MATH 4576 Rings and Fields

Additional exercises 3

Let K be the splitting field of the polynomial $x^4 - 2 \in \mathbb{Q}[x]$, $G = \mathrm{Gal}_{\mathbb{Q}}(K)$ (as in Additional exercises 2), $G = \mathrm{Gal}_{\mathbb{Q}} K$, σ, τ — elements of G such that

$$\sigma(i) = i,\ \ \sigma(\sqrt[4]{2}) = i\sqrt[4]{2},$$
$$\tau(i) = -i,\ \ \tau(\sqrt[4]{2}) = \sqrt[4]{2}.$$

(1) Prove that $G = \{\, \iota, \sigma, \sigma^2, \sigma^3, \tau, \sigma \circ \tau, \sigma^2 \circ \tau, \sigma^3 \circ \tau \,\}$.

(2) Which of the elements listed above equals $\tau \circ \sigma$?

(3) Prove that $\mathrm{ord}(\sigma \circ \tau) = 2$ and $K^{\langle \sigma \circ \tau \rangle} = \mathbb{Q}((1+i)\sqrt[4]{2})$.