times n}$, where $u_{ij} \in \mathsf{U}(A_j,B_i)$. Then $\mathsf{U}(A,B)$, where A,B are, respectively, a projective right $|\mathbf{A}|$-module and a projective left $|\mathbf{B}|$-module, can be identified with $A \otimes_{|\mathbf{A}|} |\mathsf{U}| \otimes_{|\mathbf{B}|} B$. Elements from this set are usually considered as block matrices $(U_{ij})_{m\times n}$, where the block U_{ij} is of size $l_i \times k_j$ with entries from $\mathsf{U}(A_j,B_i)$. To form a direct sum of such elements, one has to write direct sums of the corresponding blocks at each place. Certainly, some of these blocks can be "empty," if $k_j = 0$ or $l_i = 0$. An empty block is indecomposable if and only if it is of size 0×1 (in $\mathsf{U}(A_j,0)$) or 1×0 (in $\mathsf{U}(0,B_i)$); we denote it respectively by $\emptyset^j$ or by $\emptyset_i$.

In many cases the rings $|\mathbf{A}|$ and $|\mathbf{B}|$ can be identified with *tiled subrings* of rings of integer matrices. Here a *tiled subring* in $\operatorname{Mat}(n,\mathbb{Z})$ is given by an integer matrix $(d_{ij})_{n\times n}$ such that $d_{ii} = 1$ and $d_{ik} | d_{ij} d_{jk}$ for all i,j,k; the corresponding ring consists of all matrices (a_{ij}) such that $d_{ij} | a_{ij}$ for all i,j (especially $a_{ij} = 0$ if $d_{ij} = 0$).

Example 3.1. Let $\mathbf{A} \subset \mathrm{Mat}(2,\mathbb{Z})$ be the tiled ring given by the matrix

$$\begin{pmatrix} 1 & 12 \\ 0 & 1 \end{pmatrix},$$

U be the set of 2×2-matrices (u_{ij}) with $u_{ij} \in \mathbb{Z}/24$ if $i=1, j=2$, $u_{ij} \in \mathbb{Z}/2$ otherwise. We define U as an $\mathbf{A}$-$\mathbf{A}$-bimodule setting

$$\begin{pmatrix} a & 12b \\ 0 & c \end{pmatrix} \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix} = \begin{pmatrix} au_1 + bu_3 & au_2 + 12bu_4 \\ cu_3 & cu_4 \end{pmatrix};$$
$$\begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix} \begin{pmatrix} a & 12b \\ 0 & c \end{pmatrix} = \begin{pmatrix} au_1 & cu_2 + 12bu_1 \\ au_3 & cu_4 + bu_3 \end{pmatrix}.$$

If we need to indicate this action, we write

$$\begin{pmatrix} 1 & 12^* \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \mathbb{Z}/2 & \mathbb{Z}/24 \\ \mathbb{Z}/2^* & \mathbb{Z}/2 \end{pmatrix}$$

for the matrix defining the ring $\mathbf{A}$ and for the bimodule U. Thus the multiplications of the elements marked by stars is given by the $*$-*rule*:

$$(12a^*) \cdot (u \bmod 2^*) = au \bmod 2. \tag{4}$$

Example 3.2. In the classification of torsion free atoms below the following bimodule plays the crucial role. We consider the tiled rings $\mathbf{A}_2 \subset \mathrm{Mat}(2,\mathbb{Z})$ and $\mathbf{B}_2 \subset \mathrm{Mat}(7,\mathbb{Z})$ given respectively by the matrices

$$\begin{pmatrix} 1 & 12^* \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 2 & 2 & 12 & 24 & 12 & 24 \\ 1 & 1 & 1 & 12 & 24 & 6 & 24 \\ 1 & 2 & 1 & 12 & 24 & 12 & 24 \\ 0 & 0 & 0 & 1 & 2 & 12^* & 12 \\ 0 & 0 & 0 & 1 & 1 & 12 & 6 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The $\mathbf{A}_2$-$\mathbf{B}_2$-bimodule U_2 is defined as the set of matrices of the form

$$\begin{pmatrix} \mathbb{Z}/24 & 0 \\ \mathbb{Z}/12 & 0 \\ \mathbb{Z}/12 & 0 \\ \mathbb{Z}/2 & \mathbb{Z}/24 \\ 0 & \mathbb{Z}/12 \\ \mathbb{Z}/2^* & \mathbb{Z}/2 \\ 0 & \mathbb{Z}/2 \end{pmatrix}.$$

The multiplication in U_2 is given by the natural matrix multiplication, but taking into account the $*$-rule (4).

We shall use the following description of indecomposable elements in $\mathbf{El}(\mathsf{U}_2)$. Set $I_1 = \{1,2,3,4,6\}$, $I_2 = \{4,5,6,7\}$, $V = \{v \in \mathbb{N} \mid 1 \le v \le 6\}$, $V_1 = \{v \in \mathbb{N} \mid 1 \le v \le 12\}$, $V_2 = \{1,2,3\}$.

Theorem 3.3. A complete list $\mathcal{L}_2$ of non-isomorphic indecomposable objects from $\mathbf{El}(\mathsf{U}_2)$ consists of

- empty objects $\emptyset^j$ $(j = 1,2)$ and $\emptyset_i$ $(1 \le i \le 7)$;
- objects $v_i^j \in \mathsf{U}(A_j, B_i)$ ($j = 1,2$; $i \in I_j$; $v \in V_1$ if $i = 1$; $v = 1$ if $i = 6,7$ or $(ij) = (14)$; $v \in V$ otherwise);
- objects $v_{il}^j = \begin{pmatrix} v_i^j \\ 1_l^j \end{pmatrix}$ ($j = 1,2$; $i = 1,2,3$, $l = 4,6$ if $j = 1$; $i = 4,5$, $l = 6,7$ if $j = 2$; if $(il) = (26)$ or (57) then $v \in V_2$; otherwise $v \in V$);
- objects $v_{44} = (1_4^1\ v_4^2)$ with $v \in V$;
- objects $v_{4l} = \begin{pmatrix} 1_4^1 & v_4^2 \\ 0 & 1_l^2 \end{pmatrix}$ with $l = 6,7$ and $v \in V$;
- objects $v_i w_{44} = \begin{pmatrix} v_i^1 & 0 \\ 1_4^1 & w_4^2 \end{pmatrix}$ with $i = 1,2,3$ and $v, w \in V$;
- objects $v_i w_{4l} = \begin{pmatrix} v_i^1 & 0 \\ 1_4^1 & w_4^2 \\ 0 & 1_l^2 \end{pmatrix}$ with $i = 1,2,3$, $l = 6,7$ and $v, w \in V$.

Here the indices define the block containing the corresponding element.

Proof. Decompose U into 2-primary and 3-primary parts. Since for every two matrices $M_2, M_3 \in \mathrm{GL}(n,\mathbb{Z})$ there is a matrix $M \in \mathrm{GL}(n,\mathbb{Z})$ such that $M \equiv M_2 \bmod 2$ and

$M \equiv M_3 \bmod 3$, we can consider the 2-primary part and the 3-primary part separately. Note that in the 3-primary part the blocks u_4^1, u_6^1, u_6^2 and u_7^2 vanish, while the other non-zero blocks of $u \in \mathrm{ob}(\mathsf{U}_2)$ are with entries from $\mathbb{Z}/3$ and there are no restrictions on elementary transformation of the matrix u. Thus every element in the 3-primary part is a direct sum of elements 1_i^j with $j = 1, i = 1, 2, 3$ or $j = 2, i = 4, 5$.

For elements u, u' of the 2-primary part write $u < u'$ if $u' = ua$ for some non-invertible $a \in \mathbf{A}_2$. Then we have the following relations:

$$\begin{aligned}&1_1^1 < 1_3^1 < 1_2^1 < 2_1^1 < 2_3^1 < 2_2^1 < 4_1^1,\\ &1_6^1 < 1_4^1 < 4_1^1 \ \text{ and } \ 1_6^1 < 2_2^1;\\ &1_4^2 < 1_5^2 < 2_4^2 < 2_5^2 < 4_4^2,\\ &1_7^2 < 1_6^2 < 4_4^2 \ \text{ and } \ 1_7^2 < 2_5^2.\end{aligned}$$

Using them, one can easily decompose the parts

$$\tilde{u}^1 = \begin{pmatrix} u_1^1 \\ u_2^1 \\ u_3^1 \end{pmatrix} \quad \text{and} \quad \tilde{u}^2 = \begin{pmatrix} u_4^2 \\ u_5^2 \end{pmatrix}$$

into a direct sum of empty and 1×1 matrices. Now we obtain a column splitting of the remaining matrices, and with respect to the transformation that do not change $\tilde{u}^1$ and $\tilde{u}^2$, these columns are linearly ordered. Therefore, we can also split them into empty and 1×1 blocks. Together with $\tilde{u}^1$ and $\tilde{u}^2$, it splits the whole matrix u into a direct sum of matrices of the forms from the list $\mathcal{L}_2$, where v, w are powers of 2. Adding 3-primary parts, we get the result.

Example 3.4. Consider the idempotents $e = \sum_{i \in I_1} e_{ii} \in \mathbf{A}_2$ and $e' = e_{11} \in \mathbf{B}$. Set $\mathbf{A}_1 = e\mathbf{A}e$, $\mathbf{B}_1 = e'\mathbf{B}_2 e' \simeq \mathbb{Z}$ and $\mathsf{U}_1 = e'\mathsf{U}_2 e$. Then U_1 is an $\mathbf{A}_1$-$\mathbf{B}_1$-bimodule; elements from $\mathbf{El}(\mathsf{U}_1)$ can be identified with those from $\mathbf{El}(\mathsf{U}_2)$ having no second column and fifth row. Hence we get the following result.

Corollary 3.5. A complete list $\mathcal{L}_1$ of non-isomorphic indecomposable objects from $\mathbf{El}(\mathsf{U}_1)$ consists of

- empty objects $\emptyset_i$ $(i \in I_1)$;
- objects v_i $(i \in I_1;\ v \in V_1$ if $i = 1,\ v \in V$ if $i = 2, 3,\ v = 1$ if $i = 4, 6)$;
- objects $v_{il} = \begin{pmatrix} v_i \\ 1_l \end{pmatrix}$ $(i = 1, 2, 3,\ l = 4, 6$; if $(il) = (26)$ then $v \in V_2$, otherwise $v \in V)$.

Here the indices show the blocks where the corresponding elements are placed.

4 Bimodules and homotopy types

Bimodule categories arise in the following situation. Let $\mathbf{A}$ and $\mathbf{B}$ be two fully additive subcategories of the category $\mathbf{Hos}$. We denote by $\mathbf{A} \dagger \mathbf{B}$ the full subcategory of $\mathbf{Hos}$

consisting of all objects X isomorphic (in **Hos**) to the cones of morphisms $f : A \to B$ with $A \in \mathbf{A}$, $B \in \mathbf{B}$, or, the same, such that there is a cofibration sequence

$$A \xrightarrow{f} B \xrightarrow{g} X \xrightarrow{h} \Sigma A, \tag{5}$$

where $A \in \mathbf{A}$, $B \in \mathbf{B}$. Consider the **A**-**B**-bimodule H, which is the restriction on $\mathbf{A}^\circ \times \mathbf{B}$ of the "regular" **Hos**-**Hos**-bimodule Hos. If $f \in \mathrm{Hos}(A, B)$ is an element of H, it gives rise to an exact sequence like (5) with $X = Cf$. Moreover, since this sequence is a cofibration one, for every morphism $(\alpha, \beta) : f \to f'$, where $f' \in \mathbf{Hos}(A', B')$, there is a morphism $\gamma : X \to X'$, where $X' = Cf'$, such that the diagram

$$\begin{array}{ccccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & X & \xrightarrow{h} & \Sigma A & \xrightarrow{\Sigma f} & \Sigma B \\ {\scriptstyle\alpha}\downarrow & & {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\Sigma\alpha} & & \downarrow{\scriptstyle\Sigma\beta} \\ A' & \xrightarrow[f']{} & B' & \xrightarrow[g']{} & X' & \xrightarrow[h']{} & \Sigma A' & \xrightarrow[\Sigma f']{} & \Sigma B' \end{array} \tag{6}$$

commutes. In what follows we suppose that the categories **A** and **B** satisfy the following condition:

$$\mathrm{Hos}(B, \Sigma A) = 0 \quad \text{for all} \quad A \in \mathbf{A},\ B \in \mathbf{B}. \tag{7}$$

In this situation, given a morphism $\gamma : X \to X'$, we have that $h'\gamma g = 0$, hence $\gamma g = g'\beta$ for some $\beta : B \to B'$. Moreover, since the sequence

$$B \xrightarrow{g} X \xrightarrow{h} \Sigma A \xrightarrow{\Sigma f} \Sigma B$$

is cofibration as well, and $\Sigma : \mathbf{Hos}(A, B) \to \mathbf{Hos}(\Sigma A, \Sigma B)$ is a bijection, there is a morphism $\alpha : A \to A'$, which makes the diagram (6) commutative.

Note that neither γ is uniquely determined by (α, β), nor (α, β) is uniquely restored from γ. Nevertheless, we can control this non-uniqueness. Namely, if both γ and γ' fit the diagram (6) for given (α, β), their difference $\overline{\gamma} = \gamma - \gamma'$ fits an analogous diagram with $\alpha = \beta = 0$. The equality $\overline{\gamma} g = 0$ implies that $\overline{\gamma} = \sigma h$ for some $\sigma : \Sigma A \to X'$, and the equality $h'\overline{\gamma} = 0$ implies that $\overline{\gamma} = g'\tau$ for some $\tau : X \to B$. On the contrary, if $\bar{\gamma} = \sigma\sigma' = \tau'\tau$ for *some* morphisms

$$X \xrightarrow{\sigma'} \Sigma Y \xrightarrow{\sigma} X' \quad \text{and} \quad X \xrightarrow{\tau} Z \xrightarrow{\tau'} X',$$

where $Y \in \mathbf{A}$, $Z \in \mathbf{B}$, the condition (7) implies that $\overline{\gamma} g = h'\overline{\gamma} = 0$, so $\overline{\gamma}$ fits the diagram (6) with $\alpha = \beta = 0$.

Fix now γ, and let both (α, β) and (α', β') fit (6) for this choice of γ. Then the pair $(\overline{\alpha}, \overline{\beta})$, where $\overline{\alpha} = \alpha - \alpha'$, $\overline{\beta} = \beta - \beta'$, fits (6) for $\gamma = 0$. The equality $g'\overline{\beta} = 0$ implies that $\overline{\beta} = f'\sigma$ for some $\sigma : B \to A'$, and the equality $(\Sigma\overline{\alpha})h = 0$ implies that $\Sigma\overline{\alpha} = \Sigma\tau\Sigma f$, or $\overline{\alpha} = \tau f$ for some $\tau : B \to A'$. On the contrary, if $(\overline{\alpha}, \overline{\beta}) : f \to f'$ is such that $\overline{\beta} = f'\sigma$ and $\overline{\alpha} = \tau f$ with $\sigma, \tau : B \to A'$, then $g'\overline{\beta} = (\Sigma\overline{\alpha})h = 0$, hence this pair fits (6) with $\gamma = 0$.

Summarizing these considerations, we get the following statement.

Theorem 4.1. Let $\mathbf{A}, \mathbf{B}$ be fully additive subcategories of **Hos** satisfying the condition (7), $\mathbf{A} \dagger \mathbf{B}$ be the full subcategory of **Hos** consisting of all spaces such that there is a cofibration (5) with $A \in \mathbf{A}$, $B \in \mathbf{B}$. Denote by H the bimodule Hos considered as **A**-**B**-bimodule, by $\mathcal{I}$ the ideal in $\mathbf{A} \dagger \mathbf{B}$ consisting of all morphisms $\gamma : X \to X'$ that factor both through an object from $\Sigma\mathbf{A}$ and through an object from $\mathbf{B}$, and by $\mathcal{J}$ the ideal in $\mathbf{El}(\mathsf{H})$ consisting of all morphisms $(\alpha, \beta) : f \to f'$ such that β factors through f' and α factors through f. Then the factor categories $\mathbf{El}(\mathsf{H})/\mathcal{J}$ and $\mathbf{A} \dagger \mathbf{B}/\mathcal{I}$ are equivalent; an equivalence is induced by the maps $f \mapsto Cf$ and $(\alpha, \beta) \mapsto \gamma$, where γ fits a commutative diagram (6). Moreover, $\mathcal{I}^2 = 0$, thus the functor $\mathbf{A} \dagger \mathbf{B} \to \mathbf{A} \dagger \mathbf{B}/\mathcal{I}$ reflects isomorphisms.

Proof. We only have to check the last statement. But if $\gamma : X \to X'$ factors as $X \xrightarrow{\tau} B' \xrightarrow{g'} X'$ and $\gamma' : X' \to X''$ factors as $X' \xrightarrow{h'} \Sigma A \xrightarrow{\sigma} X''$, where $A \in \mathbf{A}$, $B \in \mathbf{B}$, then $\gamma'\gamma = 0$, since $h'g : B \to \Sigma A$ and $\mathrm{Hos}(B, \Sigma A) = 0$.

Corollary 4.2. In the situation of Theorem 4.1, suppose that $\mathrm{Hos}(B, A) = 0$ for each $A \in \mathbf{A}$, $B \in \mathbf{B}$. Then $\mathbf{El}(\mathsf{H}) \simeq \mathbf{A} \dagger \mathbf{B}/\mathcal{I}$. Moreover, the functor $\mathbf{A} \dagger \mathbf{B} \to \mathbf{El}(\mathsf{H})$ is a *representation equivalence*, i.e. it is dense, preserves indecomposable and reflects isomorphisms.

Note also that any *isomorphism* $f : A \xrightarrow{\sim} B$ is a zero object in $\mathbf{El}(\mathsf{H})/\mathcal{J}$, since its identity map $(1_A, 1_B)$ can be presented as $(f^{-1}f, ff^{-1})$. Obviously, the corresponding object from $\mathbf{A} \dagger \mathbf{B}$ is zero (i.e. contractible) too.

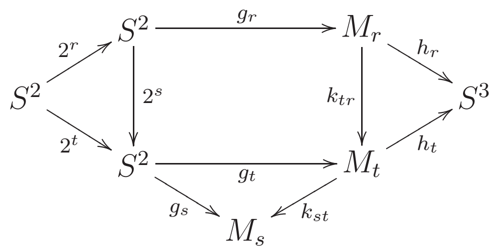
5 Small dimensions

We now use Theorem 4.1 to describe stable homotopy types of atoms of dimensions at most 5, or, the same, indecomposable objects in the categories CW_2^1 and CW_3^2.

Example 5.1. It is well known that $\pi_n(S^n) = \mathbb{Z}$ (freely generated by the identity map). It allows easily to describe atoms in CW_2^1. Such an atom X is (stably!) of the form Cf for some map $f : mS^2 \to nS^2$. Since $\mathrm{Hos}(S^n, S^{n+1}) = 0$, Theorem 4.1 can be applied. The map f is given by an integer matrix. Using automorphisms of mS^2 and nS^2, we can transform it to a diagonal form. Hence, indecomposable gluings can only be if $m = n = 1$; thus $f = q1_{S^2}$. One can see that such a gluing is indecomposable if and only if q is a power of a prime number. The corresponding atom $S^2 \cup_q B^3$ will be denoted by $M(q)$ and called *Moore atom*. It occurs in a cofibration sequence

$$S^2 \xrightarrow{q} S^2 \xrightarrow{g(q)} M(q) \xrightarrow{h(q)} S^3 \xrightarrow{q} S^3.$$

For the next section we need more information about 2-primary Moore atoms. We denote $M_t = M(2^t)$ and write g_t, h_t instead of $g(2^t), h(2^t)$. These atoms can be included into the following commutative "octahedral" diagram [16], where $t = r + s$:



(8)

Moreover, in this diagram $h_s k_{st} = 2^r h_t$.

The exact sequence (3) is here of the form

$$\pi_k^S(S^2) \xrightarrow{q} \pi_k^S(S^2) \longrightarrow \pi_k^S(M(q)) \longrightarrow \pi_k^S(S^3) \xrightarrow{q} \pi_k^S(S^3),$$

which gives the values of stable homotopy groups of the spaces $M(q)$ shown in Table 1 below. (By the way, this table implies that all Moore atoms are pairwise non-isomorphic.)

k	2	3	4
$\pi_k^S(M(q)),\ q$ odd	$\mathbb{Z}/q$	0	0
$\pi_k^S(M_t),\ t > 1$	$\mathbb{Z}/q$	$\mathbb{Z}/2$	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$
$\pi_k^S(M_1)$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/4$

Table 1

Actually, the only non-trivial case is the group $\pi_4^S(M_1)$. It can be obtained as $\pi_6(\Sigma^2 M_1)$, which is isomorphic to the 2-primary component of $\pi_6(S^3) = \mathbb{Z}/12$ (cf. [18, Lemma XI.10.2]). To prove that the sequence

$$0 \longrightarrow \pi_4^S(S^2) = \mathbb{Z}/2 \longrightarrow \pi_4^S(M_t) \longrightarrow \pi_4^S(S^3) = \mathbb{Z}/2 \longrightarrow 0$$

splits if $t > 1$, it is enough to consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_4^S(S^2) & \longrightarrow & \pi_4^S(M_1) & \longrightarrow & \pi_4^S(S_3) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 0} & & \downarrow & & \| & & \\ 0 & \longrightarrow & \pi_4^S(S^2) & \longrightarrow & \pi_4^S(M_t) & \longrightarrow & \pi_4^S(S_3) & \longrightarrow & 0 \end{array} \qquad (9)$$

arising from the diagram (8) with $r = 1$. It shows that the second row of this diagram is the pushdown of the first one along the zero map; thus it splits.

Example 5.2. Now we are able to describe atoms in CW_3^2. They are cones Cf for some $f : mS^4 \to Y$ with 2-connected Y of dimension 4. Again $\mathbf{Hos}(Y, S^5) = 0$, so Theorem 4.1 can be applied. Example 5.1 shows that Y is a bouquet of spheres S^3, S^4 and suspended Moore atoms $\Sigma M(q)$. Note that $\pi_4^S(Y) = \pi_4(Y)$ for every Y; in particular $\pi_4(S^4) = \mathbb{Z},\ \pi_4(S^3) = \mathbb{Z}/2$ (generated by the suspended Hopf map $\eta_1 = \Sigma\eta$; $\eta : S^3 \to S^2 \simeq \mathbb{CP}^1$

which is given by the rule $\eta(a,b) = (a:b)$, where $(a,b) \in \mathbb{C}^2$ are such that $|a|^2 + |b|^2 = 1$) and

$$\pi_4(\Sigma M(q)) = \pi_3^S(M(q)) = \begin{cases} \mathbb{Z}/2 & \text{if } q = 2^r \\ 0 & \text{otherwise.} \end{cases}$$

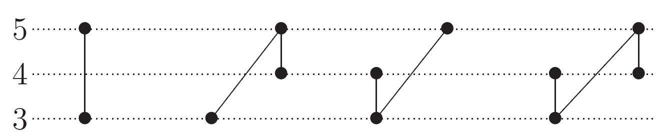
The Hopf map $\eta_2 = \Sigma^2\eta : S^4 \to S^3$ and the inclusion $j : S^2 \to M(q)$ give rise to an epimorphism $\eta_* : \pi_4(S_4) \to \pi_4(S^3)$ and to an isomorphism $j_* : \pi_4(S^3) \to \pi_4(\Sigma M_r)$, where $M_r = M(2^r)$. Moreover, if $t > r$, there is a map $M(2^r) \to M(2^t)$ that induces an isomorphism $\pi_4(M_r) \to \pi_4(M_t)$. If $Y = s_4S^4 \vee s_3S^3 \vee (\bigvee_{r=1}^{\infty} m_rM_r)$, a map $f : mS^4 \to Y$ can be given by a matrix of the form

$$\begin{pmatrix} F_4 & F_3 & G_1 & G_2 & \dots \end{pmatrix},$$

where F_i is of size $m \times s_i$ with entries from $\pi_4(S^i)$; G_r is of size $m \times m_r$ with entries from $\pi_4(\Sigma M_r)$ (some of these matrices can be "empty," containing no columns). Using automorphisms of Y and B, one can easily transform this matrix to the shape where there is at most two non-zero elements in every row (if two, one of them necessarily in the matrix F_4 and even) and at most one non-zero element in every column, as shown below:

F_4	F_3	G_r
q		
	η	
2^t	η	
		η
2^t		η

Thus X decomposes into a bouquet of the spaces $\Sigma^2M(q)$ (which are not atoms, but suspended atoms), spheres and the spaces $C(\eta)$, $C(\eta 2^t), C(2^r\eta)$ and $C(2^r\eta 2^t)$, which are gluings of the following forms:



$C(\eta)$ $C(\eta 2^t)$ $C(2^r\eta)$ $C(2^r\eta 2^t)$

Here, following Baues, we denote the cells by bullets and the attaching maps by lines; the word in brackets shows which maps are chosen to attach bigger cells to smaller ones. We do not show the fixed point, which coincide here with X^2 (since X is 2-connected); thus the lowest bullets actually describe spheres, not balls. These polyhedra are called *Chang atoms*. Again one can check that all of them are pairwise non-isomorphic.

Thus we have proved the following classical result.

Theorem 5.3 (Whitehead [23], Chang [9]). The atoms of dimension at most 5 are:
- sphere S^1 (of dimension 1);
- Moore atoms $M(q)$, where $q = p^r$, p is a prime number (of dimension 3);
- Chang atoms $C(\eta)$, $C(\eta 2^r)$, $C(2^r \eta 2^t), C(2^r \eta)$ (of dimension 5).

In what follows, we often use suspended Moore and Chang atoms. We shall denote them by the same symbols but indicating the dimension. Thus $M^d(q) = \Sigma^{d-3}M(q)$ and $C^d(w) = \Sigma^{d-5}C(w)$ for $w \in \{\eta, 2^r\eta, \eta 2^r, 2^r\eta 2^t\}$; in particular, $M(q) = M^3(q)$ and $C(w) = C^5(w)$. The same agreement will also be used for other atoms constructed below.

6 Dimension 7

We shall now consider the category CW^3. Its objects actually come from CW^3_4, so we have to classify atoms of dimension 7. Such an atom X is 3-connected, so we may suppose that $X^3 = *$. Set $B = X^5$, then X/B only has cells of dimensions 6 and 7. Therefore $X \in \Sigma^3\mathsf{CW}^1 \dagger \Sigma^2\mathsf{CW}^1 \simeq \Sigma\mathsf{CW}^1 \dagger \mathsf{CW}^1$. Consider the bifunctor $\mathsf{W}(A, B) = \mathrm{Hos}(\Sigma A, B)$ restricted to the category CW^1. Since, obviously, $\mathbf{Hos}(B, \Sigma^2 A) = 0$ for $A, B \in \mathsf{CW}^1$, we can apply Theorem 4.1. So we first classify indecomposable elements of the bimodule category $\mathbf{El}(\mathsf{W})$.

Indecomposable objects of the category CW^1 are spheres S^2, S^3 and Moore atoms $M(q)$ ($q = p^r,\ r$ prime). If q is odd, one easily sees that $\mathsf{W}(A, M(q)) = 0$ for all A, so we may only consider the spaces $M_r = M(2^r)$.

From the cofibration sequence

$$S^2 \xrightarrow{g_r} S^2 \to M_r \xrightarrow{h_r} S^3 \to S^3$$

and the diagram (5), we get the values of the Hos-groups shown in Table 2.

	S^2	S^3	M_1	M_r $(r > 1)$
S^2	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
S^3	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2^r$
M_1	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/4$	T_{1r}
M_t $(t > 1)$	$\mathbb{Z}/2^t$	$\mathbb{Z}/2$	T_{1t}	T_{tr}

Table 2

Here T_{tr} denotes the set of matrices $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$ with $a \in 2^m\mathbb{Z}/2^m$, $b \in \mathbb{Z}/2$, where $m = \min(r, t)$. The equality $\mathrm{Hos}(M_1, M_1) = \mathbb{Z}/4$ follows from the fact that this ring acts on $\pi_4^S(M_1) = \mathbb{Z}/4$, so $2\,\mathrm{Hos}(M_1, M_1) \neq 0$. The diagram (8) implies that the sequence

$$0 \longrightarrow \mathrm{Hos}(S^3, M_t) \longrightarrow \mathrm{Hos}(M_r, M_t) \longrightarrow \mathrm{Ker}\{\mathrm{Hos}(S^2, M_t) \xrightarrow{2^r} \mathrm{Hos}(S^2, M_t)\} \longrightarrow 0$$

splits if $\min(r,t) > 1$. The generator of the subgroup of diagonal matrices in T_{tr} is k_{tr}, while the matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ corresponds to the morphism $g_t \eta h_r$.

Analogous calculations, using Table 1 of the preceding section and the diagram (8), produce the following table for the values of the functor $\mathrm{Hos}(\Sigma A, B)$:

	S^2	S^3	M_1	M_r $(r>1)$
S^2	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/4$	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$
S^3	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
M_1	$\mathbb{Z}/2$	$\mathbb{Z}/4$	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$	$\mathbb{Z}/2 \oplus \mathbb{Z}/4$
M_t $(t>1)$	$\mathbb{Z}/2$	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$	$\mathbb{Z}/4 \oplus \mathbb{Z}/2$	$\mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$

Table 3

It is convenient to organize this result in the form of Table 4 below, as in [5].

	${}^1\otimes$	${}^2\otimes$	${}^3\otimes$	...	${}^\infty\otimes$	$*^\infty$	...	$*^3$	$*^2$	$*^1$
$\otimes_1$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	...	$\mathbb{Z}/2$	0	...	0	0	0
$\otimes_2$	$\mathbb{Z}/4$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	...	$\mathbb{Z}/2$	$\mathbb{Z}/2$	...	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0
$\otimes_3$	$\mathbb{Z}/4$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	...	$\mathbb{Z}/2$	$\mathbb{Z}/2$	...	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0
$\vdots$	...	...	...	...	...	...	...	...	...	...
$\otimes_\infty$	$\mathbb{Z}/4$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	...	$\mathbb{Z}/2$	$\mathbb{Z}/2$	...	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0
${}_\infty*$	0	0	0	...	$\mathbb{Z}$	$\mathbb{Z}/2$	...	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
$\vdots$	...	...	...	...	...	...	...	...	...	...
${}_3*$	0	0	0	...	0	$\mathbb{Z}/2$	...	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
${}_2*$	0	0	0	...	0	$\mathbb{Z}/2$	...	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
${}_1*$	0	0	0	...	0	$\mathbb{Z}/4$	...	$\mathbb{Z}/4$	$\mathbb{Z}/4$	$\mathbb{Z}/2$

Table 4

In this table the row marked by $\otimes_t$ (respectively, ${}_t*$) shows the part of the group $\mathrm{Hos}(\Sigma M_r, M_t)$ that comes from $\mathrm{Hos}(\Sigma M_r, S^2)$ (respectively, from $\mathrm{Hos}(\Sigma M_r, S^3)$). In the same way, the column marked by ${}^r\otimes$ (respectively, $*^r$) shows the part of this group that comes from $\mathrm{Hos}(S^3, M_t)$ (respectively, from $\mathrm{Hos}(S^4, M_t)$). The columns ${}^\infty\otimes$ and $*^\infty$ correspond, respectively, to $\mathrm{Hos}(S^4, \_)$ and $\mathrm{Hos}(S^3, \_)$; the rows $\otimes_\infty$ and ${}_\infty*$ correspond, respectively, to $\mathrm{Hos}(\_, S^2)$ and $\mathrm{Hos}(\_, S^3)$.

Therefore we consider the elements from $\mathbf{El}(\mathsf{W})$ as block matrices (W_y^x), where $x \in \{{}^r\otimes, *^r\}$, $y \in \{\otimes_t, {}^t*\}$ and the block W_y^x is with entries from the corresponding cell of Table 4. Moreover, morphisms between Moore spaces induce the following transformations of vertical stripes W^x and horizontal stripes W_y of such a matrix, which we call *admissible transformation*:

(a) replacing the stripes $M^{r\otimes}$ and M^{*^r} by $M^{r\otimes}X$ and $M^{*^r}X$;
(a′) replacing the stripes $M_{\otimes_t}$ and M_{t*} by $XM_{\otimes_t}$ and XM_{t*};
(b) replacing M^{*^r} by $M^{*^r} + M^{*^{r'}}X + M^{s\otimes}Y$, where $r' > r$, s arbitrary;
(b′) replacing $M_{\otimes_t}$ by $M_{\otimes_t} + XM_{\otimes t'} + YM_{s*}$, where $t' > t$, s arbitrary;
(c) replacing $M^{r\otimes}$ and M^{*^r} by $M^{r\otimes} + M^{r'\otimes}X$ and $M^{*^r} + 2^{r-r'}M^{*^{r'}}X$, where $r' < r$;
(c′) replacing M_{t*} and $M_{\otimes_t}$ by $M_{t*} + XM_{t'*}$ and $M_{\otimes_t} + 2^{t-t'}XM_{\otimes_{t'}}$, where $t' < t$;
(d) replacing $M^{1\otimes}$ by $M^{1\otimes} + 2M^{r\otimes}X + 2M^{*^s}Y$; r, s arbitrary;
(d′) replacing M_{1*} by $M_{1*} + 2XM_{t*} + 2YM_{\otimes_s}$; r, s arbitrary;
(e) replacing $M^{*^r}_{1*}$ by $M^{*^r}_{1*} + 2M^{s\otimes}_{\otimes_1}X$; s arbitrary;
(e′) replacing $M^{1\otimes}_{\otimes_t}$ by $M^{1\otimes}_{\otimes_t} + 2XM^{*^1}_{s*}$; s arbitrary.

Here X, Y denote arbitrary integer matrices of the appropriate size; in the transformations of types (a) and (a′) the matrix X must be invertible. Two matrices W, W' are isomorphic in $\mathbf{El}(\mathsf{W})$ if and only if W can be transformed to W' using admissible transformations.

It is convenient first to reduce the block $W^{\infty\otimes}_{\infty*}$ to a diagonal form $D = \mathrm{diag}(a_1, a_2, \dots, a_m)$ with $a_1|a_2|\dots|a_m$. Let $a_k = 2^{d_k}b_k$ with odd b_k. Denote by $W^{\infty k\otimes}$ and $W_{\infty k*}$ the parts of the stripes $W^{\infty\otimes}$ and $W_{\infty*}$ corresponding to the columns and rows with $d_k = d$ ($k = \infty$ if $d_k = 0$). Since all other matrices of these stripes are with entries from $\mathbb{Z}/2$, we can make the parts $W^{\infty 0\otimes}$ and $W_{\infty 0*}$ zero. Moreover, using admissible transformations that do not change the block D, we can replace $W^{\infty k\otimes}$ by $W^{\infty k\otimes} + W^{\infty l\otimes}X$ and $W_{\infty k*}$ by $W_{\infty k*} + YW_{\infty l*}$ for any $l < k$. In what follows we always suppose that W is already in this form.

Call two matrices of this form W, W' *2-equivalent*, if there is a matrix $W'' \simeq W$ such that $W'' \equiv W \bmod 2$. One can easily see that the problem of 2-equivalence of matrices from $\mathbf{El}(\mathsf{W})$ is actually a sort of *bunch of chains* in the sense of [8, 12]. We use the paper [12] as the source for the further discussion. Namely, we have the chain $\mathcal{E} = \{\, \otimes_t, {}_t*, {}_{\infty k}* \,\}$ for the rows and the chain $\mathcal{F} = \{\, {}^r\otimes, *^r, {}^{\infty k}\otimes \,\}$ for the columns, where

$$\begin{gathered} {}_1* < {}_2* < {}_3* < \dots < {}_{\infty\infty} < \dots < {}_{\infty 3}* < {}_{\infty 2}* < {}_{\infty 1}* < \otimes_\infty < \dots < \otimes_3 < \otimes_2 < \otimes_1, \\ {}^1\otimes < {}^2\otimes < {}^3\otimes < \dots < {}^{\infty\infty}\otimes < \dots {}^{\infty 3}\otimes < {}^{\infty 2}\otimes < {}^{\infty 1}\otimes < *^\infty < \dots < *^3 < *^2 < *^1. \end{gathered}$$

The equivalence relation $\sim$ on $\mathcal{X} = \mathcal{E} \cup \mathcal{F}$ is given by the rule

$$\otimes_t \sim {}_t* \ (t \neq \infty), \ {}^r\otimes \sim *^r \ (r \neq \infty), \ {}^{\infty k}\otimes \sim {}_{\infty k}*$$

for all possible values of t, r and $k \neq \infty$. Thus we can get a classification of our matrices up to 2-equivalence from [12]. Namely, we write $x - y$ if either $x \in \mathcal{E}$, $y \in \mathcal{F}$ or vice versa, at least one of them belongs to $\{\, \otimes_t \,\} \cup \{\, *^r \,\}$, moreover, $\{\, x, y \,\} \neq \{\, \otimes_t, *^1 \,\}$ and $\{\, x, y \,\} \neq \{\, \otimes_1, *^r \,\}$. We call an $\mathcal{X}$-*word* a sequence $w = x_1\rho_2x_2\rho_3\dots\rho_nx_n$, where $x_i \in \mathcal{X}$, $\rho_i \in \{\, \sim, - \,\}$, $\rho_i \neq \rho_{i+1}$ ($i = 2, \dots, n-1$) and $x_{i-1}\rho_ix_i$ holds in $\mathcal{X}$ for all $i = 2, \dots, n$. Such a word is called *full* if the following conditions hold:

- either $\rho_2 = \sim$ or $x_1 \not\sim y$ for all $y \in \mathcal{X}$, $y \neq x_1$;
- either $\rho_n = \sim$ or $x_n \not\sim y$ for all $y \in \mathcal{X}$, $y \neq x_n$.

w is called a *cycle* if $\rho_2 = \rho_n = -$ and $x_n \sim x_1$ in $\mathcal{X}$. If, moreover, w cannot be written in the form $v \sim v \sim \dots \sim v$ for a shorter word v, it is called *aperiodic*. We call a polynomial $f(t) \in \mathbb{Z}/2[t]$ *primitive* if it is a power of an irreducible polynomial with the leading coefficient 1. We shall identify any word w with its inverse and any cycle w with any of its cyclic shifts. Then the set of indecomposable representations of this bunch of chains is in 1-1 correspondence with the set $\mathcal{S} \cup \mathcal{B}$, where $\mathcal{S}$ is the set of full words (up to inversion) and $\mathcal{B}$ is the set of pairs (w, f), where w is an aperiodic cycle (up to a cyclic shift) and $f \neq t^d$ is a primitive polynomial. We call representations corresponding to $\mathcal{S}$ *strings* and those corresponding to $\mathcal{B}$ *bands*.

Note that an $\mathcal{X}$-word can contain at most one element ${}^{\infty k}\otimes$, at most one element $*_{\infty k}$ and at most one subword of the form $\otimes_t - *^r$ or its inverse. Replacing w by its inverse, we shall suppose that there are no words of the form $*^r - \otimes_t$ or ${}^{\infty k}\otimes \sim_{\infty k} *$. It is convenient to rewrite this answer in a modified form. Namely, we replace the subword ${}_{\infty k}* \sim^{\infty k} \otimes$, if it occurs, by ${}_k\varepsilon^k$, also omit x_1 if $\rho_2 =\sim$, omit x_n if $\rho_n =\sim$ and omit all remaining symbols $\sim$. Then we replace every subword ${}^r \otimes - \otimes_t$ by ${}^r\otimes_t$, $\otimes_t -{}^r \otimes$ by ${}_t\otimes^r$, ${}_t * - *^r$ by ${}_t*^r$, $*^r -{}_t *$ by ${}^r*_t$ and $\otimes_t - *^r$ by ${}_t\theta^r$. Note that in the last case $r \neq 1$ and $t \neq 1$. We also omit all signs $\sim$, replace any double superscript ^{rr} by ^{r} and any double subscript ${}_{tt}$ by ${}_t$. Certainly, the original word can be easily restored from such a shortened form. Now, any full word or its inverse can be written as a subword of one of the following words:

$$
\begin{aligned}
&{}^{r_1}\otimes_{t_1} *^{r_2} \otimes_{t_2} * \dots {}^{r_n}\otimes_{t_n} \qquad \text{(“usual word”)},\\
&{}_{t_{-m}}\otimes \dots {}^{r_{-2}} *_{t_{-2}} \otimes^{r_{-1}} *_{t_{-1}} \theta^{r_1} \otimes_{t_1} *^{r_2} \otimes_{t_2} \cdots *^{r_n} \qquad \text{(“theta-word”)},\\
&{}_{t_{-m}}\otimes \dots {}^{r_{-2}} *_{t_{-2}} \otimes^{r_{-1}} *_k \varepsilon^k \otimes_{t_1} *^{r_2} \otimes_{t_2} \cdots *^{r_n} \qquad \text{(“epsilon-word”)},
\end{aligned}
$$

Moreover,

- ∞ can only occur at the ends of a word, not in a theta-word or epsilon-word.
- In any theta-word $t_{-1} \neq 1$ and $r_1 \neq 1$.

Any cycle or its shift can be written as

$$
{}^{r_1}\otimes_{t_1} *^{r_2} \otimes_{t_2} * \dots {}^{r_n}\otimes_{t_n} *^{r_1}.
$$

The description of the representations in [12] also implies the following properties.

Proposition 6.1. (1) Any row (column) of a string contains at most 1 non-zero element.

(2) There are at most 2 zero rows or columns in a string, namely, they are in the following stripes:

(a) M_{t*} if w has an end $\otimes_t$ (or ${}_t\otimes$), $t \neq \infty$;
(b) M^{*^r} if w has an end ${}^r\otimes, r \neq \infty$;
(c) $M_{\otimes_t}$ if w has an end ${}_t*$;
(d) $M^{{}^r\otimes}$ if w has an end $*^r$ (or ${}^r*$);
(e) $M_{\infty k*}$ if the left end of w is ${}_k\varepsilon^k$;
(f) $M^{\infty k \otimes}$ if the right end of w is ${}_k\varepsilon^k$.

We call each end occurring in this list a *distinguished end*.

(3) The horizontal and vertical stripes of a band can be subdivided in such a way that every new horizontal or vertical band has exactly 1 non-zero block, which is invertible.

Recall that elements modulo 4 only occur in the stripes $W^{1\otimes}$ and $W_{1\otimes}$.

Corollary 6.2. Let $W \in \mathbf{El}(\mathsf{W})$ (with diagonal $W^{\infty\otimes}_{\infty*}$). Denote by $\overline{W}$ its reduction modulo 2 and by $\widetilde{W}$ the matrix obtained from W by replacing all invertible entries with 0 (thus all entries of $\widetilde{W}$ are even). Suppose that $\overline{W} = \bigoplus_{i=1}^m \overline{W}_i$, where all $\overline{W}_i$ are strings or bands. Then $W \simeq W'$, where $\overline{W}' = \overline{W}$ and the only non-zero rows and columns of $\widetilde{W}'$ can be those corresponding to the distinguished ends of types 2(a-d) of Proposition 6.1. In particular, if some of $\overline{W}_i$ is a band, a theta-string or an epsilon-string, W' has a direct summand W_i such that $\overline{W}_i \equiv W_i \bmod 2$ and $\widetilde{W}_i = 0$.

Thus we only have now to consider the case, when $W = W'$ and every $\overline{W}_i$ is a usual string. Suppose that W_i corresponds to a string w_i. It is easy to verify that if w_i and w_j have a common distinguished end, there is a sequence of distinguished transformations, which does not change $\overline{W}$ and adds the row (or column) corresponding to this end in $\widetilde{W}_i$ to the row (or column) corresponding to this end in $\widetilde{W}_j$ or vice versa. Hence, such rows (columns) are in some sense linearly ordered. As a consequence, we can transform $\widetilde{W}$ to a matrix having at most one non-zero element in every row and every column (without changing $\overline{W}$). It gives us the following description of indecomposable matrices from $\mathbf{El}(\mathsf{W})$ with $\widetilde{W} \neq 0$.

Corollary 6.3. Suppose that W is an indecomposable matrix from $\mathbf{El}(\mathsf{W})$, such that $\widetilde{W}' \neq 0$ for every matrix $W' \simeq W$. Let $\overline{W} = \bigoplus_{i=1}^m \overline{W}_i$, where each $\overline{W}_i$ is a usual string. There are, up to isomorphism, the following possibilities:

(1) $m = 1$, $\overline{W}$ corresponds to a word w and $\widetilde{W}$ has a unique non-zero element in the block W^a_b for the following choices:

$$w = {}_{t_1}*^{r_2} \otimes_{t_2} \dots,\ a = {}^1\otimes,\ b = \otimes_{t_1}\ (t_1 =\neq 1); \tag{a}$$
$$w = {}^{r_1}\otimes_{t_1} *^{r_2} \otimes \dots,\ a = *^{r_1},\ b = {}_1*\ (r_1 \neq 1);, \tag{b}$$
$$w = {}_1*^{r_1} \otimes_{t_1} *^{r_2} \dots,\ a = *^{r},\ b = {}_1*;, \tag{c}$$
$$w = {}^1\otimes_{t_1} *^{r_2} \otimes_{t_2} \dots,\ a = {}^1\otimes,\ b = \otimes_t;. \tag{d}$$

(2) $m = 2$, $\overline{W}_i$ $(i = 1, 2)$ correspond to the words w_i and $\widetilde{W}$ has a unique non-zero element in the block W^a_b, where

$$w_1 = {}_1*^{r_{-1}} \otimes_{t_{-1}} *^{r_{-2}} \otimes \dots,\ w_2 = {}^{r_1}\otimes_{t_1} *^{r_2} \otimes_{t_2} \dots,\ a = *^{r_1},\ b = {}_1*, \tag{e}$$
$$w_1 = {}^1\otimes_{t_1} *^{r_1} \otimes_{t_2} * \dots,\ w_2 = {}_{t_{-1}} *^{r_{-1}} \otimes_{t_{-2}} *^{r_{-2}} \otimes \dots,\ a = {}^1\otimes,\ b = \otimes_{t_{-1}}. \tag{f}$$

We encode these matrices by the following words w:

$$w = \dots \otimes^{r_2} *_{t_2} \otimes^{r_1} *_{t_1} \theta^1 \quad \text{in case (a)},$$
$$w = {}_1\theta^{r_1} \otimes_{t_1} *^{r_2} \otimes_{t_2} * \dots \quad \text{in case (b)},$$
$$w = \dots *_{t_2} \otimes^{r_2} *_{t_1} \otimes^{r_1} *_1 \theta^r \quad \text{in case (c)},$$
$$w = {}_t\theta^1 \otimes_{t_1} *^{r_2} *_{t_2} \otimes^{r_3} * \dots \quad \text{in case (d)},$$
$$w = \dots *^{r-2} \otimes_{t_{-2}} *^{r-1} *_1\theta^{r_1} \otimes_{t_1} *^{r_2} \otimes \dots \quad \text{in case (e)},$$
$$w = \dots * t_{-2} \otimes^{r-1} *_{t_{-1}}\theta^1 \otimes_{t_1} *^{r_2} \otimes_{t_2} * \dots \quad \text{in case (f)},$$

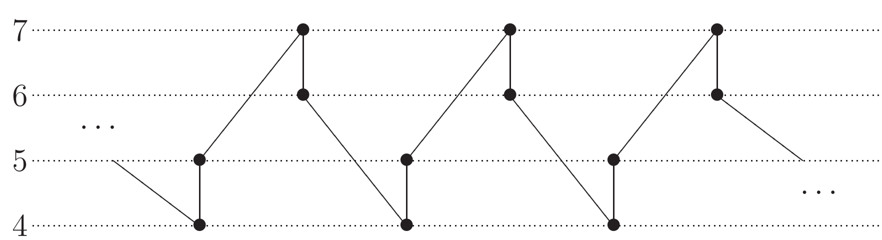
We call these words "theta-words" as well.

Obviously, cases (a-d) always give indecomposable matrices. On the other hand, one can check that in case (e) W is indecomposable if and only if $(r_{-1}+1, t_1, r_{-2}, t_2, \dots) < (r_1, t_{-2}, r_2, t_{-3}, \dots)$ with respect to the lexicographical order [5]. In case (f) W is indecomposable if and only if $(t_1+1, r_{-1}, t_2, r_{-2}, \dots) < (t_{-1}, r_2, t_{-2}, r_3, \dots)$ lexicographically. Thus we obtain a complete list of non-isomorphic indecomposable matrices from $\mathbf{El}(\mathsf{W})$. Moreover, it is easy to verify that they remain pairwise non-isomorphic and indecomposable in $\mathbf{El}(\mathsf{W})/\mathcal{I}$ as well. Thus, using Theorem 4.1, we get the following result.

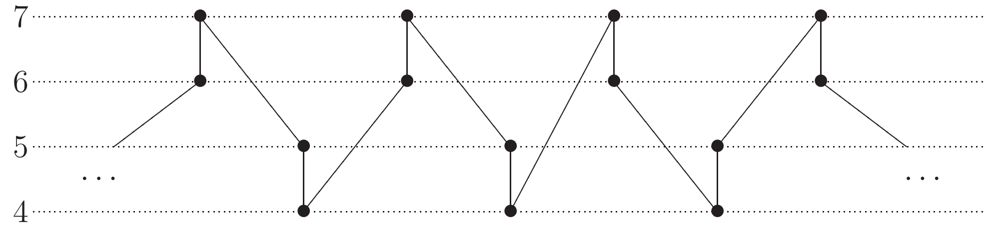
Theorem 6.4 (Baues–Hennes [7]). Indecomposable polyhedra from CW_4^3 are in 1-1 correspondence with usual words, theta-words, epsilon-words and bands defined above, with the only restriction that in a theta-word $w = \dots^{r_{-2}} *_{t_{-2}} \otimes^{r_{-1}} *_{t_{-1}} \theta^{r_1} \otimes_{t_1} *^{r_2} \otimes_{t_2} \dots$ the following conditions hold:

$$\text{if } t_{-1} = 1, \text{ then } (r_{-1}+1, t_1, r_{-2}, t_2, \dots) < (r_1, t_{-2}, r_2, t_{-3}, \dots) \text{ lexicographically},$$
$$\text{if } r_1 = 1, \text{ then } (t_1+1, r_{-1}, t_2, r_{-2}, \dots) < (t_{-1}, r_2, t_{-2}, r_3, \dots) \text{ lexicographically}.$$

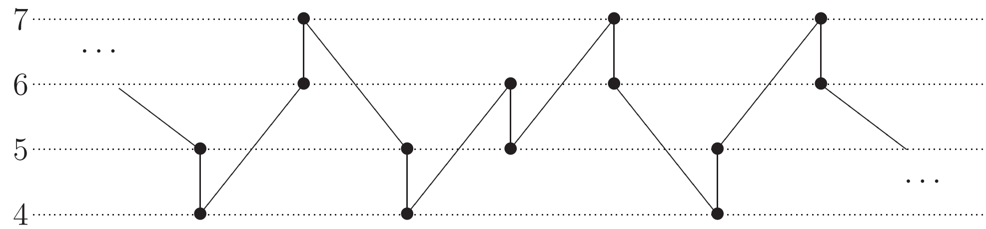
The gluings of spheres corresponding to these words can be described as follows:



for a usual word



for a theta-word



for an epsilon-word

In these diagrams vertical segments present the suspended atoms M_r, slanted lines correspond to the gluings arising from Hopf maps $S^{d+1} \to S^d$, while the long slanted line in a theta-word shows the gluing arising from the doubled Hopf map $S^6 \to S^4$.

Note that all atoms from CW_4^3 are p-primary (2-primary, except $M(q)$ with odd q). Therefore, we have the uniqueness of decomposition of spaces from CW^3 into bouquets of suspended atoms.

7 Bigger dimensions. Wildness

Unfortunately, if we pass to bigger dimensions, the calculations as above become extremely complicated. In the representations theory the arising problems are usually called "*wild*." Non-formally it means that the classification problem for a given category contains the classification of representations of arbitrary (finitely generated) algebras over a field. It is well-known, since at least 1969 [15], that it is enough to show that this problem contains the classification of pairs of linear mappings (up to simultaneous conjugacy), or, equivalently, the classification of triples of linear mappings

$$V_1 \rightrightarrows\!\!\!\to V_2 \tag{10}$$

On the other hand, problems like the one considered in the preceding section, where indecomposable objects can be parameterised by several "discrete," or combinatorial parameters (as $\mathcal{X}$-words above) and at most one "continuous" parameter (as a primitive polynomial in the description of bands), are called "*tame*." The problems, where the answer is purely combinatorial, like the classification of atoms of dimensions $d \leq 5$, are called "finite." I shall not precise these notions formally. The reader can consult, for instance, the survey [13], where it is done within the framework of representation theory. An important question in the representation theory is to distinguish finite, tame and wild cases. The following result accomplishes such an investigation for stable homotopy types.

Proposition 7.1 (Baues [5]). The classification problem for the category CW^4 is wild.

Proof. Let $\mathbf{B}$ be the category of bouquets of Moore atoms $M = M_1$, $\mathbf{A} = \Sigma^2\mathbf{B}$. Then CW^4 contains the subcategory $\Sigma^3(\mathbf{A}\dagger\mathbf{B}) \simeq \mathbf{A}\dagger\mathbf{B}$. Corollary 4.2 shows that the category $\mathbf{A}\dagger\mathbf{B}$ is representation equivalent to $\mathbf{El}(\mathsf{H})$, where H is the restriction of Hos onto $\mathbf{A}^\circ \times \mathbf{B}$. We know that $\mathrm{Hos}(M,M) = \mathbb{Z}/4$. Therefore, we only have to show that $\mathrm{Hos}(\Sigma^2 M, M) \simeq \mathcal{Z}/2 \oplus \mathcal{Z}/2 \oplus \mathcal{Z}/2$. Indeed, it implies the category $\mathbf{El}(\mathsf{H})$ is representation equivalent to the category of diagrams of the shape (10).

The cofibration sequence $S^2 \xrightarrow{2} S^2 \to M \to S^3 \xrightarrow{2} S^3$ and the Hopf map $\eta : S^5 \to S^4$ produce the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & \pi_4^S(M) & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \eta^*} & & \Big\downarrow & & \Big\downarrow{\scriptstyle \wr} & & \\
0 & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & \pi_5^S(M) & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & 0,
\end{array}$$

Since $\eta^3 = 4\nu$, where ν is the element of order 8 in $\pi_5^S(S^2) = \mathbb{Z}/24$ [21], actually $\eta^* = 0$, so the lower row splits and $\pi_5^S(M) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$. Just in the same way we show that $\mathrm{Hos}(\Sigma^2 M, S^2) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$. Now, applying the functors $\mathrm{Hos}(\_, S^2)$ and $\mathrm{Hos}(\_, M)$ to the same cofibration sequence, we get the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & \mathrm{Hos}(\Sigma^2 M, S^2) & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & 0 \\
 & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow{\scriptstyle \wr} & & \\
0 & \longrightarrow & \mathbb{Z}/2 \oplus \mathbb{Z}/2 & \longrightarrow & \mathrm{Hos}(\Sigma^2 M, M) & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & 0.
\end{array}$$

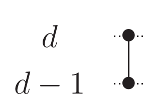
Since the upper row of this diagram splits, the lower one splits as well, hence $\mathrm{Hos}(\Sigma^2 M, M) = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$. It accomplishes the proof.

We can summarize the obtained results in the following theorem.

Theorem 7.2. The category CW^k is of finite type for $k \le 2$, tame for $k = 3$ and wild for $k \ge 4$.

8 Torsion free atoms. Dimension 9

Nevertheless, if we consider *torsion free* atoms, the situation becomes much simpler. Namely, in this case neither sphere of dimension d can be attached to the spheres of dimension $d-1$, thus in the picture describing the gluing of spheres there is no fragments of the sort



Therefore, a calculation of atoms from CWF_{k+1}^k can be organized as follows. Denote by $\mathbf{B}_k$ the full subcategory of CW consisting of bouquets of torsion free suspended atoms of dimension $2k$ and by $\mathbf{S}_k$ the category of bouquets of spheres S^{2k}. Let $\Gamma_m(X)$ denote the subgroup $\operatorname{Im}\{\pi_m^S(X^{m-1}) \to \pi_m^S(X)\}$ of $\pi_m^S(X)$. When X runs through $\mathbf{B}_k$, Γ_{2k} can be considered as an $\mathbf{S}_k$-$\mathbf{B}_k$-bimodule; we denote this bimodule by G_k. Then the following analogue of Theorem 4.1 holds (with essentially the same proof).

Proposition 8.1. Denote by $\mathcal{I}$ the ideal of the category CWF_{k+1}^k consisting of all morphisms $X \to X'$ that factor through an object from $\mathbf{B}_k$, and by $\mathcal{J}$ the ideal of the category $\mathbf{El}(\mathsf{G}_k)$ consisting of such morphisms $(\alpha, \beta) : f \to f'$ that α factors through f and β factors through f'. Then $\mathsf{CWF}_{k+1}^k/\mathcal{I} \simeq \mathbf{El}(\Gamma_k)/\mathcal{J}$. Moreover, both $\mathcal{I}^2 = 0$ and $\mathcal{J}^2 = 0$, hence the categories CWF_{k+1}^k and $\mathbf{El}(\mathsf{G}_k)$ are representation equivalent.

Proof. The only new claim here is that $\mathcal{J}^2 = 0$. But this equality immediately follows from the fact that if a morphism $X \to S^m$ factors through X^{m-1}, it is zero.

Thus a torsion free atom of dimension 7 can be obtained as a cone of a map $f : mS^6 \to Y$, where Y is a bouquet of spheres S^4, S^5 and suspended Chang atoms $C^6(\eta)$, while $f \in \Gamma_6(Y)$. Easy calculations, like above, give the following values of Γ_6:

X	S^4	S^5	$C^6(\eta)$
Γ_6	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0

(The last 0 is due to the fact that the map $\eta_* : \pi_6(S^5) \to \pi_6(S^4)$ is an epimorphism [21]). The Hopf map $\eta : S^5 \to S^4$ induces an isomorphism $\Gamma_6(S^5) \to \Gamma_6(S_4)$. Therefore, the only indecomposable torsion free atom of dimension 7 is the gluing $C(\eta^2) = C^7(\eta^2) = S^4 \cup_{\eta^2} B^7$. (Note that such an atom must contain at least one 4-dimensional cell.) Moreover, all torsion free atoms of dimensions $d \leq 7$ are 2-primary.

A torsion free atom of dimension 9 is a cone of some map $f : mS^8 \to Y$, where Y is a bouquet of spheres S^i $(5 \leq i \leq 7)$, suspended Chang atoms $C^7(\eta), C^8(\eta)$ and suspended atoms $C^8(\eta^2)$. One can calculate the following table of the groups Γ_8 for these spaces:

X	S^5	S^6	S^7	$C^7(\eta)$	$C^8(\eta)$	$C^8(\eta^2)$
Γ_8	$\mathbb{Z}/24$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/12$	0	$\mathbb{Z}/12$

Morphisms between these spaces induce epimorphims $\Gamma_8(S^5) \to \Gamma_8(C^7(\eta^2)) \to \Gamma_8(C^7(\eta))$, $\Gamma_8(S^7) \to \Gamma_8(S_6)$ and monomorphisms $\Gamma_8(S^7) \to \Gamma_8(C^7(\eta) \to \Gamma_8(S^5)$, $\Gamma_8(S^6) \to \Gamma_8(S^5)$. It can be deduced either from [21] or, perhaps easier, from the results of [22], cf. [3]. (The only non-trivial one is the monomorphism $\Gamma_8(S^7) \to \Gamma_8(C^7(\eta))$). Again we consider the map f as a block matrix

$$F = \begin{pmatrix} F_1 & F_2 & F_3 & F_4 & F_6 \end{pmatrix}^{\top}.$$

Here F_i is of size $m_i \times m$ with entries from $\Gamma_8(Y_i)$, where

$$Y_i = \begin{cases} S^5 & \text{if } i = 1, \\ C^7(\eta) & \text{if } i = 2, \\ C^8(\eta^2) & \text{if } i = 3, \\ S^6 & \text{if } i = 4, \\ S^5 & \text{if } i = 6. \end{cases}$$

We have written F_6, not F_5, in order to match the notations of the Example 3.2; so we set $I_1 = \{1,2,3,4,6\}$. Using the automorphisms of mS^7 and of Y, one can replace the matrix F by PFQ, where $P \in \mathrm{GL}(m,\mathbb{Z})$ and $Q = (Q_{ij})_{i,j\in I_1}$ is an invertible integer block matrix, where the block Q_{ij} is of size $m_i \times m_j$ with the following restrictions for the entries $a \in Q_{ij}$:

$$\begin{aligned} a =&0 \quad \text{for } i \in \{4,6\},\ j < i, \\ a \equiv&0 \bmod 2 \ \text{ for } (ij) \in \{(12),(13),(23)\}, \\ a \equiv&0 \bmod 6 \ \text{ for } (ij) = (26), \\ a \equiv&0 \bmod 12 \ \text{ for } j \in 4,6,\ i \in \{1,2,3\},\ (ij) \neq (26). \end{aligned}$$

Thus we have come to the *bimodule category* $\mathbf{El}(\mathsf{U}_1)$ considered in Example 3.4, so we can use Corollary 3.5, which describes all indecomposable objects of this category. Certainly, we are not interested in the "empty" objects $\emptyset_i$, since they correspond to the spaces with no 9-dimensional cells. Note also that the matrices $(1_4), (1_6)$ correspond not to atoms, but to suspended atoms $C^9(\eta^2)$ and $C^9(\eta)$. We use the following notation for the atoms corresponding to other indecomposable matrices F:

$$\begin{array}{lll}
A(v) & \text{for} & (v_1), \\
A(\eta v) & \text{for} & (v_2), \\
A(\eta^2 v) & \text{for} & (v_3), \\
A(v\eta) & \text{for} & \begin{pmatrix} v_1 \\ 1_6 \end{pmatrix}, \\
A(v\eta^2) & \text{for} & \begin{pmatrix} v_1 \\ 1_4 \end{pmatrix}, \\
A(\eta v\eta) & \text{for} & \begin{pmatrix} v_2 \\ 1_6 \end{pmatrix}, \\
A(\eta v\eta^2) & \text{for} & \begin{pmatrix} v_2 \\ 1_4 \end{pmatrix}, \\
A(\eta^2 v\eta) & \text{for} & \begin{pmatrix} v_3 \\ 1_6 \end{pmatrix}, \\
A(\eta^2 v\eta^2) & \text{for} & \begin{pmatrix} v_3 \\ 1_4 \end{pmatrix}.
\end{array}$$

So we have proved

Theorem 8.2 (Baues–Drozd [3]). Every torsion free atom of dimension 9 is isomorphic to one of the atoms $A(w)$ with $w \in \{ v, \eta v, \eta^2 v, v\eta, v\eta^2, \eta v\eta, \eta v\eta^2, \eta^2 v\eta, \eta^2 v\eta^2 \}$.

Using the gluing diagrams, these atoms can be described as in Table 5 below.

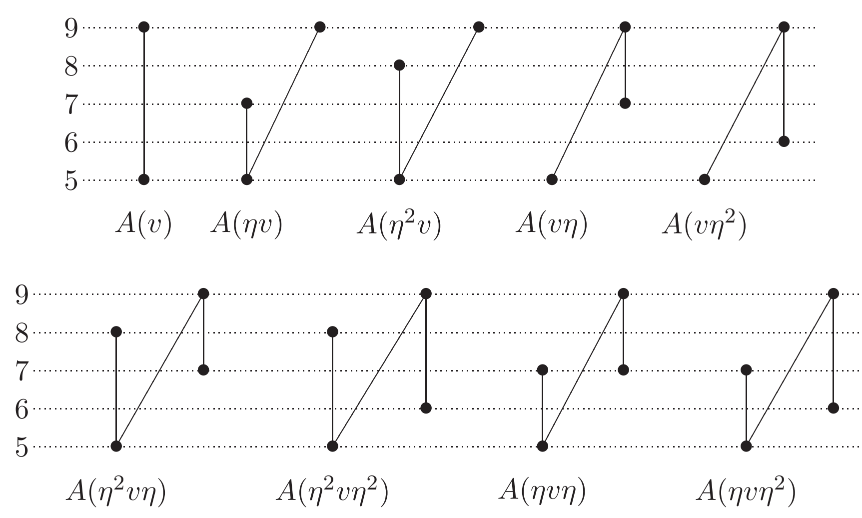


Table 5

One can also check that the 2-primary atoms in this list are those with v divisible by 3, while the only 3-primary one is $A(8)$. Thus there are altogether 29 primary suspended atoms of dimension at most 9. The congruent ones are only $A(3)$ and $A(9)$. Indeed,

$A(3) \vee S^5$ corresponds to the matrix $\begin{pmatrix} 3 \\ 0 \end{pmatrix} \bmod 24$. But the latter can be easily transformed to $\begin{pmatrix} 9 \\ 0 \end{pmatrix} \bmod 24$, which corresponds to $A(9) \vee S^5$:

$$\begin{pmatrix} 3 \\ 0 \end{pmatrix} \to \begin{pmatrix} 3 \\ 12 \end{pmatrix} \to \begin{pmatrix} -9 \\ 12 \end{pmatrix} \to \begin{pmatrix} 9 \\ 12 \end{pmatrix} \to \begin{pmatrix} 9 \\ 0 \end{pmatrix}.$$

(At the last step we add the first row multiplied by 4 to the second one; all other transformations are obvious.) One can verify that all other 2-primary atoms are pairwise non-congruent.

Corollary 8.3. The Grothendieck group $K_0(\mathsf{CWF}^4)$ is a free abelian group of rank 29.

Note that the matrix presentations allows easily to find the images in $K_0(\mathsf{CWF}^4)$ of all atoms. For instance, the equivalence of matrices

$$\begin{pmatrix} 8_1 & 0 \\ 0 & 3_1 \end{pmatrix} \sim \begin{pmatrix} 1_1 & 0 \\ 0 & 0_1 \end{pmatrix}$$

implies that $A(8) \vee A(3) \simeq A(1) \vee S^5 \vee S^9$, thus in $K_0(\mathsf{CWF}^4)$ we have

$$[A(1)] = [A(8)] + [A(3)] - [S^5] - [S^9].$$

The reader can easily make analogous calculations for all atoms of Table 5.

9 Torsion free atoms. Dimension 11

For torsion free atoms of dimension 11 analogous calculations have been done in [6]. Nevertheless, they are a bit cumbersome, so we propose here another, though rather similar, approach. Namely, denote by $\mathbf{S}'_k$ the category of bouquets of spheres S^{2k-1} and S^{2k}, by $\mathbf{B}'_k$ the category of bouquets of suspended atoms of dimension $2k-1$ and by G'_k the $\mathbf{S}'_k$-$\mathbf{B}'_k$-bimodule such that

$$\mathsf{G}'_k(S^{2k-1}, B) = \Gamma_{2k-1}(B) \quad \text{and} \quad \mathsf{G}'_k(S^{2k}, B) = \Gamma_{2k}(B).$$

Proposition 9.1. Denote by $\mathcal{I}'$ the ideal of the category CWF^k_{k+1} consisting of all morphisms $X \to X'$ that factors through an object from $\mathbf{B}'_k$, and by $\mathcal{J}'$ the ideal of the category $\mathbf{El}(\mathsf{G}'_k)$ consisting of such morphisms $(\alpha, \beta) : f \to f'$ that α factors through f and β factors through f'. Then $\mathsf{CWF}^k_{k+1}/\mathcal{I}' \simeq \mathbf{El}(\Gamma'_k)/\mathcal{J}'$. Moreover, both $(\mathcal{I}')^2 = 0$ and $(\mathcal{J}')^2 = 0$, hence the categories CWF^k_{k+1} and $\mathbf{El}(\Gamma'_k)$ are representation equivalent.

Thus we obtain torsion free atoms if dimension 11 as cones of maps $S \to Y$, where S is a bouquet of spheres of dimensions 9 and 10, while Y is a bouquet of 5-connected suspended atoms of dimensions $6 \le d \le 9$. Note that at least one of these atoms must have a cell of dimension 6 in order that such a cone be an atom.

Just as above, we have the following values of Γ_9 and Γ_{10} for such atoms:

X	S^6	$C^8(\eta)$	$C^9(\eta^2)$	S^7	$C^9(\eta)$	S^8	S^9
Γ_9	$\mathbb{Z}/24$	$\mathbb{Z}/12$	$\mathbb{Z}/12$	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	0
Γ_{10}	0	0	0	$\mathbb{Z}/24$	$\mathbb{Z}/12$	$\mathbb{Z}/2$	$\mathbb{Z}/2$

(We have arranged this table taking into account the known maps between these groups, as above.) The Hopf map $S^{10} \to S^9$ induces monomorphisms in the 4th and the 6th columns of this table, while the maps between suspended atoms induce homomorphisms analogous to those of the preceding section. Thus a morphism $f : S \to Y$ can be described by a matrix

$$F = \begin{pmatrix} F_1 & F_2 & F_3 & F_4 & 0 & F_6 & 0 \\ 0 & 0 & 0 & G_4 & G_5 & G_6 & G_7 \end{pmatrix}^{\top},$$

where the matrix F_i (G_i) has entries from the first row (respectively, second row) and the i-th column of the table above. Two matrices, F and F', define homotopic polyhedra if $F' = PFQ$, where P, Q are matrices over the tiled orders, respectively,

$$\begin{pmatrix} 1 & 2 & 2 & 12 & 24 & 12 & 24 \\ 1 & 1 & 1 & 12 & 24 & 6 & 24 \\ 1 & 2 & 1 & 12 & 24 & 12 & 24 \\ 0 & 0 & 0 & 1 & 2 & 12^* & 12 \\ 0 & 0 & 0 & 1 & 1 & 12 & 6 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 12^* \\ 0 & 1 \end{pmatrix}.$$

Here 12^* shows that the corresponding element obeys the $*$-rule (4), i.e. induces a *non-zero* map $\mathbb{Z}/2 \to \mathbb{Z}/2$ and acts as usual multiplication by 12 in all other cases.

Thus we have obtained the *bimodule category* $\mathbf{El}(\mathsf{U}_2)$ from Example 3.2, so we can use the list of indecomposable objects from Theorem 3.3. Moreover, we only have to consider the matrices having non-empty G-column and one of the parts F_1, F_2, F_3 (otherwise we have no 11-dimensional or no 6-dimensional cells). Therefore, a complete list of atoms arises from the following matrices:

$$\begin{pmatrix} v_i & 0 \\ 1_4 & w_4 \end{pmatrix}, \quad \begin{pmatrix} v_i & 0 \\ 1_4 & w_4 \\ 0 & 1_6 \end{pmatrix}, \quad \begin{pmatrix} v_i & 0 \\ 1_4 & w_4 \\ 0 & 1_7 \end{pmatrix},$$

where $i \in \{1,2,3\}$, $v, w \in \{1,2,3,4,5,6\}$. We omit the upper indices of Theorem 3.3, since here they coincide with the column number; the lower indices show to which horizontal stripe of the matrix F the corresponding elements belong. It gives the following list of 11-dimensional torsion free atoms.

Theorem 9.2 (Baues–Drozd [6]). Every torsion free atom of dimension 11 is isomorphic to one of the atoms of Table 6 below.

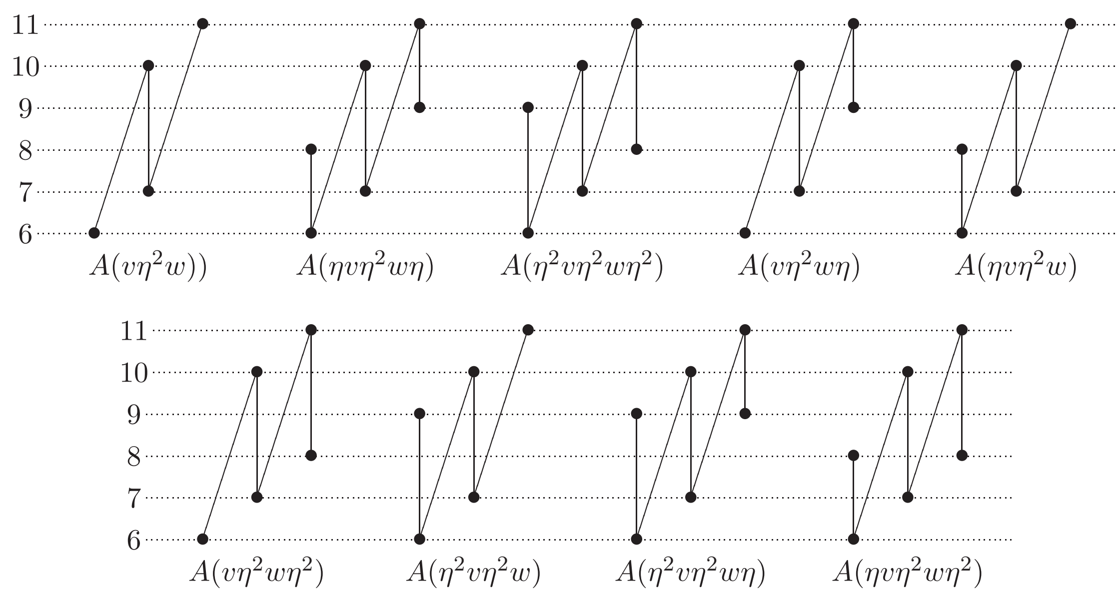


Table 6

Again 2-primary atoms are those with $v, w \in \{3,6\}$ and there are no 3-primary spaces in this table. Moreover, the new 2-primary atoms are pairwise non-congruent, therefrom we obtain the following result.

Corollary 9.3. The Grothendieck group $K_0(\mathsf{CWF}^5)$ is a free abelian group of rank 85.

We end up with the following statements about the higher dimensional torsion free spaces.

Proposition 9.4. (1) There are infinitely many non-isomorphic (even non-congruent) 2-primary atoms of dimension 13. Hence the Grothendieck group $K_0(\mathsf{CWF}^k)$ is of infinite rank for $k \geq 6$.

(2) If $k \geq 11$, the classification problem for the category CWF^k is wild.

Proof. We shall show first that $\pi_{12}^S(A^{11}(\eta^2 v)$, or, the same, $\pi_{10}^S(A(\eta^2 v)$ equals $\mathbb{Z}/2 \oplus \mathbb{Z}/2$. We consider the cofibration sequences

$$S^8 \xrightarrow{f} \Sigma C \xrightarrow{g} A \xrightarrow{h} S^9 \xrightarrow{\Sigma f} \Sigma^2 C, \qquad \text{(a)}$$

$$S^6 \longrightarrow S^4 \longrightarrow C \longrightarrow S^7 \longrightarrow S^5, \qquad \text{(b)}$$

where $A = A(\eta^2 v)$, $C = C(\eta^2)$. Note that the map f factors through S^5. From the sequence (b) we get $\pi_9^S(C) \simeq \pi_9^S(S^7) \simeq \mathbb{Z}/2$ and $\pi_9^S(\Sigma C) \simeq \pi_8^S(C) = 0$. The second equality follows from the fact that the induced map $\pi_8^S(S^7) \to \pi_8^S(S^5)$ is known to be injective [21]. Since $\pi_{10}^S(S^5) = \pi_{10}^S(S^6) = 0$, the sequence (a) gives then an exact sequence

$$0 \longrightarrow \pi_{10}^S(\Sigma C) \simeq \mathbb{Z}/2 \longrightarrow \pi_{10}^S(A) \longrightarrow \pi_{10}^S(S^9) \simeq \mathbb{Z}/2 \longrightarrow 0.$$

To show that this sequence splits, we have to check that $2\alpha = 0$ for every $\alpha \in \pi_{10}^S(A)$. In any case, 2α factors through ΣC, which gives rise to a commutative diagram

$$\begin{array}{ccccc} M^{10}(2) & \xrightarrow{\phi} & S^{10} & \xrightarrow{2} & S^{10} \\ \gamma\downarrow & & \downarrow\beta & & \downarrow\alpha \\ S^8 & \xrightarrow[f]{} & \Sigma C & \xrightarrow[g]{} & A \end{array}$$

for some β, γ (we have used the cofibration sequence for $M(2)$). Since $\pi_9^S(S^5) = \pi_{10}^S(S^5) = 0$, also $\mathrm{Hos}(M^{10}(q), S^5) = 0$. But the map $\beta\phi = f\gamma$ factors through S^5, so $\beta\phi = 0$ and $\beta = 2\sigma$ for some $\sigma \in \pi_{10}^S(\Sigma C) \simeq \mathbb{Z}/2$. Hence $\beta = 0$ and $2\alpha = 0$.

Analogous calculations show that any endomorphism of A acts as a homothety on $\pi_{10}^S(A)$. Since, obviously, $\mathrm{Hos}(A, S^{10}) = 0$, Corollary 4.2 shows that the category of spaces arising as cones of mappings $mS^{12} \to nA^{11}(\eta^2 v)$ is equivalent to the category of representations of the Kronecker quiver $\widetilde{A}_1$, or, the same, of diagrams of $\mathbb{Z}/2$-vector spaces of the shape $V_1 \rightrightarrows V_2$. But it is well-known that this quiver is of infinite type, i.e. has infinitely many non-isomorphic indecomposable representations. Obviously, all corresponding spaces are 2-primary and non-congruent, which proves the claim (1).

The claim (2) follows from the equality $\pi_{20}^S(S^{11}) \simeq (\mathbb{Z}/2)^3$. It implies that the category of spaces, which are cones of mappings $mS^{20} \to mS^{11}$, is equivalent to that of diagrams $V_1 \Rrightarrow V_2$. The latter is well-known to be wild.

Perhaps, the estimate 11 in the claim (2) of Proposition 9.4 is too big, but at the moment I do not know a better one. On the other hand, there is some evidence that the classification problem for CWF^6 is still tame.

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