HARISH–CHANDRA S-HOMOMORPHISM AND $\mathfrak{G}$-MODULES GENERATED BY A SEMIPRIMITIVE ELEMENT

Yu. A. Drozd, S. A. Ovsienko, and V. M. Futornyi

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We define Harish–Chandra S-homomorphism which generalizes the classical Harish–Chandra homomorphism and study its properties. For $\mathfrak{G}$-modules $(\mathfrak{G} \neq E_7, E_8)$, generated by semiprimitive elements we prove the existence of composition sequences.

In this article we construct a generalization of the classical Harish–Chandra homomorphism [1]. We then use our results to study the structure of modules generated by semiprimitive elements [2, 3].

1. The Harish–Chandra S-Homomorphism and Its Properties. Let $\mathfrak{G}$ be a complex simple finite-dimensional Lie algebra of rank n, $\mathfrak{H}$ its cartesian subalgebra, Δ a system of roots of $\mathfrak{G}$, π a basis of the system of roots Δ, $\Delta^+ = \Delta^+(\pi)$ the set of positive roots in Δ with respect to π, $W = W(\Delta)$ the Weyl group of the system Δ, and $\mathfrak{G}^\alpha$ the root subspace corresponding to a root α.

Let $\pi = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, $S \subsetneq \{1, 2, \ldots, n\}$, $\pi_S = \{\alpha_i \mid i \in S\}$. Denote by $\Delta_S = \Delta_S(\pi_S)$ a subsystem of roots in Δ generated by π_S. Furthermore, let $\{H_\alpha \mid \alpha \in \pi\}$ be a basis of the Cartesian subalgebra $\mathfrak{H}$, such that $\alpha(H_\alpha) = 2$ for all $\alpha \in \pi$. Let $\mathfrak{H}_S$ and $\mathfrak{H}^S$ be subalgebras $\langle H_\alpha \mid \alpha \in \pi_S\rangle$ and $\{H \in \mathfrak{H} \mid \alpha(H) = 0$ for all $\alpha \in \Delta_S\}$, respectively.

For every $\alpha \in \Delta$ choose $X_\alpha \in \mathfrak{G}^\alpha \setminus \{0\}$ and define the following subalgebras of $\mathfrak{G}$:

$$\mathfrak{G}_S = \langle X_{\pm\alpha} \mid \alpha \in \pi_S\rangle, \quad \mathfrak{N}_S^+ = \sum_{\alpha \in \Delta^+ \setminus \Delta_S} \mathfrak{G}^\alpha, \quad \mathfrak{N}_S^- = \sum_{\alpha \in \Delta^+ \setminus \Delta_S} \mathfrak{G}^{-\alpha}.$$

Let $U(\mathfrak{G})$ be the universal enveloping algebra of the algebra $\mathfrak{G}$, and let $Z(\mathfrak{G})$ be the center of $U(\mathfrak{G})$. Let Q (respectively, Q_S) be the group of radical weights of the system (respectively, Δ_S). Then the $\mathfrak{G}$-module structure with respect to adjoint representation on $U(\mathfrak{G})$ defines a Q-graduation on it: $U(\mathfrak{G}) = \bigoplus_{\lambda \in Q} U(\mathfrak{G})_\lambda$.

LEMMA 1. Let $L_S = U(\mathfrak{G})\mathfrak{N}_S^+ \cap U(\mathfrak{G})_0$. Then 1) L_S is a two-sided ideal in $U(\mathfrak{G})_0$, 2) $L_S = \mathfrak{N}_S^- U(\mathfrak{G}) \cap U(\mathfrak{G})_0$; 3) $U(\mathfrak{G})_0 = L_S \oplus U(\mathfrak{G}_S)_0 \otimes U(\mathfrak{H}^S)$.

The lemma is proven analogously to Lemma 7.4.2 in [1].

Definition: A Harish–Chandra S-homomorphism (with respect to a basis π) is a projection $\varphi_{S,\pi}: U(\mathfrak{G})_0 \to U(\mathfrak{G}_S)_0 \otimes S(\mathfrak{H}^S)$.

Remarks. 1. $\operatorname{Ker}\varphi_{S,\pi} = L_S$. 2. A Harish–Chandra S-homomorphism $\varphi_{S,\pi}$ is uniquely defined by a set $\Delta^+(\pi) \setminus \Delta_S(\pi_S)$. 3. $\varphi_{\varnothing,\pi}$ is the classical Harish–Chandra homomorphism with respect to the basis π. 4. $\operatorname{Ker} \times (\varphi_{S,\pi}|_{Z(\mathfrak{G})}) = 0$ and $\varphi_{S,\pi}(Z(\mathfrak{G})) \subset Z(\mathfrak{G}_S) \otimes S(\mathfrak{H}^S)$ where $Z(\mathfrak{G}_S)$ is the center of the universal enveloping algebra $U(\mathfrak{G}_S)$.

It is known that Harish–Chandra ϕ-homomorphisms are in a one-to-one correspondence with bases of the system of roots Δ, i.e., their number is equal to $|W|$. Harish–Chandra S-homomorphisms are described similarly.

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LEMMA 2. Let $S\subset\{1,2,\dots,n\}$, $\Omega=\{\Delta^{+}(\omega\pi)\setminus\Delta_S(w\pi_S)\mid w\in W\}$. The group W acts on the set Ω transitively.

The proof is obvious, since $\Delta^{+}(w\pi)\setminus\Delta_S(w\pi_S)=w\times(\Delta^{+}(\pi)\setminus\Delta_S(\pi_S))$ for all W ∈ W.

Proposition 1. Let $S\subset\{1,2,\dots,n\}$, W_S the Weyl group of the system of roots Δ_S, and $N(W_S)$ the normalizer of the group W_S in W. Then 1) Harish–Chandra S-homomorphisms are in one-to-one correspondence with cosets W/W_S, and 2) Harish–Chandra S-homomorphisms with a fixed root subsystem $\Delta_S(\pi)$ are in one-to-one correspondence with cosets $N(W_S)/W_S$.

Proof: Define a set $\Omega=\{\Delta^{+}(w\pi)\setminus\Delta_S(w\pi_S)\times\mid w\in W\}$. The number of different Harish–Chandra S-homomorphisms is equal to $|\Omega|$. Fix a set $\Delta^{+}(\pi)\setminus\Delta_S(\pi)\in\Omega$. Then $\mathrm{st}(\Delta^{+}(\pi)\setminus\Delta_S(\pi))=W_S$ with respect to the action of the group W on Ω. Therefore, assertion 1 follows from Lemma 2. Furthermore, every element $\Delta^{+}(w\pi)\setminus\Delta_S(w\pi_S)$ of the set Ω uniquely defines a set $\Delta_S(w\pi_S)$, but this correspondence is not injective. This means that there exist two distinct $w_1,w_2\in W$ such that $\Delta_S(w_1\pi_S)=\Delta_S(w_2\pi_S)$. Consider the natural transitive action of the group W on the set of pairs $\tilde{\Omega}=\{(\Delta^{+}(w\pi)\setminus\Delta_S(w\pi_S),\ \Delta_S(w\pi_S))\mid w\in W\}$. We see that the number of different Harish–Chandra S-homomorphisms with a fixed $\Delta_S(\pi)$ is equal to the number of different elements in $\tilde{\Omega}$ with $\Delta_S(\pi)$ at the second place, i.e., $(\mathrm{st}(\Delta_S(\pi)):\mathrm{st}(\Delta^{+}(\pi)\setminus\Delta_S(\pi)))=(N(W_S):W_S)$, which proves statement 2.

Let V be some weight $\mathfrak{G}$-module, i.e., $V=\bigoplus_{\lambda\in\mathfrak{H}^*}V_\lambda$, where $V_\lambda=\{v\in V\mid Hv=\lambda(H)v$ for all $H\in\mathfrak{H}\}$. Let $\operatorname{supp}V=\{\lambda\in\mathfrak{H}^*\mid V_\lambda\neq 0\}$. The elements of supp V are called the weights of the module V.

The following proposition plays an important role in the theory of weight $\mathfrak{H}$-modules.

LEMMA 3. 1. Let V be an irreducible weight $\mathfrak{G}$-module and $\lambda\in\operatorname{supp}V$. Then V_λ is an irreducible $U(\mathfrak{G})_0$- module.

2. Let V' be an irreducible $U(\mathfrak{G})_0$-module such that $Hv=\lambda(H)v$ for all $H\in\mathfrak{H}$, $v\in V'$. Then there exists a unique irreducible weight $\mathfrak{G}$-module V such that $V_\lambda\simeq V'$.

Proof: Let $V_\lambda\supset U$ be a proper $U(\mathfrak{G})_0$-submodule. Then, $U(\mathfrak{G})U\subsetneq V$, which contradicts the irreducibility of V. 2. A weight $\mathfrak{G}$-module $M=U(\mathfrak{G})\otimes_{U(\mathfrak{G})_0}V'$ has only one maximal submodule $\mathfrak{M}$ and $(M/\mathfrak{M})_\lambda\simeq V'$. If L is an irreducible $\mathfrak{G}$-module and $L_\lambda\simeq V'$ then there exists an epimorphism $\chi:M\to L$. Therefore, $M/\mathfrak{M}\simeq L$. Q.E.D.

LEMMA 4. Every irreducible $U(\mathfrak{G}_S)_0\otimes S(\mathfrak{H}^S)$-module V' such that $Hv=\lambda(H)v$ for all $H\in\mathfrak{H}$, $v\in V'$ extends to an irreducible weight $\mathfrak{G}$-module V such that $V_\lambda\simeq V'$.

Proof: It suffices to use the Harsh–Chandra S-homomorphism and Lemma 3 of section 2.

Lemma 4 allows us to extend irreducible weight $\mathfrak{G}$-modules to irreducible weight $\mathfrak{G}$-modules.

Definition: 1. An S-primitive element of weight λ with respect to a basis π is a nonzero element v such that $Hv=\lambda(H)v$ for all $H\in\mathfrak{H}$ and $\mathfrak{N}_S^{+}v=0$. 2. An element v ∈ V is called semi-primitive of weight λ if, for some basis π of the system of roots Δ and some $S\subset\{1,2,\dots,n\}$, v is an S-primitive element of weight λ with respect to π [3].

Remark. The definition of S-primitive elements in the case where S = ϕ coincides with the well-known definition of primitive elements.

Proposition 2. Let V be some $\mathfrak{G}$-module generated by an S-primitive element v of weight λ with respect to a basis π, θ a central character of the module V, and $\theta_S:Z(\mathfrak{G}_S)\to\mathbb{C}$, where $zv=\theta_S(z)$ for all $z\in Z(\mathfrak{G}_S)$. Let λ^S be the restriction of λ to $\mathfrak{H}^S$. Then $\theta(z)=(\theta_S\otimes\lambda^S)(\varphi_{S,\pi}(z))$ for all $z\in Z(\mathfrak{G})$.

Proof: For every $z\in Z(\mathfrak{G})$ there exist $u_1,u_2,\dots,u_k\in U(\mathfrak{G})$ and $a_1,a_2,\dots,a_k\in\mathfrak{N}_S^{+}$ such that $z=\varphi_{S,\pi}(z)+\sum_{i=1}^{k}u_ia_i$. Then $\theta(z)v=zv=\varphi_{S,\pi}(z)v+\sum_{i=1}^{k}u_ia_iv=\varphi_{S,\pi}(z)v=(\theta_S\otimes\lambda^S)(\varphi_{S,\pi}(z))v$. Since the module V is generated by the element v, we have $\theta(z)=(\theta_S\otimes\lambda^S)(\varphi_{S,\pi}(z))$.

Let $\delta=\frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha$, $\delta_S=\frac{1}{2}\sum_{\alpha\in\Delta_S^+}\alpha$, $\Delta_S^+=\Delta^+\cap\Delta_S$. Let γ be an automorphism of the algebra $S(\mathfrak{H})$ acting as $\gamma(p)(\lambda)=p(\lambda-\delta)$, where $\lambda\in\mathfrak{H}^*$, and p is a polynomial function on $\mathfrak{H}^*$. We similarly define an automorphism γ_S of the algebra $S(\mathfrak{H}_S)$ by letting $\gamma_S(p)(\lambda)=p(\lambda-\delta_S)$. Let $\gamma^S=\gamma|_{S(\mathfrak{H}^S)}$.

LEMMA 5. Suppose $\varphi_{\varnothing,\pi_S}:U(\mathfrak{G}_S)_0\to S(\mathfrak{H}_S)$ is a Harish–Chandra S-homomorphism with respect to a basis π_S. Then the following diagram commutes:

$$\begin{array}{ccc}
U(\mathfrak{G})_0 & \xrightarrow{(1\otimes\gamma^S)\circ\varphi_{S,\pi}} & U(\mathfrak{G}_S)_0\otimes S(\mathfrak{H}^S) \\
\Big\downarrow{\scriptstyle \gamma\circ\varphi_{\varnothing,\pi}} & & \Big\downarrow{\scriptstyle (\gamma_S\circ\varphi_{\varnothing,\pi_S})\otimes 1} \\
S(\mathfrak{H}) & \xleftarrow{\ m\ } & S(\mathfrak{H}_S)\otimes S(\mathfrak{H}^S)
\end{array},$$

where m is a natural isomorphism of $S(\mathfrak{H})$ onto $S(\mathfrak{H}_S)\otimes S(\mathfrak{H}^S)$.

To prove the above lemma it suffices to note that $\gamma|_{S(\mathfrak{H}_S)}=\gamma_S$. Thus, $\gamma\circ\varphi_{\varnothing,\pi}=m\circ((\gamma_S\circ\varphi_{\varnothing,\pi_S})\otimes\gamma^S)\circ\varphi_{S,\pi}$.

Let i be the restriction of a homomorphism $(1\otimes\gamma^S)\circ\varphi_{S,\pi}$ to $Z(\mathfrak{G})$ and i_S the restriction of a homomorphism $(\gamma_S\circ\varphi_{\varnothing,\pi_S})\otimes 1$ to $Z(\mathfrak{G}_S)\otimes S(\mathfrak{H}^S)$. Let j be the natural imbedding of $Z(\mathfrak{G})$ into $U(\mathfrak{G})_0$, and j_S the natural embedding of $Z(\mathfrak{G}_S)\otimes S(\mathfrak{H}^S)$ into $U(\mathfrak{G}_S)_0\otimes S(\mathfrak{H}^S)$. Consider the following commutative diagram:

$$\begin{array}{ccccc}
Z(\mathfrak{G}) & \xrightarrow{\ i\ } & Z(\mathfrak{G}_S)\otimes S(\mathfrak{H}^S) & \xrightarrow{m\circ i_S} & S(\mathfrak{H}) \\
\Big\downarrow{\scriptstyle j} & & \Big\downarrow{\scriptstyle j_S} & & \Big\| \\
U(\mathfrak{G})_0 & \xrightarrow{(1\otimes\gamma^S)\circ\varphi_{S,\pi}} & U(\mathfrak{G}_S)_0\otimes S(\mathfrak{H}^S) & \xrightarrow{m\circ((\gamma_S\circ\varphi_{\varnothing,\pi_S})\otimes 1)} & S(\mathfrak{H}).
\end{array} \qquad (1)$$

Lemma 5 implies that the image of the center $Z(\mathfrak{G})$ under the composition mapping $m\circ i_S)\circ i$ is equal to $S(\mathfrak{H})^W$.

LEMMA 6. $Z(\mathfrak{G}_S)\otimes S(\mathfrak{H}^S)\simeq S(\mathfrak{H})^{W_S}$.

Proof: The commutativity of diagram (1) implies that $Z(\mathfrak{G}_S)\otimes S(\mathfrak{H}^S)\simeq S(\mathfrak{H}_S)^{W_S}\otimes S(\mathfrak{H}^S)$. However, for all $H\in\mathfrak{H}^S$, we have $w\in W_S$ $w(H)=s_{i_1}s_{i_2}\ldots s_{i_k}(H)$, where s_{i_j} is a reflection by a root α_{i_j}, $j=\overline{1,k}$, where all $\alpha_{i_j}\in\Delta_S$. Consequently, $s_{i_j}(H)=H$ for all j = 1, k, so therefore w(H) = H. Thus, $S(\mathfrak{H}^S)=S(\mathfrak{H}^S)^{W_S}$ and $Z(\mathfrak{G}_S)\otimes S(\mathfrak{H}^S)\simeq S(\mathfrak{H}_S)^{W_S}\otimes S(\mathfrak{H}^S)^{W_S}\simeq S(\mathfrak{H})^{W_S}$. Q.E.D.

Denote an isomorphism $Z(\mathfrak{H}_S)\otimes S(\mathfrak{H}^S)\simeq S(\mathfrak{H})^{W_S}$ by ψ_S. Let $N\times(W_S)$ be the normalizer of the group W_S in W. Since for every $w\in N\times(W_S)$ we have $w(S(\mathfrak{H})^{W_S})\subset S(\mathfrak{H})^{W_S}$, for every $b\in Z(\mathfrak{G}_S)\otimes S(\mathfrak{H}^S)$ we can define

$$wb=\psi_S^{-1}(w\psi_S(b)). \qquad (2)$$

Equation (2) defines an action of $N(W_S)$ on $Z(\mathfrak{G}_S)\otimes S(\mathfrak{H}^S)$

THEOREM 1. 1) $i(Z(\mathfrak{G}))\subset(Z(\mathfrak{G}_S)\otimes S(\mathfrak{H}^S))^{N(W_S)}$; 2) $(1\otimes\gamma^S)\circ\varphi_{S,\pi}|_{Z(\mathfrak{G})}=(1\otimes\gamma^S)\circ\varphi_{S,w\pi}|_{Z(\mathfrak{G})}$ for all $w\in N(W_S)$.

Proof: Since $Z(\mathfrak{G})\simeq S(\mathfrak{H})^W$, assertion 1 of theorem follows from Lemma 6 and Eq. (2). Furthermore, since an isomorphism $\gamma\circ\varphi_{\varnothing,\pi}:Z(\mathfrak{G})\to S(\mathfrak{H})^W$ does not depend on the choice of basis π of the root system Δ and $(1\otimes\gamma^S)\circ\varphi_{S,w\pi}(Z(\mathfrak{G}))\subset Z(\mathfrak{G}_S)\otimes S(\mathfrak{H}^S)$ for all $w\in N(W_S)$, assertion 2 of the theorem from the commutativity of the following diagram:

$$\begin{array}{ccc} Z(\mathfrak{G}) & \xrightarrow{i} & Z(\mathfrak{G}_S)\otimes S(\mathfrak{H}^S) \\ \downarrow \gamma\circ\varphi_{\varnothing,\pi} & & \downarrow \psi_S \\ S(\mathfrak{H})^W & \longrightarrow & S(\mathfrak{H})^{W_S} \rightarrow S(\mathfrak{H}). \end{array} \tag{9}$$

Given an algebra A, let $\hat{A}$ be the set of isomorphism classes of irreducible representations of A. Define a natural mapping on characters $\hat{i}: Z(\mathfrak{G}_S)\widehat{\otimes} S(\mathfrak{H}^S) \rightarrow \hat{Z}(\mathfrak{G})$.

Proposition 3. For every $\theta \in \hat{Z}(\mathfrak{G})$ the set $\hat{i}^{-1}(\theta)$ is finite and $|\hat{i}^{-1}(\theta)| \leqslant (W:W_S)$.

Proof: The proof follows from the commutativity of diagram (3) and the fact that the dimension of the quotient field $S(\mathfrak{H})^{W_S}$ over the quotient field $S(\mathfrak{H})^W$ is equal to $(W:W_S)$.

2. Properties of Weight $\mathfrak{G}$-Modules Generated by Semiprimitive Elements. In this section we use results of section 1 to study $\mathfrak{G}$-modules generated by semi-primitive elements.

Recall the construction of the universal $\mathfrak{G}$-module generated by an S-primitive element of weight λ [2]. In the universal enveloping algebra $U(\mathfrak{G})$ define a subalgebra $\Lambda_S = U(\mathfrak{N}_S^+ \oplus \mathfrak{H}^S) + U(\mathfrak{G}_S)_0$.

Let (ρ_S, U) be an irreducible representation of the algebra Λ_S such that $\rho_S(a+H)u = \lambda(H)u$ for all $a \in \mathfrak{N}_S^+, H \in \mathfrak{H}, u \in U$. Defining an $\mathfrak{G}$-module $M(S,\pi,\lambda+\delta,\rho_S) = U(\mathfrak{G}) \underset{\Lambda_S}{\otimes} U$. Clearly, the module $M(S,\pi,\lambda+\delta,\rho_S)$ is a weight $\mathfrak{G}$-module, $M(S,\pi,\lambda+\delta,\rho_S)_\lambda \simeq U$, and every element of this subspace is a generating S-primitive element of weight λ.

The universality of the module is characterized by the following proposition.

Proposition 4. Suppose V is a $\mathfrak{G}$-module generated by an S-primitive element v of weight λ with respect to a basis π and V_λ is an irreducible $U(\mathfrak{G}_S)_0$-module. Then there exists a unique $\mathfrak{G}$-epimorphism $\chi: M(S,\pi,\lambda+\delta,\rho_S) \rightarrow V$ such that $\chi(1\otimes v) = v$, where (ρ_S, V_λ) is the corresponding representation of the algebra Λ_S.

The proof follows from universal properties of the tensor product.

Proposition 5. 1) In $M(S,\pi,\lambda+\delta,\rho_S)$ there exists a maximal $\mathfrak{G}$-submodule N which is different from $M(S,\pi,\lambda+\delta,\rho_S)$, and 2) if V is an irreducible $\mathfrak{G}$-module with an S-primitive element of weight λ with respect to a basis π then $V \simeq M(S,\pi,\lambda+\delta,\rho_S)/N$ where (ρ_S, V_λ) is the corresponding representation of the algebra Λ_S.

The proposition is proven in [3].

Let $L(S,\pi,\lambda+\delta,\rho_S) = M(S,\pi,\lambda+\delta,\rho_S)/N$.

Remark. If $S = \phi$ then $M(\varnothing,\pi,\lambda,\rho_\varnothing) = M(\lambda)$, where $M(\lambda)$ is a Verma module associated with either π or λ.

Proposition 6. Every simple subfactor of a module $M(S,\pi,\lambda+\delta,\rho_S)$ is isomorphic to $L(S,\pi,\mu+\delta,\tilde{\rho}_S)$ for some $\mu \in \mathfrak{H}^*$ and $\tilde{\rho}_S \in \hat{\Lambda}_S$.

The above proposition is proven analogously to a similar result for Verma modules cited in [1].

Fernando cites in [4] still another method of construction of $\mathfrak{G}$-modules generated by S-primitive elements. Let V be an irreducible weight $P = \mathfrak{G}_S \oplus \mathfrak{N}_S^+ \oplus \mathfrak{H}^S$-module such that $\mathfrak{N}_S^+ v = 0$ for all $v \in V$. Clearly, V is irreducible as a $\mathfrak{G}_S$-module. We construct a $\mathfrak{G}$-module $M_1(S,\pi,V) = U(\mathfrak{G}) \underset{U(P)}{\otimes} V$, which also contains a maximal $\mathfrak{G}$-submodule N_1. In addition, $M(S,\pi,\lambda+\delta,\rho_S)/N \simeq M_1(S,\pi,V)/N_1$, where $\lambda \in \operatorname{supp} V$, (ρ_S, V_λ) is the corresponding representation of Λ_S. Furthermore, Proposition 4 implies that there exists an epimorphim $\chi: M(S,\pi,\lambda+\delta,\rho_S) \rightarrow M_1(S,\pi,V)$.

Suppose $\mu \in \mathfrak{H}^*$. Let $P_\mu = \{\lambda \in \mathfrak{H}^* | \lambda - \mu \in Q\}$. For every $\tau \in P_\mu$ let $P_{\tau,S} = \{\lambda \in \mathfrak{H}^* | \lambda - \tau \in Q_S\} \subset P_\mu$. Suppose $\theta \in \hat{Z}(\mathfrak{G})$. Let $K_{\mu,\theta}$ be the category of weight $\mathfrak{G}$-modules V with a central character θ such that $\operatorname{supp} V \subset P_\mu$. Clearly, every module of the form $M(S, \pi, \lambda + \delta, \rho_S)$ is contained in some category $K_{\lambda,\theta}$. Given $V \in K_{\mu,\theta}$, let $T_{S,\pi}(V) = \{\lambda \in \operatorname{supp} V |$ there exists an S-primitive element $v \in V$ of weight λ with respect to $\pi\}$. Let

$$D(S, \pi, \mu, \theta) = \{P_{\tau,S} \subset P_\mu \,|\, \exists V \in K_{\mu,\theta} : T_{S,\pi}(V) \cap P_{\tau,S} \neq \varnothing\}.$$

<u>LEMMA 7</u>. $|D(S, \pi, \mu, \theta)| \leqslant (W : W_S)$ for all $S \subset \{1, 2, \ldots, n\}$, $\mu \in \mathfrak{H}^*$, $\theta \in \hat{Z}(\mathfrak{G})$.

<u>Proof</u>: Let V be an irreducible object of a category $K_{\mu,\theta}$ with an S-primitive element of weight λ with respect to the basis π. Then an algebra $Z(\mathfrak{G}_S) \otimes S(\mathfrak{H}^S)$ acts on a $\mathfrak{G}_S \oplus \mathfrak{H}^S$-module $\sum_{\chi \in P_{\lambda,S}} V_\chi$ by means of a certain character $\in Z(\mathfrak{G}_S) \widehat{\otimes} S(\mathfrak{H}^S)$. Using Proposition 3, we conclude the proof.

Now we are ready to prove the main result of this section.

<u>THEOREM 2.</u> Suppose $\mathfrak{G} \neq E_7, E_8$ and (ρ_S, U) is a finite-dimensional irreducible representation of an algebra Λ_S. Then a $\mathfrak{G}$-module $M(S, \pi, \lambda + \delta, \rho_S)$ has a composition sequence.

<u>Proof</u>: The theorem is proved by induction on $|\pi|$ for all S simultaneously. In the case $|\pi| = 1$ the theorem coincides with the corresponding result for Verma modules [1]. Suppose $|\pi| > 1$ and $|S| = p$. Let N be a maximal submodule of $M(S, \pi, \lambda + \delta, \rho_S)$. Then $N_\lambda = 0$. Since the algebra $\mathfrak{G}_S$ is of type A_p, B_p, C_p, or D_p, a $\mathfrak{G}_S$-module $\sum_{\mu \in P_{\lambda,S}} N_\mu$ and all its submodules contain a semi-primitive element [3, Theorem 3.2]. Without loss of generality we can assume that all semi-primitive elements are S'-primitive if $|S'| = p - 1$ and are not S''-primitive if $|S''| < p - 1$. Lemma 7 and the finite-dimensionality of subspaces N_λ imply that there exists an epimorphism from a finite direct sum of $\mathfrak{G}_S$-modules of the form $M(S', \pi', \lambda' + \delta_S, \rho_{S'})$ into $\sum_{\mu \in P_{\lambda,S}} N_\mu$. Applying the induction hypothesis to every module $M(S', \pi', \lambda' + \delta_S, \rho_{S'})$, we obtain the following composition sequence for the $\mathfrak{G}_S$-module $\sum_{\mu \in P_{\lambda,S}} N_\mu : \sum_{\mu \in P_{\lambda,S}} N_\mu \supset P_1 \supset \ldots \supset P_k \supset 0$. The corresponding chain of $\mathfrak{G}$-submodules of N is as follows: $N \supset N_1 \supset N_2 \supset \ldots \supset N_k$, where N_{i+1} is maximal in N_i and $\sum_{\mu \in P_{\lambda,S}} (N_i)_\mu = P_i$, $i = \overline{1, k}$. In a module N_k we choose a maximal $\mathfrak{G}$-submodule N_{k+1}. Since the $\mathfrak{G}_S$-module P_k is irreducible, we have $\operatorname{supp} N_{k+1} \cap P_{\lambda,S} = \varnothing$. Now the assertion of the theorem follows from Lemma 7. Q.E.D.

<u>Remark</u>. The authors are convinced that above theorem holds even when $\mathfrak{G} \in \{E_7, E_8\}$.

Suppose $S \subset \{1, 2, \ldots, n\}$ and $|S| = 1$. Then $\mathfrak{G}_S \simeq sl(2, \mathbb{C})$ and $\dim L(S, \pi, \lambda + \delta, \rho_S)_\lambda = 1$. Given a module $L = L(S, \pi, \lambda + \delta, \rho_S)$, there is an associated character $\beta(L) \in Z(\mathfrak{G}_S) \widehat{\otimes} S(\mathfrak{H}^S)$ by a means of which $Z(\mathfrak{G}_S) \otimes S(\mathfrak{H}^S)$ acts on a $\mathfrak{G}_S \oplus \mathfrak{H}^S$-module $\sum_{\mu \in P_{\lambda,S}} L_\mu$. However, for every $\mu \in \mathfrak{H}^*$ and $\beta \in Z(\mathfrak{G}_S) \widehat{\otimes} S(\mathfrak{H}^S)$ there exist no more than three irreducible weight $\mathfrak{G}$-modules V with S-primitive elements for which $\operatorname{supp} V \subset P_\mu$ and $\beta(B) = \beta$ (see, for example, [5]). Therefore, Proposition 3 implies the following assertion.

<u>THEOREM 3</u>. Suppose that $\mu \in \mathfrak{H}^*$, $\theta \in \hat{Z}(\mathfrak{G})$, $S \subset \{1, 2, \ldots, n\}$, and $|s| = 1$. Then the category $K_{\mu,\theta}$ contains no more than 3/2 $|W|$ irreducible objects V such that $T_{S,\pi}(V) \neq \varnothing$.

<u>Remark</u>. 1) Theorem 3 is false in the case where $|S| > 1$. This follows, for example, from results cited in [6]. 2) Under the assumptions of Theorem 3 there are infinitely many irreducible modules V in $\bigcup_\mu K_{\mu,\theta}$ such that $T_{S,\pi}(v) \neq \varnothing$.

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