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# GRADED COHEN–MACAULAY RINGS OF WILD COHEN–MACAULAY TYPE

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Abstract. We give sufficient conditions for a standard graded Cohen–Macaulay ring, or equivalently, an arithmetically Cohen–Macaulay projective variety, to be Cohen–Macaulay wild in the sense of representation theory. In particular, these conditions are applied to hypersurfaces and complete intersections.

## 1. Introduction

There is a long-standing problem to classify Cohen–Macaulay rings according the complexity of the classification of Cohen–Macaulay modules over them. The known results about such classification gave rise to the conjecture that all Cohen–Macaulay rings split into three "Cohen–Macaulay types", namely, *Cohen–Macaulay discrete* (including *Cohen–Macaulay finite*), *Cohen–Macaulay tame* and *Cohen–Macaulay wild*. It is expected to hold in general, though only proved, completely or partially, in several special cases (cf. [18, 12, 17, 10] for one-dimensional, [15, 1, 11] for two-dimensional, [14] for graded cases).

In this paper we give some sufficient conditions for a graded Cohen–Macaulay ring to be Cohen–Macaulay wild. Equivalently, it can be applied to *arithmetically Cohen–Macaulay* (*ACM*) projective varieties $X \subset \mathbb{P}^n$ and *ACM sheaves*, giving conditions for such a variety to be arithmetically Cohen–Macaulay wild. In particular, we apply the obtained results to hypersurfaces and complete intersections. Note that we consider the "*algebraical wildness*", contrary to the viewpoint of [5, 19, 20], where the wildness means "*geometrical wildness*" (we explain the difference below). Note also that, if $\dim X > 1$, arithmetically Cohen–Macaulay finiteness (tameness, wildness) is not an invariant of $X$, but of the embedding $X \hookrightarrow \mathbb{P}^n$.

We fix a field $\Bbbk$, which is supposed to be infinite, but not necessarily algebraically closed or of characteristic 0. All algebras are $\Bbbk$-algebras. A *standard graded algebra* is a graded $\Bbbk$-algebra $R = \bigoplus_{i=0}^{\infty} R_i$ such that

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$R_0 = \Bbbk$ and $R$ is generated in degree 1, i.e. $R = \Bbbk[R_1]$. We denote by $\mathfrak{m} = \bigoplus_{i=1}^{\infty}$ the maximal graded ideal in $R$. We always suppose that $R$ is Cohen–Macaulay and denote by $\mathrm{GCM}(R)$ the category of graded maximal Cohen–Macaulay $R$-modules. If we consider $R$ as a homogeneous coordinate ring of a projective variety $X = \operatorname{Proj} R$, then $X$ is an *arithmetically Cohen–Macaulay* (*ACM*) *variety* and the category $\mathrm{GCM}(R)$ is equivalent to the category of *arithmetically Cohen–Macaulay* (*ACM*) *coherent sheaves* over $X$ [5]. If $R$ is an *isolated singularity*, i.e. all rings $R_{\mathfrak{p}}$, where $\mathfrak{p} \in \operatorname{Spec} R \setminus \{\mathfrak{m}\}$, are regular, then $X$ is a regular variety and any *ACM* coherent sheaf over $X$ is locally free (or a vector bundle).

Recall definitions of *Cohen–Macaulay types* of Cohen–Macaulay algebras (in graded case).

**Definition 1.1.** Let $R$ be a graded Cohen–Macaulay algebra. It is said to be

- *Cohen–Macaulay finite* (*CM-finite*) if there is only a finite number of indecomposable graded Cohen–Macaulay $R$-modules (up to shift and isomorphism); otherwise it is said to be *Cohen–Macaulay infinite* (*CM-infinite*);
- *Cohen–Macaulay discrete* (*CM-discrete*) if, for any fixed rank $r$, there are only finitely many Cohen–Macaulay $R$-modules (up to shift and isomorphism).[1]

To give definitions of other Cohen–Macaulay types we need the notion of *families* of Cohen–Macaulay modules analogous to that used in the representation theory of algebras (cf. [9]).

**Definition 1.2.** (1) Let $\Lambda$ be a $\Bbbk$-algebra (not necessarily commutative). We consider $R \otimes \Lambda$ as a graded algebra setting $(R \otimes \Lambda)_i = R_i \otimes \Lambda$. A *family of graded* (*maximal*) *Cohen–Macaulay $R$-modules over* $\Lambda$ is a finitely generated graded $R$-$\Lambda$-bimodule $\mathcal{M}$ such that $\mathcal{M}$ is flat over $\Lambda$ and for every finite dimensional $\Lambda$-module $L$ the $R$-module $\mathcal{M} \otimes_\Lambda L$ is (maximal) Cohen–Macaulay over $R$.

We say that the $R$-modules isomorphic to $\mathcal{M} \otimes_\Lambda L$ *belong to the family* $\mathcal{M}$.

(2) A family $\mathcal{M}$ of Cohen–Macaulay $R$-modules over $\Lambda$ is said to be *strict* if, for any finite dimensional $\Lambda$-modules $L$ and $L'$,
  (a) $\mathcal{M} \otimes_\Lambda L \simeq \mathcal{M} \otimes_\Lambda L'$ implies $L \simeq L'$;
  (b) if $L$ is indecomposable, so is $\mathcal{M} \otimes_\Lambda L$.

[1] Sometimes they say *countable* instead of *discrete*, since in this case there is only a countable set of Cohen–Macaulay $R$-modules (up to isomorphism). Nevertheless, it is not very felicitous, since if the field $\Bbbk$ is countable it is always the case.

If $\mathcal{M}$ is strict and $\Lambda = \Bbbk[Y]$, where $Y$ is an affine algebraic variety over $\Bbbk$ of dimension $n$, we say that $\mathcal{M}$ is an *$n$-parameter family of graded Cohen–Macaulay $R$-modules*. Note that even in this case our definition does not coincide with the "usual" definition of a flat family of $R$-modules with the base $Y$. Obviously, a family in our sense is also a flat family with the base $Y$, but we do not know whether (or when) the contrary is true.

**Definition 1.3.** A graded Cohen–Macaulay algebra $R$ is said to be

(1) *strictly Cohen–Macaulay infinite* (*strictly CM-infinite*) if it has a strict family of graded Cohen–Macaulay modules over the polynomial algebra $\Bbbk[x]$.[2]
(2) *Cohen–Macaulay tame* (*CM-tame*) if, for any fixed rank $r$ there is a finite set $\mathfrak{F}$ of strict one-parameter families of graded Cohen–Macaulay $R$-modules such that for every indecomposable graded Cohen–Macaulay $R$-module $M$ of rank $r$ some shift $M(k)$ belongs to a family from $\mathfrak{F}$.
(3) *algebraically Cohen–Macaulay wild* (or briefly *CM-wild*) if for any finitely generated $\Bbbk$-algebra $\Lambda$ there is a strict family of graded Cohen–Macaulay $R$-modules over $\Lambda$.

The last condition means non-formally, that the classification of graded Cohen–Macaulay $R$-modules is at least as difficult as a classification of representations of all finitely generated algebras (commutative or non-commutative).

If $R$ is the homogeneous coordinate ring of a projective variety $X \subset \mathbb{P}^n$ and it is CM-finite (respectively, CM-discrete, CM-tame, strictly CM-infinite or CM-wild), we say that $X$ is *ACM-finite* (respectively, *ACM-discrete, ACM-tame, strictly ACM-infinite* or *ACM-wild*).

*Remark* 1.4.

(1) If $R$ is strictly CM-infinite, it cannot be CM-discrete (hence cannot be CM-finite), since the modules $\mathcal{M}\otimes_{\Bbbk[t]}\Bbbk[t]/(t-a)$, where $a \in \Bbbk$, are indecomposable and of the same rank (recall that we suppose the field $\Bbbk$ to be infinite).
(2) It is obvious that if $R$ is algebraically Cohen–Macaulay wild, it is also *geometrically Cohen–Macaulay wild* in the sense that for every algebraic variety $Y$ there is a flat family of graded Cohen–Macaulay $R$-modules with the base $Y$ such that all modules in this family are indecomposable and pairwise non-isomorphic. It is not known (though conjectured) whether geometrical wildness implies the algebraical one. For projective curves (that

[2] Obviously, in this case there are infinitely many ranks such that there are infinitely many indecomposable Cohen–Macaulay modules of this rank.

is for two-dimensional standard graded Cohen–Macaulay algebras) it follows from the criterion of wildness proved in [10]. There is a lot of examples of algebras known to be geometrically Cohen–Macaulay wild such that their algebraic wildness has not been proved. The simplest is that of cubic surfaces in $\mathbb{P}^3$ [5]; other examples are in [6, 19, 20].

(3) It is known (and easy to see) that in order to show that $R$ is CM-wild it is enough to construct a strict family of graded Cohen–Macaulay $R$-modules over *some* algebra $\Lambda$ which is wild in the sense of representation theory [9, Proposition 5.3]. Examples of such algebras are the free algebra $\Bbbk\langle\, x, y\,\rangle$, the polynomial algebra $\Bbbk[x, y]$ or its localization, the power series algebra $\Bbbk[[x, y]]$ [16].

## 2. The Main Theorem

From now on $R$ denotes a standard graded Cohen–Macaulay algebra of Krull dimension $d$. The following result gives some sufficient conditions for $R$ to be "bad" from the point of view of classification of Cohen–Macaulay modules.

**Theorem 2.1.** *Let $\mathbf{y} = (y_1, y_2, \dots, y_d)$ be an $R$-sequence, where $y_i$ is a homogeneous element of degree $m_i > 0$, $m = \sum_{i=1}^{d} m_i$ and $\overline{R} = R/\mathbf{y}$. Consider a homogeneous component $\overline{R}_c$, where $c > m - d + 1$.*

(1) *If $\dim \overline{R}_c > 1$, then $R$ is strictly CM-infinite.*[3]
(2) *If $\dim \overline{R}_c > 2$, then $R$ is CM-wild.*

*Equivalently, if $R$ is a homogeneous coordinate ring of a projective variety $X$ in $\mathbb{P}^n$, this variety is, respectively, strictly ACM-infinite or ACM-wild.*

The proof of this theorem grounds on the relation between some $\overline{R}$-modules and Cohen–Macaulay $R$-modules. First we prove several lemmas. We always keep the notations and suppositions of Theorem 2.1.

**Lemma 2.2.** *Let $F$ be a free graded $R$-module, $L$ be a finitely generated graded $R$-module, $\varphi : L \to F$ be such a homomorphism that the induced map $\bar\varphi : L/\mathfrak{m}L \to F/\mathfrak{m}F$ is injective. Then $\varphi$ is a split monomorphism, i.e. there is a homomorphism $\varphi' : F \to L$ such that $\varphi'\varphi = 1_L$. Thus $F = \operatorname{Im}\varphi \oplus F'$, where $F'$ is also free.*

*Proof.* There is a homomorphism $\bar\pi : F/\mathfrak{m}F \to L/\mathfrak{m}L$ such that $\bar\pi\bar\varphi = 1_{L/\mathfrak{m}L}$. It can be lifted to a homomorphism $\pi : F \to L$ such that $\operatorname{Im}(1 - \pi\varphi) \subseteq \mathfrak{m}L$. Then $\pi\varphi$ is invertible, so $\varphi$ is a split monomorphism. $\square$

[3] In [?] it has been proved that in this case $R$ is not CM discrete ("countable").

We denote by $\Omega^i(M)$ the *i-th syzygy* of the (graded) $R$-module $M$, i.e. the kernel of the map $\delta_{i-1}$ (or, the same, $\operatorname{Im}\delta_i$), where

$$(F_\bullet, \delta_\bullet): \quad ... \xrightarrow{\delta_{i+1}} F_i \xrightarrow{\delta_i} F_{i-1} \xrightarrow{\delta_{i-1}} \dots \xrightarrow{\delta_2} F_1 \xrightarrow{\delta_1} F_0 \xrightarrow{\delta_0} M \to 0$$

is the minimal graded free resolution of $M$. Note that $\Omega^d(M)$ is always a Cohen–Macaulay $R$-module. We also set $\overline{M} = M \otimes_R \overline{R} = M/\mathbf{y}M$, in particular, $\overline{\Omega}^i(M) = \Omega^i(M) \otimes_R \overline{R}$.

**Lemma 2.3.** *Let $W$ be a subspace of $n\overline{R}_c$, $\langle W \rangle$ be the $R$-submodule generated by $W$ and $M = n\overline{R}/\langle W \rangle$. Then $M(-m)$ is isomorphic to the submodule of $\overline{\Omega}^d(M)$ generated by the elements of degree $m$.*

**Corollary 2.4** (cf. [14, Theorem A])**.** *Let $W$ and $W'$ are two subspaces of $n\overline{R}_c$, $M = n\overline{R}/\langle W \rangle$, $M' = n\overline{R}/\langle W' \rangle$. The following conditions are equivalent:*

(1) *$M \simeq M'$ (as graded $R$-modules).*
(2) *$\Omega^d(M) \simeq \Omega^d(M')$.*
(3) *There is an automorphism $\sigma$ of $n\overline{R}$ such that $\sigma(W) = W'$.*

Moreover, $M$ is indecomposable if and only if so is $\Omega^d(M)$.

*Proof.* (1)⇒(2) and (3)⇒(1) are trivial.

(2)⇒(1) follows from Lemma 2.3, just as the statement about indecomposability.

(1)⇒(3). Any isomorphism $f : M \to M'$ can be lifted to a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \langle W \rangle \arrow[r] \arrow[d] & n\overline{R} \arrow[r] \arrow[d, "\sigma"] & M \arrow[r] \arrow[d, "f"] & 0 \\ 0 \arrow[r] & \langle W' \rangle \arrow[r] & n\overline{R} \arrow[r] & M' \arrow[r] & 0, \end{tikzcd}$$

where $\sigma$ is also an isomorphism. Then $\sigma(W) = W'$ since $W = \langle W \rangle_c$ and $W' = \langle W' \rangle_c$. □

Before proving Lemma 2.3 we establish the following fact. We keep the conditions of this Lemma and denote by

$(K_\bullet, \partial_\bullet)$ the Koszul resolution of $\overline{R} = R/\mathbf{y}$;

$(F_\bullet, \delta_\bullet)$ a minimal graded free resolution of $M$;

$(\mathbf{K}_\bullet, \mathbf{d}_\bullet) = n(K_\bullet, \partial_\bullet)$, the $n$-fold Koszul complex, which is a minimal graded free resolution of $n\overline{R}$;

$\varphi_\bullet : (\mathbf{K}_\bullet, \mathbf{d}_\bullet) \to (F_\bullet, \delta_\bullet)$ a morphism of complexes induced by the epimorphism $n\overline{R} \to M$.

**Lemma 2.5.** *The map* $\varphi_i$ *is a split monomorphism for each* $i$ *and* $F_i = \operatorname{Im}\varphi_i \oplus F_i'$, *where* $F_i'$ *contains no homogeneous elements of degrees less than* $c+i-1 > m$.

*Proof.* We use induction on $i$. For $i = 0, 1$ it follows from the definition of $(F_\bullet, \delta_\bullet)$. Suppose that the claim is true for all $i \le j$. We identify $\operatorname{Im}\varphi_i$ with $\mathbf{K}_i$. Since $\mathfrak{m}F_{j-1}'$ contains no elements of degree less than $c+j-2 > m$, while $\mathbf{K}_j$ is generated by elements of degree at most $m$, the matrix presentation of the homomorphism $\delta_j : \mathbf{K}_j \oplus F_j' \to \mathbf{K}_{j-1} \oplus F_{j-1}'$ is of the form

$$\delta_j = \begin{pmatrix} \mathbf{d}_j & \beta_j \\ 0 & \gamma_j \end{pmatrix}$$

Let $\delta_{j+1} : F_{j+1} \to \mathbf{K}_j \oplus F_j'$ be of the form $\left(\begin{smallmatrix}\alpha \\ \beta\end{smallmatrix}\right)$. Then $\mathbf{d}_j\alpha + \beta_j\beta = 0$. Consider any element $u \in F_{j+1}$ of degree $k < c+j$. Then $\beta(u) = 0$ since there are no elements of this degree in $\mathfrak{m}F_j'$, hence $\mathbf{d}_j\alpha(u) = 0$ and $\alpha(u) \in \operatorname{Ker}\mathbf{d}_j = \operatorname{Im}\mathbf{d}_{j+1}$. So $\alpha(u) = \mathbf{d}_{j+1}(v) = \alpha\varphi_{j+1}(v)$ for some $v \in \mathbf{K}_{j+1}$ and $u - \varphi_{j+1}(v) \in \operatorname{Ker}\delta_{j+1} \subseteq \mathfrak{m}F_{j+1}$. Therefore $F_{j+1} = \operatorname{Im}\varphi_{j+1} + \tilde{F}$, where $\tilde{F}$ is generated by elements of degrees at least $c+j$. Since $\delta_{j+1}\varphi_{j+1} = \varphi_j\mathbf{d}_{j+1}$, $\delta_{j+1}(\operatorname{Im}\varphi_{j+1}) \subseteq \mathbf{y}F_j$. Note also that $\beta\varphi_{j+1} = 0$, since $\mathbf{K}_{j+1}$ is generated by elements of degrees at most $m$.

Now consider the maps $\tilde{\varphi}_i : \mathbf{K}_i/\mathfrak{m}\mathbf{K}_i \to F_i/\mathfrak{m}F_i$ induced by $\varphi_i$. They are injective for $i \le j$. Let $v \notin \mathfrak{m}\mathbf{K}_{j+1}$ be such that $\varphi_{j+1}(v) \in \mathfrak{m}F_j$. Then $\varphi_{j+1}(v) \in \mathfrak{m}(\operatorname{Im}\varphi_{j+1})$, since all elements of $\tilde{F}$ are of degrees greater than $m$ and $\deg v \le m$. Thus $\varphi_j\mathbf{d}_{j+1}(v) = \delta_{j+1}\varphi_{j+1}(v) \in \mathbf{y}\mathfrak{m}F_j$. It is impossible, since $\mathbf{d}_{j+1}(v) \notin \mathfrak{m}\mathbf{y}\mathbf{K}_j$ and $\varphi_j$ is a split monomorphism. Therefore, $\tilde{\varphi}_{j+1}$ is a monomorphism and $\varphi_{j+1}$ is a split monomorphism by Lemma 2.2. □

*Proof of Lemma 2.3.* Consider the exact sequence

$$0 \to \Omega^d(M) \xrightarrow{\iota} F_{d-1} \to \Omega^{d-1}(M) \to 0.$$

Tensoring it with $\overline{R}$ we get an exact sequence

$$0 \to \operatorname{Ker}\bar{\iota} \to \overline{\Omega}^d(M) \xrightarrow{\bar{\iota}} \bar{F}_d \to \bar{\iota}^{d-1}(M) \to 0,$$

where $\operatorname{Ker}\bar{\iota} \simeq \operatorname{Tor}_1^R(\Omega^{d-1}(M), \overline{R}) \simeq \operatorname{Tor}_d^R(M, \overline{R})$. On the other hand,

$$\operatorname{Tor}_d^R(M, \overline{R}) \simeq \operatorname{Ker}\{1 \otimes \partial_d : M \otimes_R K_d \to M \otimes_R K_{d-1}\} \simeq M(-m),$$

since $K_d \simeq R(-m)$, $\operatorname{Im}\partial_d \subseteq \mathbf{y}K_{d-1}$ and $\mathbf{y}M = 0$. Hence $\operatorname{Ker}\bar{\iota}$ is generated by elements of degree $m$. On the other hand, all elements of degree $m$ in $\Omega^d(M)$ are in $\operatorname{Im}\mathbf{d}_d$, hence in $\mathbf{y}F_{d-1}$. Therefore, all elements of degree $m$ in $\overline{\Omega}^d(M)$ are in $\operatorname{Ker}\bar{\iota}$. It means that $\operatorname{Ker}\bar{\iota} \simeq M(-m)$ coincides with the submodule of $\overline{\Omega}^d(M)$ generated by elements of degree $m$. □

As we use $R$-$\Lambda$-bimodules, or, the same, $R\otimes\Lambda$-modules, we also need the following fact about projective resolutions, which follows from the Grothendieck's Generic Freeness Lemma [13, Theorem 14.4].

**Proposition 2.6.** *Let $\Lambda$ be a finitely generated commutative algebra without zero divisors, $\mathcal{M}$ be a finitely generated $R\otimes\Lambda$-module and $k$ be a positive integer. There is a non-zero element $a\in\Lambda$ such that there is a free resolution of $\mathcal{M}[a^{-1}]$ as of $R\otimes\Lambda[a^{-1}]$-module*

$$\cdots\to\mathcal{F}_k\xrightarrow{\Delta_k}\mathcal{F}_{k-1}\xrightarrow{\Delta_{k-1}}\cdots\to\mathcal{F}_1\xrightarrow{\Delta_1}\mathcal{F}_0\xrightarrow{\Delta_0}\mathcal{M}[a^{-1}]\to 0 \tag{2.1}$$

*such that $\mathcal{M}[a^{-1}]$ and $\operatorname{Ker}\Delta_i$ are free as $\Lambda[a^{-1}]$-module and $\operatorname{Ker}\Delta_i\subseteq\mathfrak{m}\mathcal{F}_i$ for all $i\le k$.*

*Proof.* We use induction by $r$. By the Grothendieck's Generic Freeness Lemma there is a non-zero $a\in\Lambda$ such that $\mathcal{M}[a^{-1}]$ and all homogeneous components of $\overline{\mathcal{M}}=\mathcal{M}[a^{-1}]/\mathfrak{m}\mathcal{M}[a^{-1}]$ are free over $\Lambda'=\Lambda[a^{-1}]$. Set $\mathcal{F}_0=\bigoplus_i R[-i]\otimes\overline{\mathcal{M}}_i$. It is a free $R\otimes\Lambda'$-module and there is an epimorphism $\Delta_0:\mathcal{F}_0\to\mathcal{M}[a^{-1}]$ such that $\operatorname{Ker}\Delta_0\subseteq\mathfrak{m}\mathcal{F}_0$. Obviously, $\operatorname{Ker}\Delta_0$ is also free over $\Lambda'$. It proves the statement for $k=0$. Suppose that we already have found $a$ and a resolution (2.1) for given $k$. Applying the same procedure to $\operatorname{Ker}\Delta_k$, we obtain an analogous resolution for $k+1$. □

*Proof of Theorem 2.1.* We prove the statement (2), since the proof of (1) is quite analogous. Namely, we construct a family of graded Cohen–Macaulay $R$-modules over a localization of the polynomial ring $\Lambda=\Bbbk[x,y]$.

Suppose that $\dim_{\Bbbk}\overline{R}_c>2$ and $e_1,e_2,e_3$ are linear independent elements from $\overline{R}_c$. In the $R\otimes\Lambda$-module $\overline{R}\otimes\Lambda$ consider the $\Lambda$-submodule $\mathcal{W}$ generated by $e_1\otimes 1+e_2\otimes x+e_3*y$. Note that $\mathcal{W}\simeq\Lambda$ as $\Lambda$-module. Let $\mathcal{M}=\overline{R}\otimes\Lambda/\langle\,\mathcal{W}\,\rangle$, where $\langle\,\mathcal{W}\,\rangle$ is the $\overline{R}\otimes\Lambda$-submodule generated by $\mathcal{W}$. Applying Proposition 2.6, we find a non-zero element $a\in\Lambda$ and a free resolution of the shape (2.1) for $\mathcal{M}'=\mathcal{M}[a^{-1}]$ over $R\otimes\Lambda$, where $\Lambda'=\Lambda[a^{-1}]$. Let $\varOmega=\operatorname{Ker}\Delta_{d-1}=\operatorname{Im}\Delta_d$. We prove that $\varOmega$ is a strict family of graded Cohen–Macaulay $R$-modules over $\Lambda'$. The exact sequence (2.1) splits as a sequence of $\Lambda'$-modules. Hence for any $\Lambda'$-module $L$ the sequence

$$\cdots\to\mathcal{F}_d\otimes_{\Lambda'}L\xrightarrow{\Delta_d\otimes 1}\mathcal{F}_{d-1}\otimes_{\Lambda'}L\xrightarrow{\Delta_{d-1}\otimes 1}\cdots\to$$
$$\to\mathcal{F}_1\otimes_{\Lambda'}L\xrightarrow{\Delta_1\otimes 1}\mathcal{F}_0\otimes_{\Lambda'}L\xrightarrow{\Delta_0\otimes 1}\mathcal{M}'\otimes_{\Lambda'}L\to 0$$

is exact, so is a minimal resolution of the $R$-module $\mathcal{M}'\otimes_{\Delta}L$. In particular, $\varOmega\otimes_{\Delta}L\simeq\Omega^d(\mathcal{M}'\otimes_{\Delta}L)$ is a Cohen–Macaulay $R$-module.

A finite dimensional $\Lambda'$-module $L$ is given by two square $n \times n$ matrices $A_x, A_y$ over $\Bbbk$, where $n = \dim_{\Bbbk} L$, describing respectively the action of $x$ and $y$ in a basis of $L$. Then $\mathcal{M}' \otimes_{\Lambda'} L \simeq n\overline{R}/\langle\, \mathcal{W} \otimes_{\Lambda'} L\,\rangle$ and $\langle\, \mathcal{W} \otimes_{\Lambda'} L\,\rangle$ is the subspace of $n\overline{R}_c$ generated by the columns of the matrix $Ie_1 + A_x e_2 + A_y e_3$, where $I$ is the $n \times n$ unit matrix. Suppose that $L'$ is another $La'$-modules of the same dimension given by a pair of matrices $B_x, B_y$ and $\mathcal{M}' \otimes_{\Lambda'} L \simeq \mathcal{M}' \otimes_{\Lambda'} L'$. By Lemma 2.3, there is an automorphism $\sigma$ of $n\overline{R}$ such that $\sigma(\mathcal{W} \otimes_{\Lambda'} L) = \mathcal{W} \otimes_{\Lambda'} L'$. This automorphism can be considered as invertible $n \times n$ matrix over $\Bbbk$ and $\sigma(\mathcal{W} \otimes_{\Lambda'} L) = \mathcal{W} \otimes_{\Lambda'} L'$ means that there is an invertible matrix $\tau$ such that $\sigma(Ie_1 + A_x e_2 + A_y e_3) = (Ie_1 + B_x e_2 + B_y e_3)\tau$. As the elements $e_1, e_2, e_3$ are linear independent, we obtain that $\sigma = \tau$, $B_x = \sigma A_x \sigma^{-1}$ and $B_y = \sigma A_y \sigma^{-1}$, hence $L \simeq L'$. Analogously one proves that if $L$ is indecomposable, $\Omega \otimes_{\Lambda'} L$ is indecomposable as well. □

## 3. Applications

Theorem 2.1 implies wildness conditions for special types of algebras (or varieties). In what follows we write $\Bbbk[\mathbf{x}] = \Bbbk[x_0, x_1, \dots, x_n]$.

**Corollary 3.1.** *A hypersurface of degree $e \geq 4$ in $\mathbb{P}^n$ is strictly ACM-infinite. If $n \geq 2$, it is ACM-wild.*

*Proof.* Consider the homogeneous coordinate ring $R = \Bbbk[\mathbf{x}]/(f)$ of $X$, where $\deg f = e$. It is of Krull dimension $n$. As $\Bbbk$ is infinite, we can suppose that $(x_1, x_2, \dots, x_n)$ is an $R$-sequence. Set first $y_1 = x_1^2$ and, if $n > 1$, $y_i = x_i$ for $2 \leq i \leq n$. Then $m - d + 2 = 3$ and $\dim_{\Bbbk} \overline{R}_3 = 2$, so $R$ is strictly CM-infinite.

If $n \geq 2$, set $y_1 = x_1^2$, $y_2 = x_2^2$ and, if $n > 2$, $y_i = x_i$ for $3 \leq i \leq n$. Then $m - d + 2 = 4$ and $\dim_{\Bbbk} \overline{R}_4 \geq 3$. Hence $R$ is CM-wild. □

Note that similar result for local rings was obtained by V. V. Bondarenko [2] using the technique of matrix factorizations. For graded rings it has also been proved in [7].

**Corollary 3.2.** *Let $X \subset \mathbb{P}^n$ be a complete intersection in $\mathbb{P}^n$ of codimenion $k > 1$ defined by polynomials $f_1, f_2, \dots, f_k$ of degrees $\deg f_j = d_j > 1$. If $k \geq 3$ or $d_1 \geq 3$, then $X$ is ACM-wild.*

*Proof.* We consider again the homogeneous coordinate ring $R$ of $X$, $R = \Bbbk[\mathbf{x}]/(f_1, f_2, \dots, f_k)$, and suppose that $x_k, x_{k+1}, \dots, x_n$ is an $R$-sequence. Set $y_1 = x_k^2$ and, if $n > k$, $y_i = x_{k+i-1}$ for $1 < i \leq n - k + 1$. Then $m - d + 2 = 3$ and one easily check that $\dim \overline{R}_3 \geq 4$ if either $k \geq 3$ or $d_1 \geq 3$. Therefore, $R$ is CM-wild. □

Note that cubic surfaces in $\mathbb{P}^3$ as well as intersections of two quadrics in $\mathbb{P}^4$ are known to be geometrically Cohen–Macaulay wild [5, 20]. The proof that they are algebraically Cohen–Macaulay wild, as well as their analogues of higher dimensions, is an intriguing problem. Actually, in [5, 20] it is proved even more: there are families of non-isomorphic indecomposable *Ulrich* bundles of arbitrary big dimensions. In [3] it is proved that a *local* algebra $\Bbbk[[x_0, x_1, \dots, x_n]]/(f)$ is CM-wild if $n \geq 3$ and $f$ does not contain linear and quadratic terms.

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