MATH 4576 Rings and Fields

Additional exercises 4

(1) Construct a radical extension of $\mathbb{Q}$ containing a splitting field of the polynomial $x^4 + 4x^2 - 6$.

(2) Find the Galois group of the polynomial x^4+3x^2-10.

(3) Let $\varepsilon \in \mathbb{C}$ be such that $\varepsilon^5 = 1$, $\varepsilon \neq 1$. Prove that $\mathbb{Q}(\varepsilon)$ is a normal extension of $\mathbb{Q}$ and its Galois group G is cyclic with 4 elements.

Hint: ε is a root of the irreducible polynomial $p(x) = \dfrac{x^5 - 1}{x - 1}$. Verify that the other roots of $p(x)$ are ε^k $(2 \leq k \leq 4)$. Deduce that there is an element $\sigma \in G$ such that $\sigma(\varepsilon) = \varepsilon^2$. What is $\operatorname{ord}(\sigma)$? Compare with $\#(G)$.

(The same assertion is true if we replace 5 by any prime number p and 4 by $p - 1$.)

(4) Let $K \supseteq Z_p$ (p a prime number) be a field with p^n elements, $G = \operatorname{Gal}_{Z_p} K$. Prove that

(a) The map $\phi : K \to K$ such that $\phi(a) = a^p$ is an element of G. (It is called the *Frobenius automorphism*).

(b) $\operatorname{ord}(\phi) = n$.

(c) $G = \langle \phi \rangle$.