MODULE-THEORETICAL PROPERTIES OF “GOOD” CURVE SINGULARITIES

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The purpose of this talk is to give some relations between geometric and algebraic properties of curve singularities. More precisely, we mean, firstly, their behavior under deformations, in particular their modality and, secondly, the behavior of Cohen-Macaulay modules over their local rings (in particular, their ideals).

For geometric properties, in particular, for classification of singularities of lower modalities, we refer to the classical book [1].

As the definitions related to the algebraic properties are probably less known, I shall remind some of them. Throughout this talk $\mathbf{x}$ denotes some *curve singularity* (i.e. a point of an algebraic curve, for the sake of simplicity, over complex numbers), $\mathbf{R} = \mathbf{R}_{\mathbf{x}}$ the completion of its local ring. Denote by $\mathbf{F}$ the full ring of fractions of $\mathbf{R}$, by $\mathbf{R}_0$ its normalization, i.e. the integral closure of $\mathbf{R}$ in $\mathbf{F}$ and by $\mathsf{CM}(\mathbf{R})$ the category of (maximal) Cohen-Macaulay $\mathbf{R}$-modules. If $M \in \mathsf{CM}(\mathbf{R})$, the natural homomorphism $M \to \mathbf{F} \otimes_{\mathbf{R}} M$ is an embedding. Hence, if $\mathbf{R}'$ is an over-ring of $\mathbf{R}$, i.e. $\mathbf{R} \subseteq \mathbf{R}' \subseteq \mathbf{R}_0$, we can consider the $\mathbf{R}'$-module $\mathbf{R}'M \subset \mathbf{F} \otimes_{\mathbf{R}} M$. Thus we are able to identify $\mathsf{CM}(\mathbf{R}')$ with a full subcategory of $\mathsf{CM}(\mathbf{R})$. In our case $\mathbf{R}_0 = \prod_{i=1}^{s} \mathbf{S}_i$, where $\mathbf{S}_i \simeq \mathbb{C}[[t]]$ (formal power series ring), s being the number of *branches* passing through $\mathbf{x}$. Hence $\mathbf{R}_0 M \simeq \bigoplus_{i=1}^{s} r_i \mathbf{S}_i$ for some integers r_i. Call the vector $\mathbf{rk}\, M = (r_1, r_2, \dots, r_s)$ the *rank vector* of M. Of course, if $\mathbf{R}$ is a domain, i.e. $\mathbf{x}$ lies on the unique branch, $\mathbf{rk}\, M$ is just its usual rank as of $\mathbf{R}$-module. For each vector $\mathbf{r} = (r_1, r_2, \dots, r_s)$ denote by $\mathsf{CM}_{\mathbf{r}}(\mathbf{R})$ the set of all Cohen-Macaulay $\mathbf{R}$-modules with the rank vector $\mathbf{r}$ and by $\mathsf{ind}_{\mathbf{r}}(\mathbf{R})$ its subset of indecomposable ones. Let also $\mathbf{p}_{\mathbf{r}}(\mathbf{R})$ be the maximal dimension of the families of non-isomorphic indecomposable Cohen-Macaulay $\mathbf{R}$-modules having the fixed rank vector $\mathbf{r}$. Then one defines the *Cohen-Macaulay type* of $\mathbf{R}$ (or of $\mathbf{x}$) as follows:

1. $\mathbf{R}$ *is* CM-finite *if it possesses only finitely many (up to isomorphism) indecomposable Cohen-Macaulay modules.*

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2. $\mathbf{R}$ *is* CM-tame *if it is not CM-finite, but* $\mathbf{p_r}(\mathbf{R}) \leq 1$ *for each rank vector* $\mathbf{r}$.
3. $\mathbf{R}$ *is* CM-wild *otherwise.*

Moreover, if $\mathbf{R}$ is CM-tame, let $\mathbf{n_r}(\mathbf{R})$ be the maximal number of pairwise non-intersecting one-parametre families of non-isomorphic indecomposable $\mathbf{R}$-modules of rank vector $\mathbf{r}$. Then $\mathbf{R}$ (or $\mathbf{x}$) is said to be *bounded* or *unbounded* if so are the numbers $\mathbf{n_r}(\mathbf{R})$ for all possible values of $\mathbf{r}$.

It is known (cf. [7]) that if $\mathbf{R}$ is CM-wild, then $\sup_{\mathbf{r}} \mathbf{p_r}(\mathbf{R}) = \infty$. Moreover, in this case for each (not necessarily commutative) finitely generated $\mathbb{C}$-algebra $\mathbf{A}$ there exists an exact functor $\mathsf{F} : \mathbf{A}\text{-}\mathsf{mod} \to \mathsf{CM}(\mathbf{R})$, which maps indecomposable modules to indecomposable and non-isomorphic to non-isomorphic. In this sense the complete classification of Cohen-Macaulay $\mathbf{R}$-modules seems indeed to be a "wild" problem and there is no evidence to subdivide the wild case. On the other hand, if $\mathbf{R}$ is not CM-finite, $\mathbf{p_1}(\mathbf{R}) \geq 1$, where $\mathbf{1} = (1, 1, \ldots, 1)$ (cf. [4]). In [7] it was also proved that, if $\mathbf{R}$ is CM-tame, for each rank vector $\mathbf{r}$ there exists a (finite) set of *rational* families (that is, with bases being rational curves) containing all indecomposable modules of rank vector $\mathbf{r}$. Remark that it is no longer the case for *surface* singularities (cf. [13]).

The following Table 1 presents the correlations between the geometric and algebraic properties of *plane* curve singularities (algebraically "plane" means that the maximal ideal $\mathbf{m}$ of $\mathbf{R}$ has 2 generators). In this case we have one more invariant of the singularity playing an important role, namely, the intersection quadratic form Q corresponding to its suspension. In the last column of Table 1 the *type* of this form is given as the triple (μ_+, μ_0, μ_-), where μ_+ is its positive, μ_- its negative index and μ_0 is the dimension of its kernel. We write also $\mathbf{p_1}$ for $\mathbf{p_1}(\mathbf{R})$.

If the singularity is no more plane, the results are almost the same. Namely, say that a singularity $\mathbf{x}$ *dominates* another one $\mathbf{y}$ if $\mathbf{R_x}$ is isomorphic to an over-ring of $\mathbf{R_y}$. Then, of course, the CM-type of $\mathbf{x}$ is "not worse" than that of $\mathbf{y}$, i.e. if $\mathbf{y}$ is CM-finite, so is $\mathbf{x}$, if $\mathbf{y}$ is CM-tame, $\mathbf{x}$ is CM-tame or finite, etc. The following theorem gives a complete picture.

Theorem 1. *A curve singularity* $\mathbf{x}$ *is:*

1. *CM-finite if and only if it dominates some simple plane curve singularity.*

Table 1. TYPES OF PLANE CURVE SINGULARITIES

Type of $\mathbf{x}$	CM-type of $\mathbf{R}$	Type of Q
simple $(A_n, D_n, E_{6,7,8})$	finite	negative definite
parabolic unimodal (T_{44}, T_{36})	tame bounded	(0,2,n)
hyperbolic unimodal (other T_{pq})	tame unbounded	(1,1,n)
exceptional unimodal or bimodal	wild but $\mathbf{p_1} = 1$	(2,0,n)
others	wild and $\mathbf{p_1} > 1$	

2. *CM-tame bounded if and only if it dominates some parabolic unimodal plane curve singularity (and no simple one).*
3. *CM-tame unbounded if and only if it dominates some hyperbolic unimodal plane curve singularity (and no simple or parabolic one).*
4. *CM-wild with* $\mathbf{p_1} = 1$ *if and only if it dominates some exceptional unimodal or some bimodal plane curve singularity (and no simple, parabolic or hyperbolic one).*
5. *CM-wild with* $\mathbf{p_1} > 1$ *otherwise.*

As all singularities of type T_{pq} have at least 2 branches, we get the following curious corollary.

Corollary 1. *If a curve singularity has only 1 branch (i.e. the ring* $\mathbf{R}$ *is a domain), then it is either CM-finite or CM-wild.*

Unfortunately, neither for Table 1, nor for Theorem 1, nor for at least any of the implications involved there is now any "conceptional", *a priory* proof. All known proofs are based on some more or less straightforward calculations. So the cited results, though of quite acceptable form, have at the moment rather "zoological" nature, like such claim as "*All penguins live in the south hemisphere, while all polar bears in the northern one*". A very exciting problem would be to discover a "non-calculative" proof for at least some (better, of course, for all) of these implications. The most intriguing seem now the correlations between CM-type and quadratic form Q. The thing is that such relations are widespread in the representation theory of finite-dimensional algebras, where there is some geometric evidence at least for claims of the sort "If algebra is well-behaved (with respect to representations), the corresponding form is also well-behaved (with respect to some positivity property)". It would be great if somebody could prove some similar *a priory* claim for curve singularities.

The assertion 1 of this theorem is rather old. It was proved independently by Jacobinski [12] and Drozd–Roiter [10] in quite other terms, not referring to singularities (and before Arnold's classification was started). Its relation to this classification was first marked by Greuel–Knörrer [11]. CM-type of parabolic singularities was established by Dieterich [2],[3] and the equality $\mathbf{p_1} = 1$ for all uni- and bimodal plane curve singularities by Schappert [15]. The other assertions were proved by Drozd–Greuel [8],[9]. The author's survey [6] gives a rather complete account on these and some related results with most proofs sketched.

I would like to present here a sketch of the proof for the following tameness (and hence wildness) criterion:

Let a curve singularity $\mathbf{x}$ *be not CM-finite. Then* $\mathbf{x}$ *is CM-tame if and only if it dominates some plane curve singularity of type* T_{pq} *(and wild otherwise).*

All other proofs are rather alike (and even easier).

Step 1. Remind that the singularities of type T_{pq} are given by the equation:

$$x^p + y^q + \lambda x^2 y^2 = 0$$

for some parametre $\lambda \in \mathbb{C}$. If $(pq) \neq (44)$ or (36), all of them are analytically isomorphic, i.e. all rings $\mathbf{R}$ are the same independently on λ. Consider another singularity (not plane, though complete intersection) P_{pq} given by two equations:

$$xy = 0 \quad \text{and} \quad x^p + y^q + z^2 = 0\,.$$

Let $\mathbf{P}$ be the completion of its local ring, $\mathbf{n}$ its maximal ideal and $\mathbf{P}' = \operatorname{End}(\mathbf{n})$. As P_{pq} is a complete intersection, $\mathbf{P}$ is Gorenstein, hence all its indecomposable Cohen-Macaulay modules except $\mathbf{R}$ itself are indeed $\mathbf{P}'$-modules. But the ring $\operatorname{End}(\mathbf{n}')$, where $\mathbf{n}'$ is the maximal ideal of $\mathbf{P}'$, splits into direct product of two rings of type A, whose Cohen-Macaulay modules are well known and of very simple shape. It gives us possibility, using more or less standard techniques for the calculation of Cohen-Macaulay modules, to describe all such modules for any singularity of type P_{pq}, hence check that all of them are indeed CM-tame. Consider now a family of singularities $\mathbf{x}_\lambda$ given by the equations:

$$xy = \lambda z \quad \text{and} \quad x^p + y^q + z^2 = 0\,.$$

For $\lambda \neq 0$ all of them are of type T_{pq}, while for $\lambda = 0$ we get P_{pq}. Now we use the "semi-continuity theorem" of Knörrer [14], which implies that whenever all $\mathbf{x}_\lambda$ were CM-wild, so would be also $\mathbf{x}_0$. But the latter is CM-tame and all $\mathbf{x}_\lambda$ with $\lambda \neq 0$ are isomorphic (we suppose $(pq) \neq (44), (36)$). Hence the singularity of type T_{pq} is also CM-tame. For $(pq) = (44)$ or (36) we need other arguments. Till now the only known way is that used by Dieterich and consisting also of some direct (and rather cumbersome) calculations. As we have remarked before, all singularities dominating T_{pq} are also CM-tame.

Step 2. Now we prove that whenever $\mathbf{R}$ is CM-tame, it satisfies some "over-ring conditions". Namely, denote:

$$\begin{aligned} \mathbf{m}_0 &= \operatorname{rad}\mathbf{R}_0\,, \\ \mathbf{R}' &= \mathbf{m}_0 + \mathbf{R}\,, \\ \mathbf{R}'' &= \mathbf{m}\mathbf{R}' + \mathbf{R}\,, \\ \mathbf{m}' &= \operatorname{rad}\mathbf{R}'\,, \end{aligned}$$

$\mathrm{d}(M)$ the number of generators of $\mathbf{R}$-module M.

($\mathbf{R}'$ is the "*weak normalization*" of $\mathbf{R}$, i.e. its maximal local over-ring.)

Then for any CM-tame singularity:

1. $3 \leq \mathrm{d}(\mathbf{R}_0) \leq 4$ and $\mathrm{d}(e\mathbf{R}_0) \leq 2$ for each primitive idempotent $e \in \mathbf{R}_0$.
2. $\mathrm{d}(\mathbf{R}') \leq 3$ and $\mathrm{d}((1-e)\mathbf{R}') \leq 2$ for each primitive idempotent $e \in \mathbf{R}_0$.
3. If $\mathrm{d}(\mathbf{R}_0) = 3$, then $\mathrm{d}(\mathbf{R}') = 3$ and $\mathrm{d}(\mathbf{R}'') \leq 2$.

Remark that if $\mathrm{d}(\mathbf{R}_0) = 2$ or $\mathrm{d}(\mathbf{R}_0) = 3, \mathrm{d}(\mathbf{R}') = 2$, $\mathbf{x}$ dominates some simple plane curve singularity, hence is CM-finite.

The proof of this assertion also depends on some "by hand" calculations. The only "easy" case is $\mathrm{d}(\mathbf{R}_0) > 4$ or $\mathrm{d}(\mathbf{R}') > 3$, when the wildness follows from some simple geometric observations. For instance, if $d = \mathrm{d}(\mathbf{R}) > 4$, consider all such $\mathbf{R}$-modules M that $\mathbf{R}_0 \supseteq M \supseteq \mathbf{m}_0$ and $\dim(M/\mathbf{m}_0) = 2$. They can be considered as points of the Grassmannian $\mathrm{Gr}\,(2, d)$, which has dimension $2(d-2)$, and isomorphic ones form orbits of the group $\mathbf{R}_0^*/\mathbf{R}'^*$, which is of dimension $d-1$. As $d > 4$, there exist 2-parametre families of non-isomorphic modules.

In all other cases we have to construct 2-parametre families by hand. For instance, if $d = 4$ and $\mathbf{R}_0 \simeq \mathbb{C}[[t]]$ (then only $\mathbf{m}_0^4 \subseteq \mathbf{m}\mathbf{R}_0$) such a family is obtained by considering $\mathbf{R}$-submodules in $5\mathbf{R}_0$ generated by the columns of the following matrix:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & t & 0 & t^2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & t & 0 & t^2 \\ 0 & 0 & 1 & 0 & 0 & t^2 & 0 & 0 & t^3 \\ 0 & 0 & 0 & 1 & 0 & 0 & t^2 & \alpha t^3 & \beta t^3 \\ 0 & 0 & 0 & 0 & 1 & t^3 & 0 & 0 & 0 \end{pmatrix},$$

where $\alpha, \beta \in \mathbb{C}$ are just two parametres. Knowing it, one can check that the modules corresponding to distinct values of the parametres are indeed non-isomorphic and indecomposable.

Step 3. At last, we have to prove that any singularity satisfying the over-ring conditions (1)–(3) dominates some singularity of type T_{pq}. It is also done by hand looking over case by case and using the "parametrization" of these singularities. Namely, it is known that for a singularity of type T_{pq} the ring $\mathbf{R}$ is generated (as complete local $\mathbb{C}$-subalgebra of $\mathbf{R}_0$) by two elements x, y of the following form (depending on the parity of p and q):

p	q	x	y
odd	odd	(t^{p-2}, t^2)	(t^2, t^{q-2})
odd	even	$(t, t, \alpha t^{p-2})$	$(0, t^{q/2-1}, t^2)$
even	even	$(t, 0, t, \alpha t^{p/2-1})$	$(0, t, t^{q/2-1}, t)$

Here $\alpha \in \mathbb{C} \setminus \{ 0 \}$ and $\alpha \neq 1$ if $(pq) = (44)$ or (36), while in other cases all values of α give isomorphic rings.

Over-ring Condition 1 implies that the singularity has at least 2 and at most 4 branches. Consider the case, when $s = 3$, other cases being quite alike. Condition 1 implies that $\mathbf{mR}_0$ equals either $(t) \times (t) \times (t)$ or $(t) \times (t) \times (t^2)$ (up to a permutation of branches). In the former case Condition 3 implies that $\mathbf{R}$ contains (t,t,t) and at least one element lying in $\mathbf{m}_0^2$ but not in $\mathbf{m}_0^3$. Therefore, $\mathbf{x}$ dominates some T_{3q}. In the latter case Condition 2 implies that $\mathbf{R}$ contains two elements of the shape (t,t,a) and (b,c,t^2) .Therefore, it also dominates some T_{pq}.

Altogether Steps 1–3 prove Theorem 1.

Make some additional remarks on the behavior of the ideals. Here the number $\mathrm{d} = \mathrm{d}(\mathbf{R}_0)$ plays the main role. Namely (cf. [4],[5]):

1. $\mathrm{d} - 1 = \min \{ n \mid I^n$ *is invertible for each ideal* $I \}$.
2. $\mathrm{d} = 3$ *if and only if each ideal is either invertible or dual to an invertible ideal.*

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