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# NON-COMMUTATIVE SCHEMES AND CATEGORICAL RESOLUTIONS

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## Introduction

It is now well-known that the category of coherent sheaves over the projective line is derived equivalent to that of modules over the *Kronecker algebra*, i.e. the algebra of paths of the quiver $\bullet \rightrightarrows \bullet$ [6]. The origin of this result is the existing of a *tilting sheaf* $\mathcal{T} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}(-1)_{\mathbb{P}^1}$ whose endomorphism ring is the Kronecker algebra. This result was extensively generalized to other projective varieties. On the other hand in the paper [17] it was mentioned that there is a close relations between the categories of coherent sheaves over the nodal cubic $C$ and the category of modules over the algebra $\Lambda$ given by the quiver with relations

$$\begin{tikzcd} \bullet \arrow[r, bend left, "\alpha_1"] \arrow[r, bend right, "\alpha_2"'] & \bullet \arrow[r, bend left, "\beta_1"] \arrow[r, bend right, "\beta_2"'] & \bullet \end{tikzcd} \qquad \beta_i\alpha_i = 0 \quad (i = 1, 2).$$

This time these categories cannot be derived equivalent, since $\mathrm{gl.dim}\,\Lambda = 2$ while $\mathrm{gl.dim}\,\mathcal{O}_C = \infty$. An explanation of this fact was given in [12], where the curve $C$ was extended to a non-commutative sheaf of algebras $\mathcal{A}$ such that $\mathrm{gl.dim}\,\mathcal{A} = 2$ and the derived categories of coherent sheaves over $\mathcal{A}$ and of modules over $\Lambda$ are derived equivalent. Moreover, the categories $\mathsf{Coh}\,C$ and $\mathsf{Coh}\,\mathcal{A}$ are "very close" via natural functors between them.

The main goal of this preprint is to generalize the last construction to a wide class of curves, including *non-commutative curves.* For this purpose we establish several general facts concerning non-commutative schemes. Perhaps, the most important tool is the technique of *minors*, especially, the *endomorphism construction*, which gives possibility to find a "better" non-commutative curve closely related to the original one, almost as in [12].

Using this technique we prove, in particular, the following result, giving categorical resolutions for the category of quasi-coherent sheaves over *rational* non-commutative curves.

**Theorem.** (see Corollary 4.3.5)**.** *Let $(X, \mathcal{A})$ be a rational non-commutative curve. There is a finite dimensional algebra $\Lambda$ of finite global dimension and the functors $\mathbb{F} : \mathscr{D}(\mathsf{Qcoh}\,\mathcal{A}) \to \mathscr{D}(\Lambda\text{-}\mathsf{Mod})$, $\mathbb{H} : \mathscr{D}(\mathsf{Qcoh}\,\mathcal{A}) \to \mathscr{D}(\Lambda\text{-}\mathsf{Mod})$ and $\mathbb{G} : \mathscr{D}(\Lambda\text{-}\mathsf{Mod}) \to \mathscr{D}(\mathsf{Qcoh}\,\mathcal{A})$ such that*

(1) $(\mathbb{F}, \mathbb{G})$ *and* $(\mathbb{G}, \mathbb{H})$ *are adjoint pairs.*
(2) $\mathbb{G}$ *is exact and the natural morphisms* $\mathbb{1} \to \mathbb{G} \circ \mathbb{F}$ *and* $\mathbb{G} \circ \mathbb{H} \to \mathbb{1}$ *are isomorphisms.*
(3) $\operatorname{Ker} \mathbb{G}$ *is both localizing and colocalizing subcategory of* $\mathscr{D}(\Lambda\text{-}\mathsf{Mod})$ *and* $\mathscr{D}(\mathsf{Qcoh}\,\mathcal{A}) \simeq \mathscr{D}(\Lambda\text{-}\mathsf{Mod}) / \operatorname{Ker} \mathbb{G}$.

In this situation we say that $\mathscr{D}(\mathsf{Qcoh}\,\mathcal{A})$ is a *bilocalization* of $\mathscr{D}(\Lambda\text{-}\mathsf{Mod})$.

For special type of non-commutative curves (*subhereditary* ones), generalizing the class of curves considered in [12], $\mathrm{gl.dim}\,\Lambda = 2$. Moreover, in this case $\Lambda$ arises as an extension of a *canonical algebra* in the sense of Ringel [41] (see subsection 4.4).

# 1. Preliminaries

1.1. **Categories and modules.** If the opposite is not specially mentioned, all categories $\mathscr{A}$ are supposed to be $\Bbbk$-*categories*, where $\Bbbk$ is a commutative ring. It means that all sets of morphisms $\mathscr{A}(A,B)$ are $\Bbbk$-modules and the multiplication is $\Bbbk$-bilinear. In the same way, all functors are supposed to be $\Bbbk$-*linear*.

Let $\mathscr{R}$ be a $\Bbbk$-linear category (for instance, a $\Bbbk$-algebra). We denote by $\mathscr{R}\text{-}\mathsf{Mod}$ the category of (left) $\mathscr{R}$-*modules*, i.e. functors $\mathscr{R} \to \Bbbk\text{-}\mathsf{Mod}$, the category of $\Bbbk$-modules. For an object $A \in \mathscr{R}$ we denote by $A^{\uparrow}$ the module $\mathscr{R}(A,\_)$. If $\mathscr{R}^{\mathrm{op}}$ is the opposite category of $\mathscr{R}$, we write $\mathsf{Mod}\text{-}\mathscr{R} = \mathscr{R}^{\mathrm{op}}\text{-}\mathsf{Mod}$ and call $\mathscr{R}^{\mathrm{op}}$-modules *right $\mathscr{R}$-modules*. We denote by $A^{\downarrow}$ the right module $\mathscr{R}(\_,A)$. By Yoneda's Lemma, the map $A \mapsto A^{\downarrow}$ gives a full embedding of $\mathscr{R}$ into $\mathsf{Mod}\text{-}\mathscr{R}$. We call $A^{\uparrow}$ ($A^{\downarrow}$) *free* (left) *$\mathscr{R}$-modules* (respectively, *free right $\mathscr{R}$-modules*). $\mathscr{R}$ is said to be *fully additive*[1] if it is additive and every idempotent in $\mathscr{R}$ splits, i.e. originates from a decomposition of its source into a direct sum. We denote by $\operatorname{add}\mathscr{R}$ the smallest fully additive category containing $\mathscr{R}$. It can be identified with the category $\mathsf{pro}\text{-}\mathscr{R}$ of finitely generated projective right $\mathscr{R}$-modules. For a subset $\mathbf{S}$ of objects of a fully additive category $\mathscr{R}$ we denote by $\operatorname{add}\mathbf{S}$ the full subcategory of $\mathscr{R}$ consisting of direct summands of (finite) direct sums of objects from $\mathbf{S}$. In particular, $\operatorname{add}A$ consists of direct summands of (finite) multiples $nA$ of the object $A$. Note that $\operatorname{add}\{\,A^{\uparrow} \mid A \in \operatorname{Ob}\mathscr{R}\,\}$ coincides with the category $\mathscr{R}\text{-}\mathsf{pro}$ of finitely generated projective $\mathscr{R}$-modules.

An additive category $\mathscr{R}$ is said to be *cocomplete* if it contains coproducts of arbitrary sets of objects. If $\mathbf{S}$ is a set of objects of a cocomplete category, we denote by $\operatorname{Add}\mathbf{S}$ the smallest fully additive subcategory containing all coproducts of objects from $\mathbf{S}$. If, moreover, $\mathscr{R}$ contains arbitrary filtered direct limits, we denote by $\underrightarrow{\operatorname{add}}\,\mathbf{S}$ the smallest fully additive subcategory of $\mathscr{R}$ containing all filtered direct limits of objects from $\operatorname{add}\mathbf{S}$. Note that $\underrightarrow{\operatorname{add}}\,\mathbf{S} \supseteq \operatorname{Add}\mathbf{S}$.

Any functor $\mathbf{F} : \mathscr{R} \to \mathscr{B}$, where $\mathscr{B}$ is a fully additive category can be extended to a functor $\tilde{\mathbf{F}} : \operatorname{add}\mathscr{R} \to \mathscr{B}$ and this extension is unique up to isomorphism. In particular, it gives an equivalence of categories $\mathscr{R}\text{-}\mathsf{Mod} \simeq (\operatorname{add}\mathscr{R})\text{-}\mathsf{Mod}$ and we usually identify these categories.

For a functor $\mathbf{F} : \mathscr{A} \to \mathscr{B}$ we denote by $\operatorname{Ker}\mathbf{F}$ the full subcategory of $\mathscr{A}$ consisting of such objects $A$ that $\mathbf{F}A = 0$. By $\operatorname{Im}\mathbf{F}$ we denote the *essential image* of $\mathbf{F}$, i.e. the full subcategory of $\mathscr{B}$ consisting of objects isomorphic to those of the shape $\mathbf{F}A$ for $A \in \operatorname{Ob}\mathscr{A}$. If $\mathbf{F}$ is full and $\operatorname{Im}\mathbf{F} = \mathscr{B}$, we say that $\mathbf{F}$ is *essentially surjective*.

If $\mathscr{R}$ is cocomplete, we say that an object $A \in \mathscr{R}$ is *compact* if the functor $\mathscr{R}(A,\_)$ preserves arbitrary coproducts. For instance, the modules $A^{\uparrow}$ and, more generally, those from $\mathscr{R}\text{-}\mathsf{pro}$ are compact in $\mathscr{R}\text{-}\mathsf{Mod}$. An $\mathscr{R}$-module $M$ is said to be *finitely presented* if there is an exact sequence $P_1 \to P_0 \to M \to 0$

[1] or *idempotent complete*, or *Carubian.*

with $P_i \in \mathscr{R}$-$\mathsf{pro}$. All finitely presented modules are also compact in $\mathscr{R}$-$\mathsf{Mod}$. We denote by $\mathscr{R}^{\mathrm{c}}$ the full subcategory of $\mathscr{R}$ consisting of all compact objects.

A subset $\mathbf{S} \subset \operatorname{Ob}\mathscr{R}$ is said to be a *set of generators* of $\mathscr{R}$ if for every non-zero morphism $f : A \to B$ there is a morphism $g : C \to A$, where $C \in \mathbf{S}$, such that $gf \neq 0$. If $\mathscr{R}$ is abelian and cocomplete, it means that for every $A \in \operatorname{Ob}\mathscr{R}$ there is an epimorphism $S \to X$, where $S$ is a coproduct of objects from $\mathbf{S}$. For instance, the set of free modules $\{ A^{\uparrow} \mid A \in \operatorname{Ob}\mathscr{R} \}$ is a set of compact generators for $\mathscr{R}$-$\mathsf{Mod}$.

We will say that an abelian category $\mathscr{A}$ is *of ext-dimension* $d = \operatorname{ext.dim}\mathscr{A}$ if $\operatorname{Ext}^i(A, B) = 0$ for any two objects $A, B \in \mathscr{A}$ and any $i > d$, while there are such objects $A, B$ that $\operatorname{Ext}^d(A, B) \neq 0$. If $\operatorname{ext.dim}(\mathscr{R}\text{-}\mathsf{Mod}) = d$ for some additive category $\mathscr{R}$, we also say that $\mathscr{R}$ is *of global dimension* $d$ and write $\operatorname{gl.dim}\mathscr{R} = d$.[2]

A full subcategory $\mathscr{C}$ of an abelian category $\mathscr{A}$ is called *thick* (or *Serre subcategory*) if, for any exact sequence $0 \to C' \to C \to C'' \to 0$ in $\mathscr{A}$, the object $C$ is in $\mathscr{C}$ if and only if both $C'$ and $C''$ are in $\mathscr{C}$. Then the *quotient category* $\mathscr{A}/\mathscr{C}$ is defined (see [19, Ch. III] for its definition and properties). We denote by $\Pi_{\mathscr{C}}$ the canonical functor $\mathscr{A} \to \mathscr{A}/\mathscr{C}$. It is exact, essentially surjective and $\operatorname{Ker}\Pi_{\mathscr{C}} = \mathscr{C}$. For instance, if $\mathsf{F} : \mathscr{A} \to \mathscr{B}$ is a semi-exact functor among abelian categories, its kernel is a thick subcategory of $\mathscr{A}$.

1.2. **Triangulated categories.** We denote by $\mathscr{C}(\mathscr{A})$ the category of complexes over $\mathscr{A}$ and by $\mathscr{H}(\mathscr{A})$ the quotient $\mathscr{C}(\mathscr{A})/\operatorname{ho}(\mathscr{A})$, where $\operatorname{ho}(\mathscr{A})$ is the ideal of morphisms homotopic to zero. If $\mathscr{A}$ is abelian, we denote by $\mathscr{D}(\mathscr{A})$ its derived category, i.e. the category of fractions $\mathscr{H}(\mathscr{A})[\mathcal{Q}^{-1}]$, where $\mathcal{Q}$ is the class of quasi-isomorphisms. We use the symbol $\rightsquigarrow$ for quasi-isomorphisms. As usually, we write $\mathscr{C}^\sigma(\mathscr{A})$, $\mathscr{H}^\sigma(\mathscr{A})$, $\mathscr{D}^\sigma(\mathscr{A})$, where $\sigma \in \{+, -, b\}$ for the full subcategories of the corresponding categories consisting of left, right and two-sided bounded complexes. If $\mathscr{A}$ is abelian and has enough injectives, then $\mathscr{D}^+(\mathscr{A})$ can be identified with the category $\mathscr{H}^+(\mathscr{I})$, where $\mathscr{I}$ is the full subcategory of injective objects. If, moreover, $\mathscr{A}$ is of finite global dimension, then $\mathscr{D}^b(\mathscr{A})$ can be identified with $\mathscr{H}^b(\mathscr{I})$. Dually, if $\mathscr{A}$ has enough projectives, then $\mathscr{D}^-(\mathscr{A})$ can be identified with $\mathscr{H}^-(\mathscr{P})$, where $\mathscr{P}$ denotes the full subcategory of projective objects. If, moreover, $\mathscr{A}$ is of finite global dimension, then $\mathscr{D}^b(\mathscr{A})$ can be identified with $\mathscr{H}^b(\mathscr{P})$.

A full subcategory $\mathscr{S}$ of a triangulated category $\mathscr{T}$ is said to be a *triangulated subcategory* if it is closed under shifts and extensions (or cones). We call it *thick*, if, moreover, it is closed under direct summands (in particular, under isomorphic copies). For a triangulated category $\mathscr{T}$ and a subset $\mathbf{S} \subseteq \operatorname{Ob}\mathscr{T}$ we denote by $\langle \mathbf{S} \rangle$ the smallest thick subcategory of $\mathscr{T}$ containing $\mathbf{S}$. For instance, if $\mathbf{F} : \mathscr{T} \to \mathscr{T}'$ is an *exact* (or, the same, *triangulated*) functor between triangulated categories, its kernel $\operatorname{Ker}\mathbf{F}$ is

[2] We prefer saying *ext-dimension* instead of more usual *global dimension* since otherwise, for an abelian category, the latter term becomes ambiguous.

a thick subcategory. The same is true if $\mathbf{F} : \mathscr{T} \to \mathscr{A}$ is a *homological functor* into an abelian category $\mathscr{A}$. If $\langle\, \mathbf{S} \,\rangle = \mathscr{T}$, we say that $\mathbf{S}$ *generates* $\mathscr{T}$ or is a *set of generators* of $\mathscr{T}$. If $\mathscr{T}$ is cocomplete, we denote by $\langle\, \mathbf{S} \,\rangle^{\omega}$ the smallest thick subcategory of $\mathscr{T}$ containing $\mathbf{S}$ and closed under arbitrary coproducts. If $\langle\, \mathbf{S} \,\rangle^{\omega} = \mathscr{T}$, we say that $\mathbf{S}$ *strongly generates* $\mathscr{T}$ or is a *set of strong generators* of $\mathscr{T}$. If $\mathbf{S}$ consists of *compact objects*, $\langle\, \mathbf{S} \,\rangle^{\omega} = \mathscr{T}$ if and only if for any non-zero $T \in \operatorname{Ob}\mathscr{T}$ there is a non-zero morphism $S \to T[m]$ for some $S \in \mathbf{S}$ and $m \in \mathbb{Z}$ (see [35, Lemma 1.7]). In this case $\langle\, \mathbf{S} \,\rangle = \mathscr{T}^{\mathrm{c}}$. For instance, if $\mathbf{S}$ is a set of compact generators of an abelian category $\mathscr{A}$, then $\langle\, \mathbf{S} \,\rangle^{\omega} = \mathscr{D}(\mathscr{A})$, hence $\langle\, \mathbf{S} \,\rangle = \mathscr{D}^{\mathrm{c}}(\mathscr{A})$, the subcategory of compact objects. (The latter category is often called the *perfect derived category* of $\mathscr{A}$ and denoted by $\mathcal{D}^{\mathrm{perf}}(\mathscr{A})$.) For a subcategory $\mathscr{T}'$ of $\mathscr{T}$ we write $\langle\, \mathscr{T} \,\rangle$ and $\langle\, \mathscr{T} \,\rangle^{\omega}$ instead of $\langle\, \operatorname{Ob}\mathscr{T}' \,\rangle$ and $\langle\, \operatorname{Ob}\mathscr{T}' \,\rangle^{\omega}$. We also set $\langle\, \mathbf{S}_1, \mathbf{S}_2, \dots, \mathbf{S}_m \,\rangle = \langle\, \bigcup_{i=1}^{m} \mathbf{S}_i \,\rangle$ and $\langle\, \mathbf{S}_1, \mathbf{S}_2, \dots, \mathbf{S}_m \,\rangle^{\omega} = \langle\, \bigcup_{i=1}^{m} \mathbf{S}_i \,\rangle^{\omega}$. If $\mathscr{A} = \mathscr{R}\text{-}\mathsf{Mod}$ for some additive category $\mathscr{R}$, we write $\mathscr{D}\mathscr{R}$ instead of $\mathscr{D}(\mathscr{R}\text{-}\mathsf{Mod})$. Respectively, we write $\mathscr{D}^{\sigma}\mathscr{R}$, where $\sigma \in \{\, +, -, b, \mathrm{c} \,\}$. We can choose the set of free modules $\mathscr{R}^{\uparrow} = \{\, A^{\uparrow} \mid A \in \operatorname{Ob}\mathscr{R} \,\}$ as a set of compact generators for $\mathscr{R}\text{-}\mathsf{Mod}$. Therefore, $\langle\, \mathscr{R}^{\uparrow} \,\rangle^{\omega} = \mathscr{D}\mathscr{R}$ and the perfect derived category $\mathscr{D}^{\mathrm{c}}\mathscr{R}$ consists of complexes of modules quasi-isomorphic to finite complexes of finitely generated projective modules.

If $\mathscr{S} \subset \mathscr{T}$ is a thick subcategory of a triangulated category $\mathscr{T}$, the quotient category $\mathscr{T}/\mathscr{S}$ is defined (see [37, Ch. 2] for its definition and properties). Again the canonical functor $\Pi_{\mathscr{S}} : \mathscr{T} \to \mathscr{T}/\mathscr{S}$ is exact (or, the same, triangulated), essentially surjective and $\operatorname{Ker}\Pi_{\mathscr{S}} = \mathscr{S}$.

1.3. **Adjoints and quotients.** We need the following facts from the general theory of localizations in abelian and triangulated categories proved in [19, 21, 37].

**Theorem 1.3.1.** (1) *Let $\mathscr{A}, \mathscr{B}$ be abelian categories, $\mathsf{G} : \mathscr{A} \to \mathscr{B}$ be an exact functor and $\mathsf{F} : \mathscr{B} \to \mathscr{A}$ be its left adjoint* (*right adjoint*) *such that the natural morphism $\mathbb{1}_{\mathscr{B}} \to \mathsf{G} \circ \mathsf{F}$* (*respectively, $\mathsf{G} \circ \mathsf{F} \to \mathbb{1}_{\mathscr{B}}$*) *is an isomorphism. Let $\mathscr{C} = \operatorname{Ker}\mathsf{G}$.*

(a) *$\mathsf{G} = \bar{\mathsf{G}} \circ \Pi_{\mathscr{C}}$, where $\bar{\mathsf{G}}$ is an equivalence $\mathscr{A}/\mathscr{C} \xrightarrow{\sim} \mathscr{B}$ and its quasi-inverse functor is $\bar{\mathsf{F}} = \Pi_{\mathscr{C}} \circ \mathsf{F}$.*

(b) *$\mathsf{F}$ is a full embedding and the essential image of $\mathsf{F}$ coincides with the left* (*respectively, right*) *orthogonal subcategory of $\mathscr{C}$, i.e. the full subcategory*

$$^{\perp}\mathscr{C} = \left\{\, A \in \operatorname{Ob}\mathscr{A} \mid \operatorname{Hom}(A, C) = \operatorname{Ext}^1(A, C) = 0 \text{ for all } C \in \operatorname{Ob}\mathscr{C} \,\right\}$$

(*respectively,*

$$\mathscr{C}^{\perp} = \left\{\, A \in \operatorname{Ob}\mathscr{A} \mid \operatorname{Hom}(C, A) = \operatorname{Ext}^1(C, A) = 0 \text{ for all } C \in \operatorname{Ob}\mathscr{C} \,\right\}.)$$

(c) *$\mathscr{C} = ({}^{\perp}\mathscr{C})^{\perp}$* (*respectively, $\mathscr{C} = {}^{\perp}(\mathscr{C}^{\perp})$*).

(d) *The embedding functor $\mathscr{C} \to \mathscr{A}$ has a left (respectively, right) adjoint.*

(2) *Let $\mathscr{A},\mathscr{B}$ be triangulated categories, $\mathsf{G}:\mathscr{A}\to\mathscr{B}$ be an exact (triangulated) functor and $\mathsf{F}:\mathscr{B}\to\mathscr{A}$ be its left adjoint (right adjoint) such that the natural morphism $\mathbb{1}_{\mathscr{B}}\to\mathsf{G}\circ\mathsf{F}$ (respectively, $\mathsf{G}\circ\mathsf{F}\to\mathbb{1}_{\mathscr{B}}$) is an isomorphism. Let $\mathscr{C}=\operatorname{Ker}\mathsf{G}$.*

(a) *$\mathsf{G}=\bar{\mathsf{G}}\circ\Pi_{\mathscr{C}}$, where $\bar{\mathsf{G}}$ is an equivalence $\mathscr{A}/\mathscr{C}\xrightarrow{\sim}\mathscr{B}$ and its quasi-inverse functor is $\bar{\mathsf{F}}=\Pi_{\mathscr{C}}\circ\mathsf{F}$.*

(b) *$\mathsf{F}$ is a full embedding and the essential image of $\mathsf{F}$ coincides with the left (respectively, right) orthogonal subcategory of $\mathscr{C}$, i.e. the full subcategory[3]*

$$^{\perp}\mathscr{C}=\{\,A\in\operatorname{Ob}\mathscr{A}\mid\operatorname{Hom}(A,C)=0\text{ for all }C\in\operatorname{Ob}\mathscr{C}\,\}$$

*(respectively,*

$$\mathscr{C}^{\perp}=\{\,A\in\operatorname{Ob}\mathscr{A}\mid\operatorname{Hom}(C,A)=0\text{ for all }C\in\operatorname{Ob}\mathscr{C}\,\}\,.)$$

(c) *$\mathscr{C}=(^{\perp}\mathscr{C})^{\perp}$ (respectively, $\mathscr{C}={}^{\perp}(\mathscr{C}^{\perp})$).*

(d) *The embedding functor $\mathscr{C}\to\mathscr{A}$ has a left (respectively, right) adjoint, which induces an equivalence $\mathscr{A}/^{\perp}\mathscr{C}\xrightarrow{\sim}\mathscr{C}$ (respectively, $\mathscr{A}/\mathscr{C}^{\perp}\xrightarrow{\sim}\mathscr{C}$).*

*Proof.* The statement (1a) is proved in [19, Ch. III, Proposition 5] if $\mathsf{F}$ is right adjoint of $\mathsf{G}$. The case of left adjoint is just a dualization. The proof of the statement (2a) is quite analogous. Therefore, from now on we can suppose that $\mathscr{B}=\mathscr{A}/\mathscr{C}$. Then the statements (1b) and (2b) are just [19, Ch. III, Lemma 2 et Corollaire] and [37, Theorem 9.1.16]. The statements (1c) and (2c) are [21, Corollary 2.3] and [37, Corollary 9.1.14]. Thus the statement (2d) also follows from [37, Theorem 9.1.16]. In the abelian case the left (respectively, right) adjoint $\mathsf{J}$ to the embedding $\mathscr{C}\to\mathscr{A}$ is given by the rule $A\mapsto\operatorname{Cok}\Psi(A)$ (respectively, $A\mapsto\operatorname{Ker}\Psi(A)$), where $\Psi$ is the natural morphism $\mathsf{F}\circ\mathsf{G}\to\mathbb{1}_{\mathscr{A}}$ (respectively, $\mathbb{1}_{\mathscr{A}}\to\mathsf{F}\circ\mathsf{G}$). □

*Remark.* Note that in the abelian case the composition $\Pi_{^{\perp}\mathscr{C}}\circ\mathsf{J}$ (respectively, $\Pi_{\mathscr{C}^{\perp}}\circ\mathsf{J}$) need not be an equivalence. The reason is that the subcategory $^{\perp}\mathscr{C}$ ($\mathscr{C}^{\perp}$) need not be thick (see [21]).

**Corollary 1.3.2.** *Let $\mathscr{A},\mathscr{B}$ be abelian or triangulated categories, $\mathsf{G}:\mathscr{A}\to\mathscr{B}$ be an exact functor having both left adjoint $\mathsf{F}$ and right adjoint $\mathsf{H}$. The canonical morphism $\mathbb{1}_{\mathscr{B}}\to\mathsf{G}\circ\mathsf{F}$ is an isomorphism if and only if so is the canonical morphism $\mathsf{G}\circ\mathsf{H}\to\mathbb{1}_{\mathscr{B}}$.*

*Proof.* If, say, $\mathbb{1}_{\mathscr{B}}\to\mathsf{G}\circ\mathsf{F}$ is an isomorphism, then $\mathscr{B}$ can be identified with the quotient category $\mathscr{A}/\mathscr{C}$, where $\mathscr{C}=\operatorname{Ker}\mathsf{G}$, and $\mathsf{G}$ with $\Pi_{\mathscr{C}}$. As $\mathsf{H}$ is the right adjoint of $\mathsf{G}$, the morphism $\mathsf{G}\circ\mathsf{H}\to\mathbb{1}_{\mathscr{B}}$ is an isomorphism by [19, Ch. III, Proposition 3] or [37, Lemma 9.1.7]. □

[3] Note that in the book [37] the notations for the orthogonal subcategories are opposite to ours. The latter seems more usual, especially in the representation theory, see, for instance, [2, 21]. In [19] the objects of the right orthogonal subcategory $\mathscr{C}^{\perp}$ are called *$\mathscr{C}$-closed.*

A thick subcategory $\mathscr{C}$ of an abelian or triangulated category $\mathscr{A}$ is said to be *localizing* (*colocalizing*) if the canonical functor $\mathsf{G} : \mathscr{A} \to \mathscr{A}/\mathscr{C}$ has a right (respectively, left) adjoint $\mathsf{F}$. Neeman [37] calls $\mathsf{F}$ a *Bousfield localization* (respectively, a *Bousfield colocalization*). In this case the natural morphism $\mathsf{G} \circ \mathsf{F} \to \mathbb{1}_{\mathscr{A}/\mathscr{C}}$ (respectively, $\mathbb{1}_{\mathscr{A}/\mathscr{C}} \to \mathsf{G} \circ \mathsf{F}$) is an isomorphism [19, Ch.III,Proposition 3], [37, Lemma 9.1.7]. If $\mathscr{C}$ is both localizing and colocalizing, we call it *bilocalizing* and call the category $\mathscr{A}/\mathscr{C}$ (or any equivalent) a *bilocalization* of $\mathscr{A}$.

**Corollary 1.3.3.** *Let $\mathscr{C}$ be a localizing* (*colocalizing*) *thick subcategory of an abelian category $\mathscr{A}$, $\mathscr{D}_{\mathscr{C}}(\mathscr{A})$ be the full subcategory of $\mathscr{D}(\mathscr{A})$ consisting of all complexes $C^\bullet$ such that all cohomologies $H^i(C^\bullet)$ are in $\mathscr{C}$. Suppose that the Bousfield localization* (*respectively, colocalization*) *functor $\mathsf{F}$ has right* (*respectively, left*) *derived functor. Then $\mathscr{D}_{\mathscr{C}}(\mathscr{A})$ is also a localizing* (*colocalizing*) *subcategory of $\mathscr{A}$ and $\mathscr{D}(\mathscr{A})/\mathscr{D}_{\mathscr{C}}(\mathscr{A}) \simeq \mathscr{D}(\mathscr{A}/\mathscr{C})$.*

*Proof.* We consider the case of a localizing subcategory $\mathscr{C}$, denote by $\mathsf{G}$ the canonical functor $\mathscr{A} \to \mathscr{A}/\mathscr{C}$ and by $\mathsf{F}$ its right adjoint. As $\mathsf{G}$ is exact, it induces an exact functor $\mathscr{D}(\mathscr{A}) \to \mathscr{D}(\mathscr{A}/\mathscr{C})$ acting on complexes componentwise. We denote it by $\mathsf{DG}$; it is both right and left derived of $\mathsf{G}$. Obviously, $\operatorname{Ker} \mathsf{DG} = \mathscr{D}_{\mathscr{C}}(\mathscr{A})$. Since $\mathsf{G} \circ \mathsf{F} \to \mathbb{1}_{\mathscr{A}/\mathscr{C}}$ is an isomorphism, the morphism $\mathsf{DG} \circ \mathsf{RF} \to \mathbb{1}_{\mathscr{D}(\mathscr{A}/\mathscr{C})}$ is also an isomorphism, so we can apply Theorem1.3.2 (2). □

*Remark* 1.3.4.
(1) If $\mathscr{C}$ is localizing and $\mathscr{A}$ is a Grothendieck category, the right derived functor $\mathsf{RF}$ exists [2], so $\mathscr{D}(\mathscr{A}/\mathscr{C}) \simeq \mathscr{D}_{\mathscr{C}}(\mathscr{A})$. We do not know general conditions which ensure the existence of the left derived functor $\mathsf{LF}$ in the case of colocalizing categories, though it is the case when $\mathscr{A}$ is a category of quasi-coherent modules over a quasi-compact separated non-commutative scheme and $\mathsf{F}$ is tensor product or inverse image, see Section 2.2.
(2) Miyatchi [34] proved that always $\mathscr{D}^\sigma(\mathscr{A}/\mathscr{C}) \simeq \mathscr{D}^\sigma_{\mathscr{C}}(\mathscr{A})$, where $\sigma \in \{+, -, b\}$.

## 1.4. **Categorical resolutions.**

**Definition 1.4.1.** Let $\mathscr{T}$ be a triangulated cocomplete category with a set of compact strong generators. A *resolution* of $\mathscr{T}$ is an exact functor $\mathsf{G} : \mathscr{D}(\mathscr{A}) \to \mathscr{T}$ such that

(1) $\mathscr{A}$ is locally noetherian [19, III.2].
(2) $\operatorname{ext.dim} \mathscr{A} < \infty$.
(3) $\mathsf{G}$ has both left adjoint $\mathsf{F}$ and right adjoint $\mathsf{H}$.
(4) The canonical morphisms $\mathsf{G} \circ \mathsf{F} \to \mathbb{1}_{\mathscr{T}}$ and $\mathbb{1}_{\mathscr{T}} \to \mathsf{G} \circ \mathsf{F}$ are isomorphisms. (Note that we only have to check *one* of these two conditions, see Corollary 1.3.2.)

If, moreover, the restrictions of the functors $\mathsf{F}$ and $\mathsf{G}$ onto the subcategory $\mathscr{T}^{\mathrm{c}}$ of compact objects are isomorphic, we call $\mathsf{G}$, following [28], a *weakly crepant resolution* of $\mathscr{T}$.

Theorem 1.3.1 implies the following properties of resolutions.

**Corollary 1.4.2.** *Let* $\mathsf{G} : \mathscr{D}(\mathscr{A}) \to \mathscr{T}$ *be a resolution of a triangulated category* $\mathscr{T}$*, as in Definition 1.4.1.*

(1) *The kernel* $\mathscr{S} = \operatorname{Ker} \mathsf{G}$ *is a bilocalizing subcategory of* $\mathscr{D}(\mathscr{A})$*. So* $\mathscr{T}$ *is a bilocalization of* $\mathscr{D}(\mathcal{A})$*.*
(2) $\mathsf{G} = \bar{\mathsf{G}} \circ \Pi_{\mathscr{S}}$*, where* $\bar{\mathsf{G}} : \mathscr{D}(\mathscr{A})/\mathscr{S} \xrightarrow{\sim} \mathscr{T}$*.*
(3) *The functors* $\mathsf{F}$ *and* $\mathsf{H}$ *are full embeddings with essential images, respectively,* ${}^{\perp}\mathscr{S}$ *and* $\mathscr{S}^{\perp}$*.*
(4) $\mathsf{F}(\mathscr{T}^{\mathrm{c}}) \subseteq \mathscr{D}^{\mathrm{c}}(\mathscr{A})$*.*

So we are in the situation of "*six gluing functors*" of [37, 9.2]. Note that the essential images of $\mathsf{F}$ and $\mathsf{H}$ usually do not coincide.

*Proof.* We only have to prove the statement (4). As $\mathsf{G}$ has right adjoint, it preserves coproducts. Then, for a compact object $A \in \mathscr{T}$ and an arbitrary coproduct $B = \coprod_{i\in I} B_i$ in $\mathscr{B} = \mathscr{D}(\mathscr{A})$,

$$\begin{aligned} \mathscr{B}(\mathsf{F}A, B) &\simeq \mathscr{T}(A, \mathsf{G}B) \simeq \mathscr{T}(A, \textstyle\coprod_{i\in I} \mathsf{G}B_i) \simeq \\ &\simeq \textstyle\coprod_{i\in I} \mathscr{T}(A, \mathsf{G}B_i) \simeq \coprod_{i\in I} \mathscr{B}(\mathsf{F}A, B_i), \end{aligned}$$

so $\mathsf{F}A$ is compact. □

1.5. **Realization theorem.** We recall the following "realization theorem" (see [25, 4.3] or [31, 2.6]).

**Theorem 1.5.1.** *Let* $\mathscr{A}$ *be an abelian cocomplete category with enough injective objects,* $\mathbf{S}$ *be a set of compact generators of* $\mathscr{A}$ *and* $I^{\bullet}_A$ *be an injective resolution of* $A \in \mathbf{S}$*. Denote by* $\mathbb{A}$ *the DG-category with the set of objects* $\mathbf{S}$*, the sets of morphisms* $\mathbb{A}^n(A, B) = \prod_k \mathscr{A}(I^k_A, I^{k+n}_B)$ *and the differential*

$$d(f_k) = (d_B \circ f_k - (-1)^n f_{k+1} \circ d_A),$$

*where* $d_A$ *is the differential in the complex* $I^{\bullet}_A$*. Then* $\mathscr{D}(\mathscr{A}) \simeq \mathscr{D}(\mathbb{A})$*, the derived category of the DG-category* $\mathbb{A}$ *in the sense of* [25]*, and* $\mathscr{D}^{\mathrm{c}}(\mathscr{A}) \simeq \mathscr{D}^{\mathrm{c}}(\mathbb{A})$*.*[4]

Certainly, when we speak about equivalences of triangulated categories, we always mean that they are compatible with the triangulated structure.

A cocycle from $\mathbb{A}^n(I^{\bullet}_A, I^{\bullet}_B)$ is evidently a morphism of complexes $I^{\bullet}_A \to I^{\bullet}_B[n]$, while coboundaries are morphisms homotopic to zero. Therefore $H^*(\mathbb{A})$ coincides with the category $\mathbb{E} = \operatorname{Ext}^*(\mathbf{S})$ whose set of objects is $\mathbf{S}$ and the set of morphisms $A \to B$ of degree $i$ is $\operatorname{Ext}^i(A, B)$. The latter is a graded category (with zero differential), so its derived category in the sense of Keller is just the derived category $\mathscr{D}(\mathbb{E}^{\mathrm{op}}\text{-}\mathsf{GrMod})$ of graded (left) $\mathbb{E}^{\mathrm{op}}$-modules. Usually this derived category is *not equivalent* to $\mathscr{D}(\mathbb{A})$. One

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[4] $\mathscr{D}^{\mathrm{c}}(\mathbb{A})$ is equivalent to the category $\mathscr{H}^b_p(\mathbb{A})$ in Keller's notations [25, 4.2]. Note also that Keller considers *right* $\mathbb{A}$-modules when constructing $\mathscr{D}(\mathbb{A})$.

needs to consider $\mathbb{E}$ as an $A_\infty$-*algebra* in order to obtain a derived equivalence [24, 33]. Nevertheless, the following simple partial case often occurs in calculations.

**Corollary 1.5.2.** *Suppose that there is a subcategory $\mathbb{H} \subseteq \mathbb{A}$ such that, for every $i$ and every $A, B \in \mathbb{S}$, $Z^i\mathbb{A}(I_A^\bullet, I_B^\bullet) = \mathbb{H}^i(I_A^\bullet, I_B^\bullet) \oplus B^i\mathbb{A}(I_A^\bullet, I_B^\bullet)$. Then $\mathscr{D}(\mathscr{A}) \simeq \mathscr{D}(\mathbb{E}^{\mathrm{op}}\text{-}\mathsf{GrMod})$ and $\mathscr{D}^{\mathrm{c}}(\mathscr{A}) \simeq \mathscr{D}^{\mathrm{c}}(\mathbb{E}^{\mathrm{op}}\text{-}\mathsf{GrMod})$.*

*Proof.* Under these conditions the embedding $\mathbb{H} \to \mathbb{A}$ is a quasi-isomorphism and $\mathbb{H} \simeq \mathbb{E}$. □

Obviously, there are dual results concerning projective resolutions. We prefer injectives, since our main goal is the study of categories of coherent sheaves.

1.6. **Rouquier dimension.** We recall the definition of the dimension of a triangulated category $\mathscr{T}$ by Rouquier [42].

Let $\mathbf{S}$ be a set of objects from a triangulated category $\mathscr{T}$. We define $\langle \mathbf{S} \rangle_k$ recursively, setting

$$\langle \mathbf{S} \rangle_1 = \mathrm{add}\,\{\, A[n] \mid A \in \mathbf{S},\, n \in \mathbb{Z} \,\},$$
$$\langle \mathbf{S} \rangle_{k+1} = \mathrm{add}\,\{\, Cf \mid f : A \to B,\, B \in \langle \mathbf{S} \rangle_1,\, A \in \langle \mathbf{S} \rangle_k \,\},$$

where $Cf$ is the *cone* of $f$, i.e. the third object of a triangle $A \xrightarrow{f} B \to C \to A[1]$. We write $\langle C \rangle_k$, where $C \in \mathrm{Ob}\,\mathscr{T}$, instead of $\langle \{S\} \rangle$.

The *dimension* $\dim \mathscr{T}$ of $\mathscr{T}$ equals $d$ if $d$ is the minimal integer such that $\langle C \rangle_{d+1} = \mathscr{T}$ for an object $C$. If such $d$ does not exist, we set $\dim \mathscr{T} = \infty$. For instance, if $\boldsymbol{A}$ is a hereditary (but not semi-simple) ring, then $\dim \mathscr{D}^b(\boldsymbol{A}\text{-}\mathsf{mod}) = 1$. If $\Lambda$ is a finite dimensional algebra over a perfect field and $\mathrm{gl.dim}\,\Lambda = d$, then $\dim \mathscr{D}^b(\Lambda\text{-}\mathsf{mod}) \le d$ [42, Proposition 7.4]. If $X$ is a smooth quasi-projective variety of dimension $n$, then $\dim \mathscr{D}^b(\mathsf{Qcoh}\,X) \le 2n$ [42, Proposition 7.9]. If $X$ is a smooth curve, then $\dim \mathscr{D}^b(\mathsf{Qcoh}\,X) = 1$ [38].

**Proposition 1.6.1.** *Let $\mathscr{A}, \mathscr{B}$ be abelian categories, $\mathsf{G} : \mathscr{A} \to \mathscr{B}$ be an exact functor, $\mathsf{F} : \mathscr{B} \to \mathscr{A}$ be a functor such that $\mathsf{G} \circ \mathsf{F} \simeq 1_{\mathscr{B}}$. Then $\dim \mathscr{D}^b(\mathscr{B}) \le \dim \mathscr{D}^b(\mathscr{A})$.*

*Proof.* Let $\mathscr{D}^b(\mathscr{A}) = \langle C^\bullet \rangle_{d+1}$. As $\mathsf{G}$ is exacts, $\mathsf{G}(\langle C^\bullet \rangle_k) \subseteq \langle \mathsf{G}(C^\bullet) \rangle_k$. Consider any complex $B^\bullet$ from $\mathscr{D}^b(\mathscr{B})$. Then the complex $\mathsf{F}(B^\bullet)$ obtained by componentwise action of $\mathsf{F}$ belongs to $\mathscr{D}^b(\mathscr{A})$, hence to $\langle C^\bullet \rangle_{d+1}$. Therefore, the complex $\mathsf{G} \circ \mathsf{F}(B^\bullet) \simeq B^\bullet$ belongs to $\langle \mathsf{G}(C^\bullet) \rangle_{d+1}$. Hence $\langle \mathsf{G}(C^\bullet) \rangle_{d+1} = \mathscr{D}^b(\mathscr{B})$. □

1.7. **Semi-orthogonal decompositions.** Let $\mathscr{T}$ be a triangulated cocomplete category with a set of compact generators. A sequence of subsets $(\mathbf{S}_1, \mathbf{S}_2, \dots, \mathbf{S}_m)$ of $\mathrm{Ob}\,\mathscr{T}^{\mathrm{c}}$ is said to be a *semi-orthogonal collection* if $\mathbf{S}_j \subseteq \mathbf{S}_i^\perp$ for $j < i$. If, moreover, $\langle \mathbf{S}_1, \mathbf{S}_2, \dots, \mathbf{S}_m \rangle = \mathscr{T}^{\mathrm{c}}$, they say that $\langle \mathbf{S}_1, \mathbf{S}_2, \dots, \mathbf{S}_m \rangle$ is a *semi-orthogonal decomposition* of $\mathscr{T}$. Certainly, then $\langle \mathbf{S}_1, \mathbf{S}_2, \dots, \mathbf{S}_m \rangle^\omega = \mathscr{T}$.

Suppose that $\mathscr{T}$ is a $\Bbbk$-category for some field $\Bbbk$. A compact object $C$ is said to be *exceptional* if $\mathscr{T}(C, C[n]) = 0$ for $n \neq 0$, while $\mathscr{T}(C, C) = \Bbbk$. A semi-orthogonal collection $(C_1, C_2, \dots, C_m)$ of exceptional objects is called an *exceptional collection*. If, moreover, $\langle\, C_1, C_2, \dots, C_m \,\rangle = \mathscr{T}^{\mathrm{c}}$, they say that this exceptional collection is *full*, or $\langle\, C_1, C_2, \dots, C_m \,\rangle$ is an *exceptional decomposition* of $\mathscr{T}$.

1.8. **Tilting.** Recall that a triangulated category $\mathscr{T}$ is said to be *algebraic* if there is a fully faithful exact functor $\mathscr{T} \to \mathscr{H}(\mathscr{A})$ for some additive category $\mathscr{A}$ [25, 27]. A set of compact objects $\mathbf{T}$ of an algebraic triangulated cocomplete category is said to be *pre-tilting* if $\mathscr{T}(T, T'[n]) = 0$ for all objects $T, T' \in \mathbf{T}$ and all integers $n \neq 0$. If, moreover, $\mathbf{T}$ strongly generates $\mathscr{T}$, the set $\mathscr{T}$ is said to be *tilting*. The results of [25, Section 4.3] (see also [27, Section 7.5]) give the following property of such sets.

**Theorem 1.8.1.** *Let $\mathscr{T}$ be an algebraic triangulated cocomplete category and $\mathbf{T}$ be a pre-tilting set of objects from $\mathscr{T}$. We consider $\mathbf{T}$ as a full subcategory of $\mathscr{T}$. There is an exact equivalence $\langle\, \mathbf{T} \,\rangle^{\omega} \simeq \mathscr{D}\mathbf{T}^{\mathrm{op}}$. In particular, if $\mathbf{T}$ is a tilting set, $\mathscr{T} \simeq \mathscr{D}\mathbf{T}^{\mathrm{op}}$.*

Note that if a pre-tilting set $\mathbf{T}$ is finite, then $\mathbf{T}^{\mathrm{op}}\text{-}\mathsf{Mod} \simeq \Lambda\text{-}\mathsf{Mod}$, where $\Lambda = (\mathrm{End} \bigoplus_{T \in \mathbf{T}} T)^{\mathrm{op}}$. Note also that if $\mathrm{gl.dim}\, \mathbf{T} < \infty$, then $\langle\, \mathbf{T} \,\rangle \simeq \mathscr{D}^b\mathbf{T}^{\mathrm{op}}$.

## 2. Non-commutative schemes and their derived categories

2.1. **Non-commutative schemes.** Let $X$ be a scheme and $\mathcal{A}$ be an $\mathcal{O}_X$-*algebra*, i.e. a sheaf of $\mathcal{O}_X$-algebras, which is quasi-coherent as a sheaf of $\mathcal{O}_X$-modules. We call the pair $(X, \mathcal{A})$ a *non-commutative scheme*. We call this non-commutative scheme *affine* (*separated, quasi-compact, projective*) if so is the scheme $X$. We call it *reduced* if $\mathcal{A}$ has no nilpotent ideals. For instance, an affine non-commutative scheme is just a sheafification $(\operatorname{Spec} \boldsymbol{R}, \boldsymbol{A}\tilde{})$ of an algebra $\boldsymbol{A}$ over a commutative ring $\boldsymbol{R}$. A *morphism* of non-commutative schemes $f : (X, \mathcal{A}) \to (Y, \mathcal{B})$ is a pair $(f_X, f^{\#})$, where $f_X : X \to Y$ is a morphism of schemes and $f^{\#} : f_X^{-1}\mathcal{B} \to \mathcal{A}$ is a morphism of $f_X^{-1}\mathcal{O}_Y$-algebras.

In what follows we always suppose that *all considered non-commutative schemes are quasi-compact and separated.*

We denote by $\mathsf{Qcoh}\,\mathcal{A}$ and $\mathsf{Coh}\,\mathcal{A}$ the categories, respectively, of quasi-coherent and coherent $\mathcal{A}$-modules. In this context $\mathsf{Qcoh}\,\mathcal{A}$ consists of the sheaves of $\mathcal{A}$-module which are quasi-coherent as sheaves of $\mathcal{O}_X$-modules. If $\mathcal{A}$ is coherent as an $\mathcal{O}_X$-module, then $\mathsf{Coh}\,\mathcal{A}$ consists of $\mathcal{A}$-modules which are coherent as $\mathcal{O}_X$-modules. As we suppose $X$ quasi-compact and separated, the category $\mathsf{Qcoh}\,\mathcal{A}$ is a *Grothendieck category*. In particular, every quasi-coherent $\mathcal{A}$-module has a quasi-coherent injective envelop. A morphism of non-commutative schemes $f : (X, \mathcal{A}) \to (Y, \mathcal{B})$ induces the functors of *direct image* $f_* : \mathsf{Qcoh}\,\mathcal{A} \to \mathsf{Qcoh}\,\mathcal{B}$ and *inverse image* $f^* : \mathsf{Qcoh}\,\mathcal{B} \to \mathsf{Qcoh}\,\mathcal{A}$. Namely, $f^*\mathcal{F} = \mathcal{A} \otimes_{f_X^{-1}\mathcal{B}} f_X^{-1}\mathcal{F}$. As usually, $(f^*, f_*)$ is an adjoint pair, i.e. there is an isomorphism of functors $\operatorname{Hom}_{\mathcal{A}}(f^*\mathcal{F}, \mathcal{G}) \simeq \operatorname{Hom}_{\mathcal{B}}(\mathcal{F}, f_*\mathcal{G})$. Note that $f^*$ does not coincide with the inverse image under the underlying morphism $X \to Y$. We write $\mathsf{Qcoh}\,X$, $\mathsf{Coh}\,X$ etc. instead of $\mathsf{Qcoh}\,\mathcal{O}_X$, $\mathsf{Coh}\,\mathcal{O}_X$ etc. If $(X, \mathcal{A}) = (\operatorname{Spec} \boldsymbol{R}, \boldsymbol{A}\tilde{})$ is affine, we usually identify $\mathsf{Qcoh}\,\mathcal{A}$ with the category $\boldsymbol{A}$-$\mathsf{Mod}$ of $\boldsymbol{A}$-modules. If $\boldsymbol{A}$ is noetherian, $\mathsf{Coh}\,\mathcal{A}$ is identified with the category $\boldsymbol{A}$-$\mathsf{mod}$ of finitely generated $\boldsymbol{A}$-modules.

**Proposition 2.1.1.** *Let $\mathcal{C} = \operatorname{cen}(\mathcal{A})$, the center of $\mathcal{A}$, $X' = \operatorname{Spec}\mathcal{C}$ be the spectrum of the (commutative) $\mathcal{O}_X$-algebra $\mathcal{C}$, $\phi : X' \to X$ be the structural morphism, and $\mathcal{A}' = \phi^{-1}\mathcal{A}$.*

(1) *$\mathcal{A}'$ is an $\mathcal{O}_{X'}$-algebra, so $(X', \mathcal{A}')$ is a non-commutative scheme.*
(2) *For any $\mathcal{F} \in \mathsf{Qcoh}\,\mathcal{A}$ the natural map $\mathcal{F} \to \phi_*\phi^{-1}\mathcal{F}$ an isomorphism.*
(3) *For any $\mathcal{F}' \in \mathsf{Qcoh}\,\mathcal{A}'$ the natural map $\phi^{-1}\phi_*\mathcal{F}' \to \mathcal{F}'$ is an isomorphism.*
(4) *The functors $\phi^{-1}$ and $\phi_*$ establish an equivalence of the categories $\mathsf{Qcoh}\,\mathcal{A}$ and $\mathsf{Qcoh}\,\mathcal{A}'$ as well as of $\mathsf{Coh}\,\mathcal{A}$ and $\mathsf{Coh}\,\mathcal{A}'$.*

Thus we can suppose, when necessary, that our non-commutative scheme $(X, \mathcal{A})$ is *central*, i.e. $\mathcal{O}_X = \operatorname{cen}\mathcal{A}$, in particular, that $\mathcal{A}$ is a *sincere* $\mathcal{O}_X$-module, i.e. $\operatorname{ann}_{\mathcal{O}_X}\mathcal{A} = 0$. In this case, if $(X, \mathcal{A})$ is reduced, so is also the scheme $X$.

*Proof.* All claims are obviously local, so we can suppose that $X = \operatorname{Spec} R$ and $X' = \operatorname{Spec} R'$, where $R'$ is the center of the $R$-algebra $A = \Gamma(X, \mathcal{A})$. Then all claims are trivial. $\square$

We say that a central non-commutative scheme $(X, \mathcal{A})$ is *noetherian* if $X$ is noetherian and $\mathcal{A}$ is coherent as an $\mathcal{O}_X$-module. If $(X, \mathcal{A})$ is not necessarily central, we say that it is *noetherian* if so is the central scheme $(X', \mathcal{A}')$, where $X' = \operatorname{Spec}(\operatorname{cen} \mathcal{A})$ and $\mathcal{A}' = \phi^{-1}(\mathcal{A})$, where $\phi : X' \to X$ is the structural morphism. In particular, an affine non-commutative scheme $(\operatorname{Spec} \boldsymbol{R}, \boldsymbol{A}\tilde{})$ is noetherian if and only if $\boldsymbol{A}$ is a *noetherian algebra*, i.e. $\boldsymbol{C} = \operatorname{cen} \boldsymbol{A}$ is noetherian and $\boldsymbol{A}$ is a finitely generated $\boldsymbol{C}$-module.

We call $\operatorname{ext.dim} \mathsf{Qcoh}\, \mathcal{A} = \operatorname{ext.dim} \mathsf{Coh}\, \mathcal{A}$ the (*left*) *global dimension* of the non-commutative scheme $(X, \mathcal{A})$ and denote it by $\operatorname{gl.dim} \mathcal{A}$. If $(X, \mathcal{A})$ is noetherian, then $\operatorname{gl.dim} \mathcal{A} = \sup \{ \operatorname{gl.dim} \mathcal{A}_x \mid x \in X \}$ and $\operatorname{gl.dim} \mathcal{A} = \operatorname{gl.dim} \mathcal{A}^{\mathrm{op}}$ [11, § 8.3].

Let $(X, \mathcal{A})$ be noetherian. We denote by $\mathsf{lp}\, \mathcal{A}$ the full subcategory of $\mathsf{Coh}\, \mathcal{A}$ consisting of *locally projective* modules, i.e. such $\mathcal{A}$-modules $\mathcal{P}$ that $\mathcal{P}_x$ is a finitely generated projective $\mathcal{A}_x$-module for every point $x \in X$. For instance, if $\mathcal{F}$ is a locally free coherent $\mathcal{O}_X$-module, then the $\mathcal{A}$-module $\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{F}$ is locally projective and locally finitely generated. We say that $(X, \mathcal{A})$ *has enough locally projective modules* if for every coherent $\mathcal{A}$-module $\mathcal{M}$ there is an epimorphism $\mathcal{P} \to \mathcal{M}$ for some module $\mathcal{P} \in \mathsf{lp}\, \mathcal{A}$. Since any quasi-coherent module is the sum of its coherent submodules, then for any quasi-coherent $\mathcal{A}$-module $\mathcal{M}$ there is an epimorphim $\mathcal{P} \to \mathcal{M}$, where $\mathcal{P}$ is a coproduct of modules from $\mathsf{lp}\, \mathcal{A}$. An important example arises as follows.

We say that a non-commutative scheme $(X, \mathcal{A})$ is *quasi-projective* if it is noetherian and there is an ample $\mathcal{O}_X$-module $\mathcal{L}$ [22, Section 4.5]. Note that in this case $X$ is indeed a quasi-projective scheme over the ring $R = \bigoplus_{n=0}^{\infty} \Gamma(X, \mathcal{L}^{\otimes n})$.

**Proposition 2.1.2.** *Any quasi-projective non-commutative scheme* $(X, \mathcal{A})$ *has enough locally projective modules.*

*Proof.* Let $\mathcal{L}$ be an ample $\mathcal{O}_X$-module, $\mathcal{M}$ be any coherent $\mathcal{A}$-module. There is an epimorphism of $\mathcal{O}_X$-modules $n\mathcal{O}_X \to \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes m}$ for some $m$, hence also an epimorphism $\mathcal{F} = n\mathcal{L}^{\otimes(-m)} \to \mathcal{M}$. Since $\operatorname{Hom}_{\mathcal{A}}(\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{F}, \mathcal{M}) \simeq \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{M})$, it gives an epimorphism of $\mathcal{A}$-modules $\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{M}$, where $\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{F} \in \mathsf{lp}\, \mathcal{A}$. $\square$

We define an *invertible* $\mathcal{A}$*-module* as an $\mathcal{A}$-module $\mathcal{I}$ such that $\operatorname{End}_{\mathcal{A}} \mathcal{I} \simeq \mathcal{A}^{\mathrm{op}}$ and the natural map $\mathcal{H}om_{\mathcal{A}}(\mathcal{I}, \mathcal{A}) \otimes_{\mathcal{A}} \mathcal{I} \to (\operatorname{End}_{\mathcal{A}} \mathcal{I})^{\mathrm{op}} \simeq \mathcal{A}$ is an isomorphism. For instance, the modules constructed in the preceding proof are invertible. On the contrary, one easily proves that, if $\mathcal{A}$ is noetherian and $\operatorname{cen} \mathcal{A} = \mathcal{O}_X$, any invertible $\mathcal{A}$-module $\mathcal{I}$ is isomorphic to $\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{L}$, where $\mathcal{L} = \mathcal{H}om_{\mathcal{A}\text{-}\mathcal{A}}(\mathcal{I}, \mathcal{I})$ and $\mathcal{L}$ is an invertible $\mathcal{O}_X$-module. (We will not use this fact.)

We call a noetherian non-commutative scheme $(X,\mathcal{A})$ *strongly Gorenstein*[5] if $\operatorname{inj.dim}_{\mathcal{A}}\mathcal{A}=\dim X$. Since a quasi-coherent sheaf is injective if and only if all its stalks are injective and a localization of an injective module is injective,

$$\operatorname{inj.dim}_{\mathcal{A}}\mathcal{A}=\sup\left\{\operatorname{inj.dim}_{\mathcal{A}_x}\mathcal{A}_x \mid x \text{ is a closed point of } X\right\}.$$

So $(X,\mathcal{A})$ is strongly Gorenstein if and only if all localizations $\mathcal{A}_x$ are strongly Gorenstein for the closed points of $X$. Moreover, we can replace $\mathcal{A}_x$ here by its completion $\hat{\mathcal{A}}_x$ in the $\mathfrak{m}_x$-adic topology.

2.2. **Derived categories and functors.** We denote by $\mathscr{C}\mathcal{A}$, $\mathscr{H}\mathcal{A}$ and $\mathscr{D}\mathcal{A}$ respectively the category of complexes over $\mathsf{Qcoh}\,\mathcal{A}$, the category of complexes modulo homotopy and the derived category of $\mathsf{Qcoh}\,\mathcal{A}$. If $(X,\mathcal{A})=(\operatorname{Spec}\boldsymbol{R},\boldsymbol{A}\tilde{\ })$ is affine, we also write $\mathscr{C}\boldsymbol{A}$, $\mathscr{H}\boldsymbol{A}$ and $\mathscr{D}\boldsymbol{A}$ instead of $\mathscr{C}\mathcal{A}$, $\mathscr{H}\mathcal{A}$ and $\mathscr{D}\mathcal{A}$. They are equivalent to the corresponding categories for complexes of $\boldsymbol{A}$-modules and usually identified with them. By $\mathscr{D}^{\mathrm{c}}\mathcal{A}$ we denote the full subcategory of *compact* objects $C^\bullet$ from $\mathscr{D}\mathcal{A}$. If $X$ is quasi-compact and separated, $\mathscr{D}^{\mathrm{c}}\mathcal{A}$ consists of *perfect complexes*, i.e. the complexes locally quasi-isomorphic to finite complexes of finitely generated locally projective modules. It is proved in [36] for the commutative case. The proof in the non-commutative case is the same, since we have the crucial ingredient:

**Proposition 2.2.1.** *Let $\boldsymbol{R}$ be a commutative ring, $\boldsymbol{A}$ be an $\boldsymbol{R}$-algebra and $\boldsymbol{I}=(f_1,f_2,\dots,f_m)$ be a finitely generated ideal of $\boldsymbol{R}$. Denote by $\boldsymbol{A}\text{-}\mathsf{Mod}_{\boldsymbol{I}}$ the full subcategory of $\boldsymbol{A}\text{-}\mathsf{Mod}$ consisting of all modules $M$ such that for every element $v\in M$ there is an integer $k$ such that $\boldsymbol{I}^k v=0$. Let $\mathscr{D}_{\boldsymbol{I}}\boldsymbol{A}$ be the full subcategory of $\mathscr{D}\boldsymbol{A}$ consisting of all complexes $C^\bullet$ such that $H^i(C^\bullet)\in\boldsymbol{R}\text{-}\mathsf{Mod}_{\boldsymbol{I}}$ for all $i$. Then the Koszul complex $K^\bullet(f_1,f_2,\dots,f_m)=\bigotimes_{i=1}^m(\boldsymbol{A}\xrightarrow{f_i}\boldsymbol{A})$ is a compact generator of $\mathscr{D}_{\boldsymbol{I}}\boldsymbol{A}$.*

Here $\otimes$ denotes $\otimes_{\boldsymbol{A}}$ and $\boldsymbol{A}$ is naturally considered as $\boldsymbol{A}$-bimodule.

*Proof.* It just repeats the proof of [7, Proposition 6.1], since the elements $f_1,f_2,\dots,f_m$ act centrally on $\boldsymbol{A}$. □

If $(X,\mathcal{A})$ is noetherian and has enough locally projective modules (for instance, is quasi-projective), the set $\mathbf{P}$ of coherent locally projective modules is a set of compact generators for $\mathsf{Qcoh}\,\mathcal{A}$. Therefore, $\langle\,\mathbf{P}\,\rangle^\omega=\mathscr{D}\mathcal{A}$, hence $\mathscr{D}^{\mathrm{c}}\mathcal{A}$ consists of complexes quasi-isomorphic to finite complexes of locally projective modules.

Recall that a complex $\mathcal{I}^\bullet$ is said to be *K-injective* [43] if for every acyclic complex $\mathcal{C}^\bullet$ the complex $\operatorname{Hom}^\bullet(\mathcal{C}^\bullet,\mathcal{I}^\bullet)$ is acyclic too. We denote by $\mathsf{K}\text{-}\mathsf{inj}\,\mathcal{A}$ the full subcategory of $\mathscr{H}\mathcal{A}$ consisting of K-injective complexes and by $\mathsf{K}\text{-}\mathsf{inj}_0\,\mathcal{A}$ its full subcategory consisting of acyclic K-injective complexes.

[5] We use this term to avoid the ambiguity, since there are several versions of "Gorenstein" in non-commutative case. Note that according to our definition $\mathcal{A}^{\mathrm{op}}$ is strongly Gorenstein if so is $\mathcal{A}$.If $\mathcal{A}=\mathcal{O}_X$, strongly Gorenstein is the same as Gorenstein in the usual sense of the commutative algebra.

**Corollary 2.2.2.** *Let $(X, \mathcal{A})$ be a non-commutative scheme.*

(1) *For every complex $\mathcal{C}^\bullet$ in $\mathscr{C}\mathcal{A}$ there is a* K-injective resolution*, i.e. a K-injective complex $\mathcal{I}^\bullet \in \mathscr{C}\mathcal{A}$ together with a quasi-isomorphism $\mathcal{C}^\bullet \rightsquigarrow \mathcal{I}^\bullet$.*
(2) $\mathscr{D}\mathcal{A} \simeq \mathsf{K\text{-}inj}\,\mathcal{A}/\,\mathsf{K\text{-}inj}_0\,\mathcal{A}$.

*Proof.* As the category $\mathsf{Qcoh}\,\mathcal{A}$ is a Grothendieck category, (1) follows immediately from [2, Proposition 5.3] (see also [43, Lemma 3.7 and Proposition 3.13]). Then (2) follows from [43, Proposition 1.5]. □

A complex $\mathcal{F}^\bullet$ is said to be *K-flat* [43] if for every acyclic complex $\mathcal{S}^\bullet$ of *right* $\mathcal{A}$-modules the complex $\mathcal{F}^\bullet \otimes_{\mathcal{A}} \mathcal{S}^\bullet$ is acyclic. The next result is quite analogous to [1, Proposition 1.1] and the proof just repeats that of the cited paper.

**Proposition 2.2.3.** *Let $(X, \mathcal{A})$ be a non-commutative scheme. Then for every complex $\mathcal{C}^\bullet$ in $\mathscr{C}\mathcal{A}$ there is a* K-flat replica*, i.e. a K-flat complex $\mathcal{F}^\bullet$ quasi-isomorphic to $\mathcal{C}^\bullet$.*

*Remark* 2.2.4. If $(X, \mathcal{A})$ is noetherian and has enough locally projective modules, every complex from $\mathscr{C}^-\mathcal{A}$ has a locally projective (hence flat) resolution. Then [43, Theorem 3.4] implies that for every complex $\mathcal{C}$ from $\mathscr{A}\mathcal{A}$ there is an *Lp-resolution*, i.e. a K-flat complex $\mathcal{F}^\bullet$ consisting of locally projective modules together with a quasi-isomorphism $\mathcal{F}^\bullet \rightsquigarrow \mathcal{C}^\bullet$. For instance, it is the case if $(X, \mathcal{A})$ is *quasi-projective* (Proposition 2.1.2).

A complex $\mathcal{I}^\bullet$ is said to be *weakly K-injective* if for every acyclic K-flat complex $\mathcal{F}^\bullet$ the complex $\mathrm{Hom}^\bullet(\mathcal{F}^\bullet, \mathcal{I}^\bullet)$ is exact.

**Proposition 2.2.5** ([43, Propositions 5.4 and 5.15])**.** *Let $f : (X, \mathcal{A}) \to (Y, \mathcal{B})$ be a morphism of non-commutative scheme.*

(1) *If $\mathcal{F}^\bullet \in \mathscr{C}\mathcal{B}$ is K-flat, then so is also $f^*\mathcal{F}^\bullet$. If, moreover, $\mathcal{F}$ is flat and acyclic, then $f^*\mathcal{F}$ is acyclic too.*
(2) *If $\mathcal{I} \in \mathscr{C}\mathcal{A}$ is weakly K-injective, then $f_*\mathcal{I}$ is weakly K-injective. If, moreover, $\mathcal{I}$ is weakly K-injective and acyclic, then $f_*\mathcal{I}$ is acyclic too.*

**Proposition 2.2.6** (cf. [43, Section 6])**.** *Let $(X, \mathcal{A})$ be a non-commutative scheme.*

(1) *The derived functors $\mathrm{RHom}^\bullet_{\mathcal{A}}(\mathcal{F}^\bullet, \mathcal{G}^\bullet)$ and $\mathrm{R}\mathcal{H}om^\bullet_{\mathcal{A}}(\mathcal{F}^\bullet, \mathcal{G}^\bullet)$ exist and can be calculated using a K-injective resolution of $\mathcal{G}^\bullet$ or a weakly K-injective resolution of $\mathcal{G}^\bullet$ and a K-flat replica of $\mathcal{F}^\bullet$.*
(2) *The derived functor $\mathcal{F}^\bullet \overset{\mathsf{L}}{\otimes}_{\mathcal{A}} \mathcal{G}^\bullet$, where $\mathcal{G}^\bullet \in \mathscr{D}\mathcal{A}^{\mathrm{op}}$, exists and can be calculated using a K-flat replica either of $\mathcal{F}$ or of $\mathcal{G}$. Moreover, if $\mathcal{G}$ is an $\mathcal{A}$-$\mathcal{B}$-bimodule, where $\mathcal{B}$ is another sheaf of $\mathcal{O}_X$-algebras, there*

*are isomorphisms of functors*

$$\mathrm{RHom}_{\mathcal{B}}(\mathcal{F}^\bullet \overset{\mathsf{L}}{\otimes}_{\mathcal{A}} \mathcal{G}^\bullet, \mathcal{M}^\bullet) \simeq \mathrm{RHom}_{\mathcal{A}}(\mathcal{F}, \mathrm{R}\mathcal{H}om_{\mathcal{B}}(\mathcal{G}^\bullet, \mathcal{M}^\bullet))$$

$$\mathrm{R}\mathcal{H}om_{\mathcal{B}}(\mathcal{F}^\bullet \overset{\mathsf{L}}{\otimes}_{\mathcal{A}} \mathcal{G}^\bullet, \mathcal{M}^\bullet) \simeq \mathrm{R}\mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathrm{R}\mathcal{H}om_{\mathcal{B}}(\mathcal{G}^\bullet, \mathcal{M}^\bullet)).$$

(3) *For every morphism $f : (X, \mathcal{A}) \to (Y, \mathcal{B})$ the derived functors $\mathrm{L}f^* : \mathscr{D}\mathcal{B} \to \mathscr{D}\mathcal{A}$ and $\mathrm{R}f_* : \mathscr{D}\mathcal{A} \to \mathscr{D}\mathcal{B}$ exist. They can be calculated using, respectively, K-flat replicas in $\mathscr{C}\mathcal{B}$ and weakly K-injective resolutions in $\mathscr{C}\mathcal{A}$. Moreover, there are isomorphisms of functors*

$$\mathrm{RHom}^\bullet_{\mathcal{B}}(\mathcal{F}^\bullet, \mathrm{R}f_*\mathcal{G}^\bullet) \simeq \mathrm{RHom}^\bullet_{\mathcal{A}}(\mathrm{L}f^*\mathcal{F}^\bullet, \mathcal{G}^\bullet)$$
$$\mathrm{R}\mathcal{H}om^\bullet_{\mathcal{B}}(\mathcal{F}^\bullet, \mathrm{R}f_*\mathcal{G}^\bullet) \simeq \mathrm{R}f_*\, \mathrm{R}\mathcal{H}om^\bullet_{\mathcal{A}}(\mathrm{L}f^*\mathcal{F}^\bullet, \mathcal{G}^\bullet).$$

(4) *If $g : (Y, \mathcal{B}) \to (Z, \mathcal{C})$ is another morphism of non-commutative schemes, then $\mathrm{L}(g \circ f)^* \simeq \mathrm{L}f^* \circ \mathrm{L}g^*$ and $\mathrm{R}(g \circ f)_* \simeq \mathrm{R}g_* \circ \mathrm{R}f_*$.*

*If the considered non-commutative schemes have enough locally projective modules (for instance, are quasi-projective), one can replace in these statements K-flat replicas by Lp-resolutions.*

In particular, let $f : \boldsymbol{A} \to \boldsymbol{B}$ be a homomorphism of rings. We consider $\boldsymbol{B}$ as an algebra over a subring $\boldsymbol{S}$ (an arbitrary one) of its center and $\boldsymbol{A}$ as an algebra over the subring $\boldsymbol{R} = \mathrm{cen}\, \boldsymbol{A} \cap f^{-1}(\boldsymbol{S})$. Then we can identify $f$ with its sheafification $f^{\sim} : (\mathrm{Spec}\, \boldsymbol{S}, \boldsymbol{B}^{\sim}) \to (\mathrm{Spec}\, \boldsymbol{R}, \boldsymbol{A}^{\sim})$. In this context the functors $(f^{\sim})^*$ and $(f^{\sim})_*$ are just sheafifications of the, respectively, the "back-up" functor ${}_BM \mapsto {}_AM$ and the "change-of-scalars" functor ${}_AN \mapsto {}_BB \otimes_A N$.

### 2.3. **Minors.**

**Definition 2.3.1.** Let $(X, \mathcal{B})$ be a non-commutative scheme, $\mathcal{P}$ be a locally projective and locally finitely generated $\mathcal{B}$-module, $\mathcal{A} = (\mathcal{E}nd_{\mathcal{B}}\, \mathcal{P})^{\mathrm{op}}$. The non-commutative scheme $(X, \mathcal{A})$ is called a *minor* of the non-commutative scheme $(X, \mathcal{B})$.[6]

In this situation we consider $\mathcal{P}$ as $\mathcal{B}$-$\mathcal{A}$-bimodule (left over $\mathcal{B}$, right over $\mathcal{A}$). Let $\mathcal{P}^\vee = \mathcal{H}om_{\mathcal{B}}(\mathcal{P}, \mathcal{B})$. It is an $\mathcal{A}$-$\mathcal{B}$-bimodule, locally projective and locally finitely generated over $\mathcal{B}$. The following statements are evidently local, then they are well-known.

**Proposition 2.3.2.** *The natural homomorphism $\mathcal{P} \to \mathcal{H}om_{\mathcal{B}}(\mathcal{P}^\vee, \mathcal{B})$ is an isomorphism. Moreover, $\mathcal{A} \simeq \mathcal{E}nd_{\mathcal{B}}\, \mathcal{P}^\vee \simeq \mathcal{P}^\vee \otimes_{\mathcal{B}} \mathcal{P}$.*

We consider the following functors:

$$\begin{aligned} \mathsf{F} &= \mathcal{P} \otimes_{\mathcal{A}} \_ : \mathsf{Qcoh}\, \mathcal{A} \to \mathsf{Qcoh}\, \mathcal{B}, \\ \mathsf{G} &= \mathcal{H}om_{\mathcal{B}}(\mathcal{P}, \_) : \mathsf{Qcoh}\, \mathcal{B} \to \mathsf{Qcoh}\, \mathcal{A}, \\ \mathsf{H} &= \mathcal{H}om_{\mathcal{A}}(\mathcal{P}^\vee, \_) : \mathsf{Qcoh}\, \mathcal{A} \to \mathsf{Qcoh}\, \mathcal{B}. \end{aligned} \tag{2.3.1}$$

[6] In the affine case this notion was introduced in [16]. Actually, the main results of this section are just global analogues of those from [16].

Note that $\mathsf{G}$ is exact and $\mathsf{G} \simeq \mathcal{P}^\vee \otimes_{\mathcal{B}} \_$, so both $(\mathsf{F}, \mathsf{G})$ and $(\mathsf{G}, \mathsf{H})$ are adjoint pairs of functors. If the non-commutative scheme $(X, \mathcal{B})$ is noetherian, so is also $(X, \mathcal{A})$ and these functors map coherent sheaves to coherent ones.

**Theorem 2.3.3.** (1) *$\mathsf{Qcoh}\,\mathcal{A} \simeq \mathsf{Qcoh}\,\mathcal{B}/\mathscr{C}$, where $\mathscr{C} = \operatorname{Ker}\mathsf{G} = \mathcal{P}^\perp$ is a bilocalizing subcategory of $\mathsf{Qcoh}\,\mathcal{B}$. Thus $\mathsf{Qcoh}\,\mathcal{A}$ is a bilocalization of $\mathsf{Qcoh}\,\mathcal{B}$.*
(2) *The natural morphism $\phi : \mathbb{1}_{\mathsf{Qcoh}\,\mathcal{A}} \to \mathsf{G} \circ \mathsf{F}$ is an isomorphism.*
(2′) *The natural morphism $\phi' : \mathsf{G} \circ \mathsf{H} \to \mathbb{1}_{\mathsf{Qcoh}\,\mathcal{A}}$ is an isomorphism.*
(3) *The functor $\mathsf{F}$ is a full embedding and its essential image is ${}^\perp\mathscr{C}$. So the pair $(\mathsf{F}, \mathsf{G})$ induces an equivalence between $\mathsf{Qcoh}\,\mathcal{A}$ and ${}^\perp\mathscr{C}$.*
(3′) *The functor $\mathsf{H}$ is a full embedding and its essential image is $\mathscr{C}^\perp$. So the pair $(\mathsf{H}, \mathsf{G})$ induces an equivalence between $\mathsf{Qcoh}\,\mathcal{A}$ and $\mathscr{C}^\perp$.*
(4) *${}^\perp\mathscr{C}$ coincides with the full subcategory $\mathsf{Qcoh}\,\mathcal{P}$ of $\mathsf{Qcoh}\,\mathcal{B}$ consisting of all modules $\mathcal{M}$ such that for every point $x \in X$ there is an exact sequence $P_1 \to P_0 \to \mathcal{M}_x \to 0$, where $P_0, P_1$ are multiples (maybe, infinite) of $\mathcal{P}_x$.*

*Proof.* Theorem 1.3.1 and Corollary 1.3.2 show that it is enough to prove the following statement.

**Proposition 2.3.4.** *The natural morphism $\phi : \mathbb{1}_{\mathsf{Qcoh}\,\mathcal{A}} \to \mathsf{G} \circ \mathsf{F}$ is an isomorphism and the essential image of $\mathsf{F}$ coincides with $\mathsf{Qcoh}\,\mathcal{P}$.*

As both claims are local, we can suppose that the non-commutative scheme $(X, \mathcal{B})$ is affine, so replace $\mathsf{Qcoh}\,\mathcal{B}$ by $\boldsymbol{B}$-$\mathsf{Mod}$, where $\boldsymbol{B} = \Gamma(X, \mathcal{B})$. Then $\mathcal{P} = P^\sim$ for some finitely generated projective $\boldsymbol{B}$-module and $\mathcal{A} = \boldsymbol{A}^\sim$, where $\boldsymbol{A} = (\operatorname{End}_{\boldsymbol{B}} P)^{\mathrm{op}}$. Hence we can also replace $\mathsf{Qcoh}\,\mathcal{A}$ by $\boldsymbol{A}$-$\mathsf{Mod}$ and $\mathsf{Qcoh}\,\mathcal{P}$ by $P$-$\mathsf{Mod}$, the full subcategory of $\boldsymbol{B}$-$\mathsf{Mod}$ consisting of all modules $N$ such that there is an exact sequence $P_1 \to P_0 \to N \to 0$, where $P_i$ are multiples (maybe infinite) of $P$.

Obviously, $\phi(\boldsymbol{A})$ is an isomorphism. Since $\mathsf{F}$ and $\mathsf{G}$ preserve arbitrary coproducts, $\phi(F)$ is an isomorphism for any free $\boldsymbol{A}$-module $F$. Let $M \in \mathcal{A}$-$\mathsf{Mod}$. There is an exact sequence $F_1 \to F_0 \to M \to 0$, which gives rise to a commutative diagram with exact rows

$$\begin{array}{ccccccc}
F_1 & \longrightarrow & F_0 & \longrightarrow & M & \longrightarrow & 0 \\
\downarrow{\scriptstyle \phi(F_1)} & & \downarrow{\scriptstyle \phi(F_0)} & & \downarrow{\scriptstyle \phi(M)} & & \\
\mathsf{G}\circ\mathsf{F}(F_1) & \longrightarrow & \mathsf{G}\circ\mathsf{F}(F_0) & \longrightarrow & \mathsf{G}\circ\mathsf{F}(M) & \longrightarrow & 0
\end{array}$$

As the first two vertical arrows are isomorphisms, so is $\phi(M)$, which proves the first claim. Moreover, we get an exact sequence $\mathsf{F}(F_1) \to \mathsf{F}(F_0) \to \mathsf{F}(N) \to 0$, where $\mathsf{F}(F_i)$ are direct multiples (maybe infinite) of $\mathsf{F}(\boldsymbol{A}) = P$. Therefore, $\mathsf{F}(\boldsymbol{A}) \in P$-$\mathsf{Mod}$.

Consider now the natural morphism $\psi : \mathsf{F} \circ \mathsf{G} \to \mathbb{1}_{B\text{-}\mathsf{Mod}}$. This time $\psi(P)$ is an isomorphism. Let now $N$ be a $\boldsymbol{B}$-module such that there is an exact sequence $P_1 \to P_0 \to N$, where $P_i$ are multiples (maybe infinite) of $P$. Hence there is a commutative diagram with exact rows

$$\begin{tikzcd} \mathsf{F}\circ\mathsf{G}(P_1) \arrow[r] \arrow[d, "\psi(P_1)"'] & \mathsf{F}\circ\mathsf{G}(P_0) \arrow[r] \arrow[d, "\psi(P_0)"'] & \mathsf{F}\circ\mathsf{G}(N) \arrow[r] \arrow[d, "\psi(N)"'] & 0 \\ P_1 \arrow[r] & P_0 \arrow[r] & N \arrow[r] & 0 \end{tikzcd}$$

The first two vertical arrows are isomorphisms, so $\psi(N)$ is also an isomorphism. Thus the essential image of $\mathsf{F}$ is indeed $P$-$\mathsf{Mod}$. $\square$

Actually, we can describe the kernel of this bilocalization explicitly.

**Theorem 2.3.5.** *Let* $\mathcal{I}_{\mathcal{P}} = \operatorname{Im}\{\mu_{\mathcal{P}} : \mathcal{P}\otimes_{\mathcal{A}}\mathcal{P}^\vee \to \mathcal{B}\}$, *where* $\mu(p\otimes\gamma) = \gamma(p)$. *Then* $\operatorname{Ker}\mathsf{G} = \{\,\mathcal{M}\in \mathsf{Qcoh}\,\mathcal{B} \mid \mathcal{I}_{\mathcal{P}}\mathcal{M} = 0\,\} \simeq \mathsf{Qcoh}(\mathcal{B}/\mathcal{I}_{\mathcal{P}})$.

*Proof.* Again the statement is local, so we only have to prove it for a ring $\boldsymbol{B}$, a finitely generated projective $\boldsymbol{B}$-module $P$ and the ideal $I_P = \operatorname{Im}\mu_P$. It follows from [13, Proposition VII.3.1] that $I_P P = P$. Therefore, if $f : P \to M$ is non-zero, then $I_P \operatorname{Im} f = \operatorname{Im} f \neq 0$, hence $I_P M \neq 0$. On the contrary, if $I_P M \neq 0$, there is an element $u \in M$, elements $p_i \in P$ and homomorphisms $\gamma_i : P \to B$ such that $\sum_i \gamma_i(p_i)u \neq 0$. Let $\beta : B \to M$ maps 1 to $u$ and $\gamma_i^u = \beta\gamma_i$. Then at least one of the homomorphisms $\gamma_i^u$ is non-zero. $\square$

The functor $\mathsf{G}$ is exact, so it induces a functor $\mathsf{DG} : \mathscr{D}\mathcal{B} \to \mathscr{D}\mathcal{A}$ mapping a complex $\mathcal{F}^\bullet$ to $\mathsf{G}\mathcal{F}^\bullet$. It is both left and right derived functor of $\mathsf{G}$. We can also consider the left derived functor $\mathsf{LF}$ of $\mathsf{F}$ and the right derived functor $\mathsf{RH}$ of $\mathsf{H}$, both being functors $\mathscr{D}\mathcal{A} \to \mathscr{D}\mathcal{B}$. Obviously, $\mathsf{DG}$ maps $\mathscr{D}^\sigma\mathcal{B}$ to $\mathscr{D}^\sigma\mathcal{A}$, where $\sigma \in \{+,-,b\}$, $\mathsf{LF}$ maps $\mathscr{D}^\sigma\mathcal{A}$ to $\mathscr{D}^\sigma\mathcal{B}$ for $\sigma\in\{-,b\}$, and $\mathsf{RH}$ maps $\mathscr{D}^\sigma\mathcal{A}$ to $\mathscr{D}^\sigma\mathcal{B}$ for $\sigma\in\{+,b\}$.

**Theorem 2.3.6.**
1. *The functors* $(\mathsf{LF},\mathsf{DG})$ *and* $(\mathsf{DG},\mathsf{RH})$ *form adjoint pairs.*
2. $\mathscr{D}\mathcal{A} \simeq \mathscr{D}\mathcal{B}/\mathscr{D}_{\mathscr{C}}\mathcal{B}$, *where* $\mathscr{C} = \operatorname{Ker}\mathsf{G} = \mathcal{P}^\perp$, *as in Theorem 2.3.3. Moreover,* $\mathscr{D}_{\mathscr{C}}\mathcal{B}$ *is a bilocalizing subcategory of* $\mathscr{D}\mathcal{B}$, *so* $\mathscr{D}\mathcal{A}$ *is a bilocalization of* $\mathscr{D}\mathcal{B}$.
3. *The natural map* $\mathbb{1}_{\mathscr{D}\mathcal{A}} \to \mathsf{DG}\circ\mathsf{LF}$ *is an isomorphism.*

(3′) *The natural map* $\mathsf{DG}\circ\mathsf{RH} \to \mathbb{1}_{\mathscr{D}\mathcal{A}}$ *is an isomorphism.*

4. *The functor* $\mathsf{LF}$ *is a full embedding and its essential image is* ${}^\perp\mathscr{C}$. *So the pair* $(\mathsf{LF},\mathsf{DG})$ *defines an equivalence* $\mathscr{D}\mathcal{A}\simeq {}^\perp\mathscr{C}$.

(4′) *The functor* $\mathsf{RH}$ *is a full embedding and its essential image is* $\mathscr{C}^\perp$. *So the pair* $(\mathsf{RH},\mathsf{DG})$ *defines an equivalence* $\mathscr{D}\mathcal{A}\simeq\mathscr{C}^\perp$.

5. *The functor* $\mathsf{LF}$ *maps* $\mathscr{D}^{\mathrm{c}}\mathcal{A}$ *to* $\mathscr{D}^{\mathrm{c}}\mathcal{B}$.
6. ${}^\perp\mathscr{C}$ *coincides with the full subcategory* $\mathscr{D}\underrightarrow{\mathcal{P}}$ *of* $\mathscr{D}\mathcal{B}$ *consisting of complexes quasi-isomorphic to K-flat complexes* $\mathcal{F}^\bullet$ *such that for every* $x\in X$ *and every component* $\mathcal{F}^i$ *the localization* $\mathcal{F}^i_x$ *belongs to* $\underrightarrow{\operatorname{add}}\,\mathcal{P}_x$. *The same is true if we replace* $\mathscr{D}$ *by* $\mathscr{D}^-$.

(7) *If $\mathcal{A}$ and $\mathcal{B}$ have enough locally projective modules (for instance, $X$ is quasi-projective), ${}^{\perp}\mathscr{C}$ coincides with the full subcategory $\mathscr{D}\mathcal{P}$ of $\mathscr{D}\mathcal{B}$ consisting of complexes quasi-isomorphic to K-flat complexes $\mathcal{F}^\bullet$ such that $\mathcal{F}^i_x \in \operatorname{Add}\mathcal{P}_x$ for every $i \in \mathbb{Z}$ and every point $x \in X$. The same is true if we replace $\mathscr{D}$ by $\mathscr{D}^-$.*

*Proof.* (1) Let $\mathcal{F}^\bullet$ be a K-flat replica of $\mathcal{M}^\bullet \in \mathscr{D}\mathcal{A}$ and $\mathcal{I}^\bullet$ be an injective resolution of $\mathcal{N}^\bullet \in \mathscr{D}\mathcal{B}$. Then $\mathsf{LF}\mathcal{M}^\bullet = \mathsf{F}\mathcal{F}^\bullet$ and $\mathsf{DG}\mathcal{N}^\bullet = \mathsf{G}\mathcal{I}^\bullet$. As $\mathcal{P} \in \mathsf{lp}\,\mathcal{B}$, the complex $\mathsf{F}\mathcal{F}^\bullet$ is K-flat and the complex $\mathsf{G}\mathcal{I}^\bullet$ is K-injective. By Proposition 2.2.6 (2),

$$\begin{aligned}\operatorname{Hom}_{\mathcal{B}}(\mathsf{F}\mathcal{F}^\bullet, \mathcal{I}^\bullet) = \operatorname{RHom}_{\mathcal{B}}(\mathsf{F}\mathcal{F}^\bullet, \mathcal{I}^\bullet) \simeq& \\ \simeq \operatorname{RHom}_{\mathcal{A}}(\mathcal{F}^\bullet, \mathsf{G}\mathcal{I}^\bullet) &= \operatorname{Hom}_{\mathcal{A}}(\mathcal{F}^\bullet, \mathsf{G}\mathcal{I}^\bullet).\end{aligned}$$

Hence $\mathsf{LF}$ is left adjoint to $\mathsf{DG}$.

Choose now a K-flat replica $\mathcal{G}^\bullet$ of $\mathcal{N}^\bullet$ and a K-injective resolution $\mathcal{J}^\bullet$ of $\mathcal{M}^\bullet$. Then $\mathsf{DG}\mathcal{N}^\bullet = \mathsf{G}\mathcal{G}^\bullet$ and $\mathsf{RH}\mathcal{M}^\bullet = \mathsf{H}\mathcal{J}^\bullet$. By [43, Proposition 5.14], $\mathsf{H}\mathcal{J}^\bullet$ is weakly K-injective. By Proposition 2.2.6 (2) and [43, Proposition 6.1],

$$\begin{aligned}\operatorname{Hom}_{\mathcal{A}}(\mathsf{G}\mathcal{G}^\bullet, \mathcal{J}^\bullet) = \operatorname{RHom}_{\mathcal{A}}(\mathsf{G}\mathcal{G}^\bullet, \mathcal{J}^\bullet) \simeq& \\ \simeq \operatorname{RHom}_{\mathcal{B}}(\mathcal{G}^\bullet, \mathsf{H}\mathcal{J}^\bullet) &= \operatorname{Hom}_{\mathcal{B}}(\mathcal{G}^\bullet, \mathsf{H}\mathcal{J}^\bullet).\end{aligned}$$

Hence $\mathsf{RH}$ is right adjoint to $\mathsf{DG}$.

The statements (3) and (3′) follow from the statements (2) and (2′) of Theorem 2.3.3. Then the statements (2),(4) and (4′) follow from Theorem 1.3.1 and Corollary 1.3.2.

(5) As the right adjoint $\mathsf{DG}$ of $\mathsf{LF}$ preserves arbitrary coproducts, $\mathsf{LF}$ maps compact objects to compact ones.

(6) The construction of [1, Proposition 1.1] gives for any complex $\mathcal{M}^\bullet \in \mathscr{D}\mathscr{A}$ a quasi-isomorphic K-flat complex $\mathcal{F}^\bullet$ such that all its components $\mathcal{F}^i$ are flat. Moreover, $\mathcal{F}^\bullet$ is left bounded if so is $\mathcal{M}^\bullet$. By [11, Ch. X, § 1, Théorème 1], $\mathcal{F}^i_x \simeq \varinjlim \mathcal{L}^i_n$, where $\mathcal{L}^i_n$ are projective finitely generated $\mathcal{A}_x$-modules, hence belong to $\operatorname{add}\mathcal{A}_x$. Then $\mathsf{LF}\mathcal{M}^\bullet \simeq \mathsf{F}\mathcal{F}^\bullet$. As $\mathsf{F}$ preserves direct limits and $\mathsf{F}\mathcal{A} \simeq \mathcal{P}$, $\mathsf{F}\mathcal{F}^i_x \simeq \varinjlim \mathsf{F}\mathcal{L}^i_n$ and $\mathsf{F}\mathcal{L}^i_n \in \operatorname{add}\mathcal{P}_x$. Hence $\mathcal{M}^\bullet \in \mathscr{D}\underrightarrow{\mathcal{P}}$.

On the contrary, let $\mathcal{N}^\bullet \in \mathscr{D}\underrightarrow{\mathcal{P}}$. We can suppose that it is K-flat and for every $i \in \mathbb{Z}$ and every $x \in X$ we can present $\mathcal{N}^i_x$ as $\varinjlim \mathcal{N}^i_n$, where $\mathcal{N}^i_n \in \operatorname{add}\mathcal{P}_x$. Then the complex $\mathsf{G}\mathcal{N}^\bullet$ is also K-flat [43, Proposition 5.4], so $\mathsf{LF} \circ \mathsf{DG}(\mathcal{N}^\bullet) \simeq \mathsf{FG}(\mathcal{N}^\bullet)$. As the natural map $\mathsf{FG}(\mathcal{P}) \to \mathcal{P}$ is an isomorphism, the same is true for any module from $\underrightarrow{\operatorname{add}}\,\mathcal{P}_x$. Therefore, the natural map $\mathsf{LF} \circ \mathsf{DG}(\mathcal{N}) \to \mathcal{N}$ is also an isomorphism.

The proof of (7) is quite analogous to the proof of (6), taking into account that in this situation every complex is quasi-isomorphic to a K-flat complex of locally projective modules. $\square$

## 3. Non-commutative curves

3.1. **Non-commutative curves.** We call a *curve* a noetherian excellent reduced scheme such that all its irreducible components are of dimension 1. We call a *non-commutative curve* a reduced non-commutative scheme $(X, \mathcal{A})$ such that $X$ is a curve and $\mathcal{A}$ is a torsion free finitely generated $\mathcal{O}_X$-module. Note that, due to Proposition 2.1.1, we can suppose, without loss of generality, that the $\mathcal{O}_X$-module $\mathcal{A}$ is sincere. In what follows $(X, \mathcal{A})$ always denotes a non-commutative scheme and we suppose that $\mathcal{A}$ is a sincere $\mathcal{O}_X$-module. We denote by $X_{\mathrm{cl}}$ the set of closed points of $X$, by $X_{\mathrm{reg}}$ and $X_{\mathrm{sg}}$, respectively, its subsets of regular and singular points. As $X$ is excellent and reduced, the set $X_{\mathrm{sg}}$ is finite.

We denote by $\mathsf{tf}\,\mathcal{A}$ the full subcategory of $\mathsf{Coh}\,\mathcal{A}$ consisting of modules which are torsion free or, the same, maximal Cohen–Macaulay over $\mathcal{O}_X$. Recall that modules from $\mathsf{tf}\,\mathcal{A}$ can be defined locally. Namely, let $\mathcal{K} = \mathcal{K}_X$ be the $\mathcal{O}_X$-algebra which is locally the full ring of quotients of $\mathcal{O}_X$ and $\mathcal{KM} = \mathcal{K} \otimes_{\mathcal{O}_X} \mathcal{M}$. Then $\mathcal{KA}$ is a sheaf of semi-simple $\mathcal{K}$-algebras and $\mathcal{A}$ is an $\mathcal{O}_X$*-order* in $\mathcal{KA}$. It means that $\mathcal{A}$ is an $\mathcal{O}_X$-subalgebra of $\mathcal{KA}$ coherent as $\mathcal{O}_X$-module and such that it generates $\mathcal{KA}$ as $\mathcal{K}$-module. If $\mathcal{V}$ is a coherent $\mathcal{KA}$-module, $\mathcal{M} \subset \mathcal{V}$ is its coherent $\mathcal{A}$-submodule and $\mathcal{KM} = \mathcal{V}$, we say that $\mathcal{M}$ is an $\mathcal{A}$*-lattice* in $\mathcal{V}$. Obviously, then $\mathcal{M} \in \mathsf{tf}\,\mathcal{A}$, and any $\mathcal{M} \in \mathsf{tf}\,\mathcal{A}$ can (and will) be considered as an $\mathcal{A}$-lattice in $\mathcal{KM}$. If $\mathcal{M}$ is an $\mathcal{A}$-lattice in $\mathcal{V}$ and $x$ is a closed point of $X$, then $\mathcal{M}_x$ is an $\mathcal{A}_x$-lattice in $\mathcal{V}_x$ and $\mathcal{M}$ is completely defined by the set of $\mathcal{A}_x$-lattices $\{\, \mathcal{M}_x \mid x \in X_{\mathrm{cl}} \,\}$.

**Lemma 3.1.1.** (1) *If $\mathcal{M}$ and $\mathcal{N}$ are lattices in a coherent $\mathcal{KA}$-module $\mathcal{V}$, then $\mathcal{M}_x = \mathcal{N}_x$ for almost all closed points $x \in X$.*

(2) *Let $\mathcal{M} \in \mathsf{tf}\,\mathcal{A}$, $\mathcal{V} = \mathcal{KM}$, $S$ be a finite set of closed points of $X$ and for each $x \in S$ be given a $\mathcal{KA}_x$-lattice $N(x) \subset \mathcal{V}_x$. Then there is an $\mathcal{A}$-lattice $\mathcal{N} \subset \mathcal{V}$ such that $\mathcal{N}_x = N(x)$ for all $x \in S$ and $\mathcal{N}_x = \mathcal{M}_x$ for all $x \notin S$.*

*Proof.* The first statement is evident. The second one is easily reduced to the affine case. Then one can repeat the proof of [10, Ch. 7, §4, Théorème 3] almost without changes. □

**Lemma 3.1.2.** *Any non-commutative curve $(X, \mathcal{A})$ has enough invertible modules. Namely, the set*

$$\mathbf{L}_{\mathcal{A}} = \{\, \mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{L} \mid \mathcal{L} \text{ is an invertible ideal of } \mathcal{O}_X \,\}$$

*is a set of generators for $\mathsf{Qcoh}\,\mathcal{A}$.*

*Proof.* It is enough to prove this statement for the curve $X$. Moreover, we can suppose $X$ irreducible. We have to show that for any quasi-coherent $\mathcal{O}_X$-module $\mathcal{M}$ there is a non-zero homomorphism $\mathcal{L} \to \mathcal{M}$, where $\mathcal{L}$ is an invertible submodule of $\mathcal{O}_X$. Evidently, we can suppose that $\mathcal{M}$ is coherent and either torsion or torsion free. If $\mathcal{M}$ is torsion and $x \in \operatorname{supp} \mathcal{M}$, there is a non-zero map $\phi : \mathcal{O}_{X,x} \to \mathcal{M}_x$. There is an ideal $\mathcal{N} \subset \mathcal{O}_X$ such that

$\mathcal{N}_x = \operatorname{Ker}\phi$ and $\mathcal{N}_y = \mathcal{O}_{X,y}$ for $y \neq x$. Then $\mathcal{O}_X/\mathcal{N} \simeq \operatorname{Im}\phi \subseteq \mathcal{M}$ which gives a non-zero map $\mathcal{O}_X \to \mathcal{M}$.

Let now $\mathcal{M}$ be torsion free, so $\mathcal{M} \subset \mathcal{K}\mathcal{M}$. Replacing $\mathcal{M}$ by its intersection with a simple submodule of $\mathcal{K}\mathcal{M}$, we can suppose that $\mathcal{M} \subset \mathcal{K}'$, where $\mathcal{K}'$ is a simple direct summand of $\mathcal{K}$. Let $\mathcal{O}'$ be the projection of $\mathcal{O}_X$ onto $\mathcal{K}'$. Then both modules $\mathcal{M}/\mathcal{M}\cap\mathcal{O}'$ and $\mathcal{O}'/\mathcal{M}\cap\mathcal{O}'$ are torsion. Let $S = (\operatorname{supp}\mathcal{M}/\mathcal{M}\cap\mathcal{O}') \cup (\operatorname{supp}\mathcal{O}'/\mathcal{M}\cap\mathcal{O}')$. It is a finite set. For each $x \in S$ there is a submodule $L'(x) \subseteq \mathcal{M}_x \cap \mathcal{O}'_x$ isomorphic to $\mathcal{O}'_x$. We can find a submodule $L(x) \subseteq \mathcal{O}_{X,x}$ such that $L'(x)$ is the projection of $L(x)$ onto $\mathcal{K}'$ and $L(x) \simeq \mathcal{O}_{X,x}$. Using Lemma 3.1.1, construct a submodule $\mathcal{L} \subset \mathcal{M}$ such that $\mathcal{L}_x = L(x)$ if $x \in S$ and $\mathcal{L}_x = \mathcal{O}_{X,x}$ if $x \notin S$. Then $\mathcal{L}$ is an invertible ideal and the projection of $\mathcal{L}$ onto $\mathcal{K}'$ gives a non-zero homomorphism $\mathcal{L} \to \mathcal{M}$. □

We will use the duality for left and right coherent torsion free $\mathcal{A}$-modules established in the following theorem.

**Theorem 3.1.3.** (1) *There is a* canonical $\mathcal{A}$-module*, i.e. such a module* $\omega_{\mathcal{A}} \in \mathsf{tf}\,\mathcal{A}$ *that* $\operatorname{inj.dim}_{\mathcal{A}} \omega_{\mathcal{A}} = 1$ *and* $\mathcal{E}nd_{\mathcal{A}}\,\omega_{\mathcal{A}} \simeq \mathcal{A}^{\mathrm{op}}$ *(so* $\omega_{\mathcal{A}}$ *can be considered as an* $\mathcal{A}$-bimodule*). Moreover,* $\omega_{\mathcal{A}}$ *is isomorphic as a bimodule to an ideal of* $\mathcal{A}$*.*

We denote by $\mathcal{M}^*$, where $\mathcal{M} \in \mathsf{Qcoh}\,\mathcal{A}$ (or $\mathcal{M} \in \mathsf{Qcoh}\,\mathcal{A}^{\mathrm{op}}$) the $\mathcal{A}^{\mathrm{op}}$-module (respectively, $\mathcal{A}$-module) $\mathcal{H}om_{\mathcal{A}}(\mathcal{M}, \omega_{\mathcal{A}})$ (respectively, $\mathcal{H}om_{\mathcal{A}^{\mathrm{op}}}(\mathcal{M}, \omega_{\mathcal{A}})$).

(2) *The natural map* $\mathcal{M} \to \mathcal{M}^{**}$ *is an isomorphism for every* $\mathcal{M} \in \mathsf{tf}\,\mathcal{A}$ *(or* $\mathcal{M} \in \mathsf{tf}\,\mathcal{A}^{\mathrm{op}}$*) and the functors* $\mathcal{M} \mapsto \mathcal{M}^*$ *establish an exact duality of the categories* $\mathsf{tf}\,\mathcal{A}$ *and* $\mathsf{tf}\,\mathcal{A}^{\mathrm{op}}$*. Moreover, if* $\mathcal{M} \in \mathsf{Coh}\,\mathcal{A}$*, then* $\mathcal{M}^{**} \simeq \mathcal{M}/\operatorname{tors}\mathcal{M}$*, where* $\operatorname{tors}\mathcal{M}$ *denotes the maximal* $\mathcal{O}_X$*-periodic submodule of* $\mathcal{M}$*.*

*Proof.* Each local ring $\mathcal{O}_x = \mathcal{O}_{X,x}$ is excellent, so its integral closure in $\mathcal{K}_x$ is finitely generated and its completion $\hat{\mathcal{O}}_x$ is reduced. Therefore $\mathcal{O}_x$ has a canonical module $\omega_x$ which can be considered as an ideal in $\mathcal{O}_x$ [23, Korollar 2.12]. Moreover, $\mathcal{O}_x$ is normal for almost all $x \in X_{\mathrm{cl}}$ and in this case we can take $\omega_x = \mathcal{O}_{X,x}$. By Lemma 3.1.1, there is an ideal $\omega_X \subseteq \mathcal{O}_X$ such that $\omega_{X,x} = \omega_x$ for each $x \in X$. Then $\operatorname{inj.dim}_{\mathcal{O}_X} \omega_X = \sup\left\{ \operatorname{inj.dim}_{\mathcal{O}_{X,x}} \omega_x \right\} = 1$. As the natural map $\mathcal{O}_{X,x} \to \operatorname{End}_{\mathcal{O}_{X,x}} \omega_x$ is an isomorphism for each $x \in X$, the natural map $\mathcal{O}_X \to \mathcal{E}nd_{\mathcal{O}_X}\omega_X$ is an isomorphism too. Therefore, $\omega_X$ is a canonical $\mathcal{O}_X$-module. Then it is known that the functor $\mathcal{M} \mapsto \mathcal{M}^* = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \omega_X)$ is an exact self-duality of $\mathsf{tf}\,\mathcal{O}_X$ and the natural map $\mathcal{M} \to \mathcal{M}^{**}$ is an isomorphism. Set now $\omega_{\mathcal{A}} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{A}, \omega_X)$. Then $\mathcal{H}om_{\mathcal{A}}(\mathcal{M}, \omega_{\mathcal{A}}) \simeq \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \omega_X)$ for any $\mathcal{A}$-module $\mathcal{M}$, whence all statements of the theorem follow. □

3.2. **Minors for curves.** In this subsection $(X, \mathcal{B})$ denotes a non-commutative curve, $\mathcal{P}$ is a locally projective coherent $\mathcal{A}$-module and $\mathcal{A} = (\mathcal{E}nd_{\mathcal{B}}\,\mathcal{P})^{\mathrm{op}}$ is the corresponding minor of $\mathcal{B}$. We supplement the results of Section 2.3

for the functors $\mathsf{F}, \mathsf{G}, \mathsf{F}$ defined by the rules (2.3.1) and their derived functors. All these functors map coherent sheaves to coherent ones. The functor $\mathsf{G}$ maps $\mathsf{tf}\,\mathcal{B}$ to $\mathsf{tf}\,\mathcal{A}$ and $\mathsf{H}$ maps $\mathsf{tf}\,\mathcal{A}$ to $\mathsf{tf}\,\mathcal{B}$. It is not true for $\mathsf{F}$, so we modify it setting $\mathsf{F}^\dagger\mathcal{M} = (\mathsf{F}\mathcal{M})^{**}$. Then $\mathsf{F}^\dagger$ also maps $\mathsf{tf}\,\mathcal{A}$ to $\mathsf{tf}\,\mathcal{B}$. We denote by $\mathcal{P}'$ the $\mathcal{B}$-module $(\mathcal{P}^\vee)^*$.

**Theorem 3.2.1.** (1) *The functors* $\mathsf{H}$ *and* $\mathsf{G}$ *establish an equivalence of* $\mathsf{tf}\,\mathcal{A}$ *and* $\mathsf{tf}^*\,\mathcal{P}$*, where* $\mathsf{tf}^*\,\mathcal{P}$ *is the full subcategory of* $\mathsf{tf}\,\mathcal{B}$ *consisting of all torsion free* $\mathcal{B}$*-modules* $\mathcal{M}$ *such that for every point* $x \in X$ *there is an exact sequence* $0 \to \mathcal{M}_x \to Q \to N \to 0$*, where* $Q$ *is a multiple of the* $\mathcal{B}_x$*-module* $\mathcal{P}'_x$ *and* $N \in \mathsf{tf}\,\mathcal{A}_x$*.*

(2) *The restriction of functor* $\mathsf{F}^\dagger$ *onto* $\mathsf{tf}\,\mathcal{A}$ *is left adjoint to the restriction of* $\mathsf{G}$ *onto* $\mathsf{tf}\,\mathcal{B}$*. Moreover, if* $\mathcal{M} \in \mathsf{tf}\,\mathcal{A}$*, the natural map* $\mathsf{G} \circ \mathsf{F}^\dagger(\mathcal{M})$ *is an isomorphism and the functors* $\mathsf{F}^\dagger$ *and* $\mathsf{G}$ *define an equivalence of the categories* $\mathsf{tf}\,\mathcal{A}$ *and* $\mathsf{tf}\,\mathcal{P}$*, where* $\mathsf{tf}\,\mathcal{P}$ *is the full subcategory of* $\mathsf{tf}\,\mathcal{B}$ *consisting of all sheaves* $\mathcal{M}$ *such that for every point* $x \in X$ *there is an epimorphism* $n\mathcal{P}_x \to \mathcal{M}$*.*[7]

*Proof.* (1) This statement is local, so we can suppose that $X = \operatorname{Spec} \boldsymbol{R}$, where $\boldsymbol{R}$ is an excellent local reduced ring of Krull dimension 1, $\mathcal{B} = \boldsymbol{B}^\sim$ for some $\boldsymbol{R}$-order $\boldsymbol{B}$, i.e. an $\boldsymbol{R}$-algebra $\boldsymbol{B}$ without nilpotent ideals which is finitely generated and torsion free as an $\boldsymbol{R}$-module, $\mathcal{P} = P^\sim$ for some finitely generated projective $\boldsymbol{B}$-module $P$. Moreover, we can suppose that $P$ is sincere as $\boldsymbol{B}$-module. Then $\mathcal{A} = \boldsymbol{A}^\sim$, where $\boldsymbol{A} = \operatorname{End}_{\boldsymbol{B}} P$. If $L \in \mathsf{tf}\,\boldsymbol{A}$, there is an exact sequence $m\boldsymbol{A} \to n\boldsymbol{A} \to L^* \to 0$, which gives an exact sequence

$$0 \to L \to n\boldsymbol{A}^* \to m\boldsymbol{A}^*. \tag{3.2.1}$$

We denote by $\psi$ the natural morphism $\mathbb{1}_{\boldsymbol{B}\text{-}\mathsf{Mod}} \to \mathsf{H} \circ \mathsf{G}$. For $M = \boldsymbol{A}^*$ we have

$$\begin{aligned}
\mathsf{H}(\boldsymbol{A}^*) &= \operatorname{Hom}_{\boldsymbol{A}}(P^\vee, \operatorname{Hom}_{\boldsymbol{R}}(\boldsymbol{A}, \omega_{\boldsymbol{R}})) \simeq \\
&\simeq \operatorname{Hom}_{\boldsymbol{R}}(P^\vee, \omega_{\boldsymbol{R}}) = P', \\
\mathsf{G}(P') &= \operatorname{Hom}_{\boldsymbol{B}}(P, \operatorname{Hom}_{\boldsymbol{R}}(P^\vee, \omega_{\boldsymbol{R}})) \simeq \\
&\simeq \operatorname{Hom}_{\boldsymbol{R}}(P^\vee \otimes_{\boldsymbol{B}} P, \omega_{\boldsymbol{R}}) \simeq \boldsymbol{A}^*,
\end{aligned}$$

since $P^\vee \otimes_{\boldsymbol{B}} P \simeq \boldsymbol{A}$. Hence $\psi(P')$ is an isomorphism. The exact sequence (3.2.1) gives an exact sequence $0 \to \mathsf{H}(L) \to nP' \to mP'$, which shows that $\mathsf{H}(L) \in \mathsf{tf}^*\,P$. Let now $M \in \mathsf{tf}^*\,P$. An exact sequence $0 \to M \to nP' \to N \to 0$, where $N \in \mathsf{tf}\,\boldsymbol{B}$, gives an exact sequence $0 \to \mathsf{G}M \to \mathsf{G}(nP') \to \mathsf{G}N \to 0$. For any $\boldsymbol{B}$-module $N$, $\psi(N)$ maps $x \in N$ to the homomorphism

$$N \to \operatorname{Hom}_{\boldsymbol{A}}(P^\vee, \operatorname{Hom}_{\boldsymbol{B}}(P, N)) \simeq \operatorname{Hom}_{\boldsymbol{B}}(P \otimes_{\boldsymbol{A}} P^\vee, N)$$

mapping $x \in N$ to the homomorphism $P \otimes_{\boldsymbol{A}} P^\vee \to N$ which maps $\alpha \otimes \gamma$ to $\gamma(\alpha)x$. Tensoring with the full ring of quotient $\boldsymbol{K}$ of $\boldsymbol{B}$, we obtain the

[7] In affine case, under some restrictions, it was also established in [16].

map $\boldsymbol{K}N \to \mathrm{Hom}_{\boldsymbol{KB}}(\boldsymbol{K}P \otimes_{\boldsymbol{KA}} \boldsymbol{K}P^\vee, \boldsymbol{K}N)$. As $\boldsymbol{KB}$ is semi-simple and $\boldsymbol{K}P$ is sincere, the natural map $\boldsymbol{K}P \otimes_{\boldsymbol{KA}} \boldsymbol{K}P^\vee \to \boldsymbol{B}$ is surjective, hence $\boldsymbol{K}\psi(N)$ is injective. If $N$ is torsion free, hence embeds into $\boldsymbol{K}N$, it implies that $\psi(N)$ is injective. So we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, "\psi(M)"] & nP' \arrow[r] \arrow[d, "\psi(nP')"] & N \arrow[d, "\psi(N)"] \\
0 \arrow[r] & \mathsf{H}\circ\mathsf{G}(M) \arrow[r] & \mathsf{H}\circ\mathsf{G}(nP') \arrow[r] & \mathsf{H}\circ\mathsf{G}(N).
\end{tikzcd}$$

Since $\psi(nP')$ is an isomorphism and $\psi(N)$ is a monomorphism, $\psi(M)$ is an isomorphism, so $M \in \mathrm{Im}\,\mathsf{H}$.

(2) If $\mathcal{M} \in \mathsf{tf}\,\mathcal{A}$ and $\mathcal{N} \in \mathsf{tf}\,\mathcal{B}$, then also $\mathsf{G}\mathcal{N} \in \mathsf{tf}\,\mathcal{A}$, so

$$\mathrm{Hom}_{\mathcal{B}}(\mathsf{F}^\dagger\mathcal{M}, \mathcal{N}) \simeq \mathrm{Hom}_{\mathcal{B}}(\mathsf{F}\mathcal{M}, \mathcal{N}) \simeq \mathrm{Hom}_{\mathcal{B}}(\mathcal{M}, \mathsf{G}\mathcal{N}),$$

which proves the first claim. Consider now the functors

$$\mathsf{H}' = \mathcal{H}om_{\mathcal{A}}(\mathcal{P}, \_) : \mathsf{Qcoh}\,\mathcal{A}^{\mathrm{op}} \to \mathsf{Qcoh}\,\mathcal{B}^{\mathrm{op}},$$
$$\mathsf{G}' = \mathcal{H}om_{\mathcal{B}}(\mathcal{P}^\vee, \_) : \mathsf{Qcoh}\,\mathcal{B}^{\mathrm{op}} \to \mathsf{Qcoh}\,\mathcal{A}^{\mathrm{op}}.$$

As we have just proved, they establish an equivalence between the categories $\mathsf{tf}\,\mathcal{A}^{\mathrm{op}}$ and $\mathsf{tf}^*\,\mathcal{P}^\vee$, where $\mathsf{tf}^*\,\mathcal{P}^\vee$ consists of all right $\mathcal{B}$-modules $\mathcal{N}$ such that for every point $x \in X$ there is an exact sequence $0 \to \mathcal{N}_x \to Q \to N' \to 0$, where $Q$ is a multiple of $\mathcal{P}_x^*$ and $N' \in \mathsf{tf}\,\mathcal{B}_x$. Equivalently, there is an epimorphism $Q^* \to \mathcal{N}_x^*$, i.e. $\mathcal{N}^* \in \mathsf{tf}\,\mathcal{P}$. On the other hand,

$$\mathsf{H}'\mathcal{M}^* = \mathcal{H}om_{\mathcal{A}}(\mathcal{P}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \omega_X)) \simeq$$
$$\simeq \mathcal{H}om_{\mathcal{O}_X}(\mathcal{P} \otimes_{\mathcal{A}} \mathcal{M}, \omega_X) = (\mathcal{P} \otimes_{\mathcal{A}} \mathcal{M})^* = \mathsf{F}^\dagger\mathcal{M}.$$

Hence the statement about $\mathsf{F}^\dagger$ and $\mathsf{G}$ follows by duality. □

3.3. **Morita equivalence.** We call two non-commutative schemes $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ *Morita equivalent* if $\mathsf{Qcoh}\,\mathcal{A} \simeq \mathsf{Qcoh}\,\mathcal{B}$. From Theorem 2.3.3 it follows that $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ are Morita equivalent if there is a locally projective and locally finitely generated $\mathcal{A}$-module $\mathcal{P}$ such that $\mathcal{B} \simeq (\mathcal{E}nd_{\mathcal{A}}\,\mathcal{P})^{\mathrm{op}}$ and $\mathcal{P}_x$ is a generator of $\mathcal{A}_x$-$\mathsf{Mod}$ for every point $x \in X$. In this case we say that $\mathcal{P}$ is a *local progenerator* for $\mathcal{A}$. It so happens that for curves we can check Morita equivalent locally.

First of all we recall some facts concerning *normal curves*. An *over-ring* of a non-commutative curve $(X, \mathcal{A})$ is, by definition, a non-commutative curve $(X, \mathcal{A}')$, where $\mathcal{A} \subseteq \mathcal{A}' \subset \mathcal{K}\mathcal{A}$ and $\mathcal{A}'$ is coherent as $\mathcal{O}_X$-module (or, the same, as $\mathcal{A}$-module). Note that, due to Lemma 3.1.1, over-rings can be constructed locally. Namely, given a finite subset $S \subset X_{\mathrm{cl}}$ and an over-ring $\mathcal{A}'(x)$ of $\mathcal{A}_x$ for every $x \in S$, there is an over-ring $\mathcal{A}'$ of $\mathcal{A}$ such that $\mathcal{A}'_x = \mathcal{A}_x$ if $x \notin S$, while $\mathcal{A}'_x = \mathcal{A}'(x)$ for all $x \in S$.

We say that a non-commutative curve $(X, \mathcal{A})$ is *normal* if it has no proper over-rings. Then, if $(X, \mathcal{A})$ is central, $X$ is also normal, hence regular. If

$(X,\mathcal{A})$ is normal, every module $\mathcal{M} \in \mathsf{tf}\,\mathcal{A}$ is locally projective [15, Theorem 26.12]. If it is sincere, it is a local progenerator for $\mathcal{A}$, so $\mathcal{A}$ is Morita equivalent to $\mathcal{B} = (\mathcal{E}nd_{\mathcal{A}}\,\mathcal{M})^{\mathrm{op}}$.

As a non-commutative curve $(X,\mathcal{A})$ is noetherian, excellent and reduced, there is a *normal over-ring* $(X,\mathcal{A}')$ of $\mathcal{A}$.

**Theorem 3.3.1.** *Let $(X,\mathcal{A})$ and $(X,\mathcal{B})$ be non-commutative curves such that the rings $\mathcal{A}_x$ and $\mathcal{B}_x$ are Morita equivalent for every point $x \in X_{\mathrm{cl}}$. Then these non-commutative curves are Morita equivalent.*

*Proof.* From the Morita theorem for rings it follows that for every point $x \in X_{\mathrm{cl}}$ there is a progenerator $P_x$ for $\mathcal{A}_x$ such that $\mathcal{B}_x \simeq (\mathrm{End}_{\mathcal{A}_x} P_x)^{\mathrm{op}}$. There is a $\mathcal{KA}$-module $\mathcal{V}$ such that $\mathcal{V}_x = \mathcal{K}_x P_x$ for all $x$. In particular, $\mathcal{V}_x$ is a sincere $\mathcal{KA}$-module. Choose a normal over-ring $\mathcal{A}'$ of $\mathcal{A}$ an $\mathcal{A}$-lattice $\mathcal{M}$ in $\mathcal{V}$. Then $\mathcal{M}$ is a local progenerator for $\mathcal{A}'$ and $\mathcal{B}' = (\mathcal{E}nd_{\mathcal{A}'}\,\mathcal{M}$ is Morita equivalent to $\mathcal{A}'$. We set $P_x = \mathcal{M}_x$ for all points $x$ such that $\mathcal{A}_x = \mathcal{A}'_x$ and $\mathcal{B}_x = \mathcal{B}'_x$. These are almost all points of $X$. By Lemma cur-1, there is an $\mathcal{A}$-lattice $\mathcal{P}$ such that $\mathcal{P}_x = P_x$ for all $x$. Then $\mathcal{P}$ is a local progenerator for $\mathcal{A}$ and $\mathcal{B} \simeq (\mathcal{E}nd_{\mathcal{A}}\,\mathcal{P})^{\mathrm{op}}$. □

**Corollary 3.3.2.** *Let $(X,\mathcal{A})$ and $(X,\mathcal{B})$ be two normal non-commutative curves. If the $\mathcal{K}$-algebras $\mathcal{KA}$ and $\mathcal{KB}$ are Morita equivalent, so are $\mathcal{A}$ and $\mathcal{B}$.*

*Proof.* If $\mathcal{KA}$ and $\mathcal{KB}$ are Morita equivalent, there is a sincere $\mathcal{KA}$-module $\mathcal{V}$ such that $\mathcal{KB} \simeq (\mathrm{End}_{\mathcal{KA}}\,\mathcal{V})^{\mathrm{op}}$. Then it is known [15] that $\mathcal{A}_x$ and $\mathcal{B}_x$ are Morita equivalent for all points $x \in X$. Thus $\mathcal{A}$ and $\mathcal{B}$ are Morita equivalent. □

Let $\Gamma(X,\mathcal{KA}) = \prod_{i=1}^{s} \mathrm{Mat}(m_i, \boldsymbol{D}_i)$, where $\boldsymbol{D}_i$ are skew fields. Corollary 3.3.2 implies that a normal non-commutative curve $(X,\mathcal{A})$ is defined up to Morita equivalence by the set $\{\,\boldsymbol{D}_1, \boldsymbol{D}_2, \dots, \boldsymbol{D}_s\,\}$.

3.4. **Hereditary non-commutative curves.** We call a non-commutative curve $(X,\mathcal{A})$ *hereditary* if $\mathrm{gl.dim}\,\mathcal{A}_x = 1$ for every $x \in X_{\mathrm{cl}}$. Then $\mathcal{E}xt^i_{\mathcal{A}}(\mathcal{M},\mathcal{N}) = 0$ if $i > 1$ or $i = 1$ and $\mathcal{M} \in \mathsf{tf}\,\mathcal{A}$. Hence $\mathcal{E}xt^i_{\mathcal{A}}(\mathcal{M},\mathcal{N}) = 0$ for $i > 2$ and $\mathcal{E}xt^1_{\mathcal{A}}(\mathcal{M},\mathcal{N})$ has finite support for any coherent modules $\mathcal{M}$ and $\mathcal{N}$. Therefore, $\mathrm{Ext}^i_{\mathcal{A}} = 0$ for $i > 1$, so $\mathrm{gl.dim}\,\mathcal{A} = 1$, and $\mathrm{Ext}^1_{\mathcal{A}}(\mathcal{M},\mathcal{N}) = 0$ if $\mathcal{M} \in \mathsf{tf}\,\mathcal{A}$ and $\mathcal{N}$ is torsion. It means that any coherent $\mathcal{A}$-module is a direct sum of a torsion free module and a torsion one. A torsion free coherent $\mathcal{A}$-module $\mathcal{L}$ is said to be *irreducible* if $\mathcal{KA}$ is a simple $\mathcal{KA}$-module.

As we have already mentioned, a normal non-commutative curve is *hereditary* [15, Theorem 26.12]. Moreover, any hereditary $(X,\mathcal{A})$ coincides with the intersection of its maximal over-rings [18, Theorem 1.1]. Therefore, if $(X,\mathcal{A})$ is hereditary and central, $X$ is normal.

Existence of normal over-rings implies important homological properties of non-commutative curves.

**Proposition 3.4.1.** *Let $(X,\mathcal{A})$ be a non-commutative curve, $\mathcal{M}$ and $\mathcal{N}$ be coherent $\mathcal{A}$-modules.*

(1) *If $i>1$, then $\mathrm{Ext}^i_{\mathcal{A}}(\mathcal{M},\mathcal{N}) = \mathrm{H}^0(X,\mathcal{E}xt^i_{\mathcal{A}}(\mathcal{M},\mathcal{N}))$ for $i>1$. In particular, $\mathrm{Ext}^i_{\mathcal{A}}(\mathcal{M},\mathcal{N})=0$ if $i>1$ and $\mathcal{M}$ is locally projective.*

(2) *If $\mathcal{M}$ is locally projective, then $\mathrm{Ext}^1_{\mathcal{A}}(\mathcal{M},\mathcal{N})\simeq \mathrm{H}^1(X,\mathcal{H}om_{\mathcal{A}}(\mathcal{M},\mathcal{N}))$.*

*Proof.* We use the spectral sequence

$$\mathrm{H}^p(X,\mathcal{E}xt^q_{\mathcal{A}}(\mathcal{M},\mathcal{N}))\Rightarrow \mathrm{Ext}^{p+q}_{\mathcal{A}}(\mathcal{M},\mathcal{N}).$$

Let $\mathcal{A}'$ be a normal over-ring of $\mathcal{A}$. As it is hereditary and $\mathcal{A}_x=\mathcal{A}'_x$ for almost all $x\in X$, the modules $\mathcal{E}xt^i_{\mathcal{A}}(\mathcal{M},\mathcal{N})$ have finite support for any $i>0$, which implies (1). If $\mathcal{M}$ is locally projective, $\mathcal{E}xt^i_{\mathcal{A}}(\mathcal{M},\mathcal{N})=0$ for $i>0$, whence (2). □

3.5. **Endomorphism construction.** We use the results on minors in the following situation. Let $(X,\mathcal{A})$ be a non-commutative curve, $\mathcal{F}$ be a coherent torsion free $\mathcal{A}$-module, $\mathcal{F}_+=\mathcal{A}\oplus\mathcal{F}$ and $\mathcal{A}_{\mathcal{F}}=(\mathcal{E}nd_{\mathcal{A}}\,\mathcal{F}_+)^{\mathrm{op}}$. If $\mathcal{E}=(\mathcal{E}nd_{\mathcal{A}}\,\mathcal{F})^{\mathrm{op}}$, then $\mathcal{A}_{\mathcal{F}}$ can be identified with the algebra of matrices of the form

$$\begin{pmatrix}\mathcal{A} & \mathcal{F}\\ \mathcal{F}' & \mathcal{E}\end{pmatrix},\quad \text{where } \mathcal{F}'=\mathcal{H}om_{\mathcal{A}}(\mathcal{F},\mathcal{A}).$$

Thus $\mathcal{F}_+$ is identified with the locally projective right $\mathcal{A}_{\mathcal{F}}$-module $e\mathcal{A}_{\mathcal{F}}=\begin{pmatrix}\mathcal{A} & \mathcal{F}\end{pmatrix}$, where $e=\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$, and $\mathcal{F}_+^\vee=\mathcal{H}om_{\mathcal{A}_{\mathcal{F}}}(\mathcal{F}_+,\mathcal{A}_{\mathcal{F}})$ is identified with the locally projective $\mathcal{A}_{\mathcal{F}}$-module $\mathcal{A}_{\mathcal{F}}e=\left(\begin{smallmatrix}\mathcal{A}\\ \mathcal{F}'\end{smallmatrix}\right)\simeq\mathcal{H}om_{\mathcal{A}}(\mathcal{F}_+,\mathcal{A})$. Moreover,

$$\mathcal{A}\simeq e\mathcal{A}_{\mathcal{F}}e\simeq(\mathcal{E}nd_{\mathcal{A}_{\mathcal{F}}}\,\mathcal{F}_+^\vee)^{\mathrm{op}}$$

Let also $\mathcal{P}=\mathcal{A}_{\mathcal{F}}(1-e)=\left(\begin{smallmatrix}\mathcal{F}\\ \mathcal{E}\end{smallmatrix}\right)$. Then $\mathcal{P}'=\mathcal{H}om_{\mathcal{A}_{\mathcal{F}}}(\mathcal{P},\mathcal{A}_{\mathcal{F}})\simeq(1-e)\mathcal{A}_{\mathcal{F}}\simeq\begin{pmatrix}\mathcal{F}' & \mathcal{E}\end{pmatrix}$ and $(\mathcal{E}nd_{\mathcal{A}_{\mathcal{F}}}\,\mathcal{P})^{\mathrm{op}}\simeq(1-e)\mathcal{A}_{\mathcal{F}}(1-e)\simeq\mathcal{E}$. So we can apply the previous results to the functors

$$\begin{aligned}\mathsf{F}=\mathcal{F}_+^\vee\otimes_{\mathcal{A}}\text{\_}:\mathsf{Qcoh}\,\mathcal{A}\to\mathsf{Qcoh}\,\mathcal{A}_{\mathcal{F}},\\ \mathsf{G}=\mathcal{H}om_{\mathcal{A}_{\mathcal{F}}}(\mathcal{F}_+^\vee,\text{\_}):\mathsf{Qcoh}\,\mathcal{A}_{\mathcal{F}}\to\mathsf{Qcoh}\,\mathcal{A},\\ \mathsf{H}=\mathcal{H}om_{\mathcal{A}}(\mathcal{F}_+,\text{\_}):\mathsf{Qcoh}\,\mathcal{A}\to\mathsf{Qcoh}\,\mathcal{A}_{\mathcal{F}},\end{aligned}$$

as well as to the functors

$$\begin{aligned}\mathsf{F}'=\mathcal{P}\otimes_{\mathcal{E}}\text{\_}:\mathsf{Qcoh}\,\mathcal{E}\to\mathsf{Qcoh}\,\mathcal{A}_{\mathcal{F}},\\ \mathsf{G}'=\mathcal{H}om_{\mathcal{A}_{\mathcal{F}}}(\mathcal{P},\text{\_}):\mathsf{Qcoh}\,\mathcal{A}_{\mathcal{F}}\to\mathsf{Qcoh}\,\mathcal{E},\\ \mathsf{H}'=\mathcal{H}om_{\mathcal{E}}(\mathcal{P}',\text{\_}):\mathsf{Qcoh}\,\mathcal{E}\to\mathsf{Qcoh}\,\mathcal{A}_{\mathcal{F}}\end{aligned}$$

and for modifications $\mathsf{F}^\dagger$ and $\mathsf{F}'^\dagger$ of the functors $\mathsf{F}$ and $\mathsf{F}'$, which map torsion free sheaves to torsion free ones. $\mathsf{Qcoh}\,\mathcal{A}$ is a bilocalization of $\mathsf{Qcoh}\,\mathcal{B}$ and $\mathscr{D}\mathcal{A}$ is a bilocalization of $\mathscr{D}\mathcal{B}$. Namely, $\mathsf{Qcoh}\,\mathcal{A}\simeq\mathsf{Qcoh}\,\mathcal{A}_{\mathcal{F}}/\mathscr{C}$, where $\mathscr{C}=(\mathcal{F}_+^\vee)^\perp$ is bilocalizing in $\mathsf{Qcoh}\,\mathcal{A}_{\mathcal{F}}$, and $\mathscr{D}\mathcal{A}\simeq\mathscr{D}\mathcal{A}_{\mathcal{F}}/\mathscr{D}_{\mathscr{C}}\mathcal{A}_{\mathcal{F}}$, where $\mathscr{D}_{\mathscr{C}}\mathcal{A}_{\mathcal{F}}$ is bilocalizing in $\mathscr{D}\mathcal{A}_{\mathcal{F}}$. The functors $\mathsf{F}$ and $\mathsf{F}'$ define full embeddings of the categories, respectively, $\mathsf{Qcoh}\,\mathcal{A}$ and $\mathsf{Qcoh}\,\mathcal{E}$ ($\mathsf{Coh}\,\mathcal{A}$ and $\mathsf{Coh}\,\mathcal{E}$) into $\mathsf{Qcoh}\,\mathcal{A}_{\mathcal{F}}$

($\mathsf{Coh}\,\mathcal{A}_{\mathcal{F}}$). Their left derived functors $\mathsf{LF}$ and $\mathsf{LF}'$ define full embeddings of the categories, respectively, $\mathscr{D}\mathcal{A}$ and $\mathscr{D}\mathcal{E}$ ($\mathscr{D}^{\mathrm{c}}\mathcal{A}$ and $\mathscr{D}^{\mathrm{c}}\mathcal{E}$) into $\mathscr{D}\mathcal{A}_{\mathcal{F}}$ ($\mathscr{D}^{\mathrm{c}}\mathcal{A}_{\mathcal{F}}$). The next statement is an immediate consequence of Theorem 2.3.5.

**Proposition 3.5.1.** *The category $\mathscr{C} = (\mathcal{F}_+^{\vee})^{\perp}$ is equivalent to $\mathsf{Qcoh}\,\bar{\mathcal{E}}$, where $\bar{\mathcal{E}} = \mathcal{E}/\mathcal{I}$, where $\mathcal{I}$ is the image of the natural map $\mu_{\mathcal{F}} : \mathcal{F}' \otimes_{\mathcal{A}} \mathcal{F} \to \mathcal{E}$.*

*Proof.* Indeed, the components $\mathcal{A}$, $\mathcal{F}$ and $\mathcal{F}'$ of the matrix presentation of $\mathcal{A}_{\mathcal{F}}$ are contained in $\operatorname{Im}\mu_{\mathcal{P}}$ and $\operatorname{Im}\mu_{\mathcal{P}} \cap \mathcal{E} = \operatorname{Im}\mu_{\mathcal{F}}$. □

**Lemma 3.5.2.** (1) *Let $\mathcal{M} \in \mathsf{Qcoh}\,\mathcal{A}_{\mathcal{F}}$, $\phi_{\mathcal{M}} : \mathsf{F} \circ \mathsf{G}(\mathcal{M}) \to \mathcal{M}$ and $\phi'_{\mathcal{M}} : \mathsf{F}' \circ \mathsf{G}'(\mathcal{M}) \to \mathcal{M}$ be the natural homomorphisms arising from the adjunction. Then $\operatorname{Im}\phi + \operatorname{Im}\phi' = \mathcal{M}$.*

(2) *$\boldsymbol{\Sigma} = \mathsf{F}(\mathbf{L}_{\mathcal{A}}) \cup \mathsf{F}'(\mathbf{L}_{\mathcal{E}})$ is a set of compact generators of $\mathsf{Qcoh}\,\mathcal{A}_{\mathcal{F}}$, hence a set of compact strong generators for $\mathscr{D}\mathcal{A}_{\mathcal{F}}$. Therefore, $\mathscr{D}^{\mathrm{c}}\mathcal{A}_{\mathcal{F}}$ consists of complexes quasi-isomorphic to finite complexes of modules from* add $\boldsymbol{\Sigma}$.

(3) *If $\mathcal{L}$ is an ample $\mathcal{O}_X$-module, the set*

$$\boldsymbol{\Sigma}_{\mathcal{L}} = \left\{ \mathcal{F}_+^{\vee} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n} \mid n \in \mathbb{Z} \right\} \cup \left\{ \mathcal{P} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n} \mid n \in \mathbb{Z} \right\}.$$

*is a set of compact generators of $\mathsf{Qcoh}\,\mathcal{A}_{\mathcal{F}}$, hence a set of compact strong generators of $\mathscr{D}\mathcal{A}_{\mathcal{F}}$. Therefore, $\mathscr{D}^{\mathrm{c}}\mathcal{A}_{\mathcal{F}}$ consists of complexes quasi-isomorphic to finite complexes of modules from* add $\boldsymbol{\Sigma}_{\mathcal{L}}$.

*Proof.* Let $\mathcal{M}$ be a quasi-coherent $\mathcal{A}_{\mathcal{F}}$-module, $\mathcal{M}_1 = e\mathcal{M} \simeq \mathsf{G}\mathcal{M}$ and $\mathcal{M}_2 = (1-e)\mathcal{M} \simeq \mathsf{G}'\mathcal{M}$. Then $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$ (as $\mathcal{O}_X$-module), $\mathsf{F}\mathcal{M}_1 = \mathcal{M}_1 \oplus \mathcal{F}' \otimes_{\mathcal{A}} \mathcal{M}_1$, $\mathsf{F}'\mathcal{M}_2 = \mathcal{M}_2 \oplus \mathcal{F} \otimes_{\mathcal{E}} \mathcal{M}_2$, $\operatorname{Im}\phi_{\mathcal{M}} = \mathcal{M}_1 \oplus \operatorname{Im}\mu_1$ and $\operatorname{Im}\phi'\mathcal{M} = \operatorname{Im}\mu_2 \oplus \mathcal{M}_2$, where $\mu_1 : \mathcal{F}' \otimes_{\mathcal{A}} \mathcal{M}_1 \to \mathcal{M}_2$ and $\mu_2 : \mathcal{F} \otimes_{\mathcal{E}} \mathcal{M}_2 \to \mathcal{M}_1$ arise from the multiplication $\mathcal{A}_{\mathcal{F}} \times \mathcal{M} \to \mathcal{M}$. It proves (1). Then (2) follows immediately from Lemma 3.1.2 and (3) from Proposition 2.1.2. □

**Theorem 3.5.3.** *If $\mathcal{A}$ is strongly Gorenstein, the restrictions of $\mathsf{LF}$ and $\mathsf{RH}$ onto $\mathscr{D}^{\mathrm{c}}\mathcal{A}$ are isomorphic. Hence the restriction of $\mathsf{LF}$ onto $\mathscr{D}^{\mathrm{c}}\mathcal{A}$ is both left and right adjoint to the restriction of $\mathsf{DG}$ onto $\mathscr{D}^{b}\mathcal{A}_{\mathcal{F}}$.*

*Proof.* Note first that if $\mathcal{M}$ is a locally projective coherent $\mathscr{A}$-module, then the natural map $\mathcal{F}_+^{\vee} \otimes_{\mathcal{A}} \mathcal{M} \to \mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{M})$ is an isomorphism. Indeed, this claim is local, so it is enough to verify it for $\mathcal{M} = \mathcal{A}$, and we have already mentioned that $\mathcal{F}_+^{\vee} \simeq \mathcal{H}om_{\mathcal{A}}(\mathcal{F}_+, \mathcal{A})$.

As $\mathcal{A}$ has enough locally projective modules, $\mathscr{D}^{\mathrm{c}}\mathcal{A}$ consists of complexes quasi-isomorphic to finite complexes of locally projective coherent modules. Let $\mathcal{F}^{\bullet}$ be such a complex. As $\mathcal{K}\mathcal{A}$ is injective as $\mathcal{A}$-module, all modules $\mathcal{K}\mathcal{F}^i$ are injective. As $\operatorname{inj.dim}_{\mathcal{A}} \mathcal{A} = 1$ and hence $\operatorname{inj.dim}_{\mathcal{A}} \mathcal{F}^i = 1$ too, the quotient modules $\mathcal{K}\mathcal{F}^i/\mathcal{F}^i$ are also injective. It gives an exact sequence of complexes

$$0 \to \mathcal{F}^{\bullet} \to \mathcal{K}\mathcal{F}^{\bullet} \xrightarrow{f} \mathcal{K}\mathcal{F}^{\bullet}/\mathcal{F}^{\bullet} \to 0.$$

Thus $\mathcal{F}^{\bullet} \simeq Cf[-1]$, where $Cf$ is the cone of $f$. All components of $Cf$ are injective, hence $\mathsf{RH}(\mathcal{F}^{\bullet}) = \mathsf{H}(Cf[-1])$. Applying $\mathsf{H}$, we obtain an exact sequence of complexes

$$0 \to \mathsf{H}(\mathcal{F}^\bullet) \to \mathsf{H}(\mathcal{K}\mathcal{F}^\bullet) \xrightarrow{\mathsf{H}(f)} \mathsf{H}(\mathcal{K}\mathcal{F}^\bullet/\mathcal{F}^\bullet) \to \mathcal{E}xt^1_{\mathcal{A}}(\mathcal{P},\mathcal{F}^\bullet)$$

and $\mathcal{E}xt^1_{\mathcal{A}}(\mathcal{P},\mathcal{F}^\bullet)=0$, since $\operatorname{inj.dim}_{\mathcal{A}}\mathcal{F}^i=1$ and $\mathcal{P}$ is torsion free. Therefore, $\mathsf{H}(\mathcal{F}^\bullet)$ is quasi-isomorphic to $C\mathsf{H}(f)[-1]$ which coincides with $\mathsf{H}(Cf[-1])=\mathsf{RH}(\mathcal{F}^\bullet)$. As $\mathsf{H}(\mathcal{F}^\bullet)\simeq\mathsf{F}(\mathcal{F}^\bullet)=\mathsf{LF}(\mathcal{F}^\bullet)$, we have proved the statement. $\square$

3.6. **Hereditary over-rings.** Let now $(X,\mathcal{H})$ be any hereditary (for instance, normal) over-ring of $\mathcal{A}$, $\mathcal{I}$ be the *conductor* of $\mathcal{H}$ in $\mathcal{A}$, i.e. the annihilator of the *right* $\mathcal{A}$-module $\mathcal{H}/\mathcal{A}$. Then

$$\mathcal{I}(U)=\{\,\lambda\in\mathcal{K}(U)\mid \mathcal{H}(U)\lambda\subseteq\mathcal{A}(U)\,\}$$

for every open $U\subseteq X$, so $\mathcal{I}$ can be identified with $\mathcal{H}om_{\mathcal{A}}(\mathcal{H},\mathcal{A})$. Apply the construction of the preceding subsection to the $\mathcal{A}$-module $\mathcal{F}=\mathcal{H}$. Then (using the corresponding notations) $\mathcal{E}=\mathcal{H}$,

$$\mathcal{A}_{\mathcal{H}}=\begin{pmatrix}\mathcal{A}&\mathcal{H}\\ \mathcal{I}&\mathcal{H}\end{pmatrix},$$

$\mathcal{H}_+=\left(\begin{smallmatrix}\mathcal{A}\\ \mathcal{I}\end{smallmatrix}\right)$ and $\mathcal{P}=\begin{pmatrix}\mathcal{H}&\mathcal{H}\end{pmatrix}$. Since $\mathcal{P}$ and $\mathcal{P}^\vee=\begin{pmatrix}\mathcal{I}&\mathcal{H}\end{pmatrix}$ are locally projective as $\mathcal{H}$-modules, the functors $\mathsf{F}'$ and $\mathsf{G}'$ establish an equivalence of the category $\mathsf{lp}\,\mathcal{H}=\mathsf{tf}\,\mathcal{H}$ and the full subcategory $\mathsf{lp}\,\mathcal{P}$ of $\mathsf{lp}\,\mathcal{A}_{\mathcal{H}}$ consisting of all modules $\mathcal{N}$ such that $\mathcal{N}_x\in\operatorname{add}\mathcal{P}_x$ for all $x\in X$. Moreover, these functors are exact, so there derived functors are just calculated by applying $\mathsf{F}'$ and $\mathsf{G}'$ to each component of a complex. They establish an equivalence between $\mathscr{D}\mathcal{H}$ and the full subcategory $\mathscr{D}\mathcal{P}\subset\mathscr{D}\mathcal{A}_{\mathcal{H}}$ consisting of all complexes that are quasi-isomorphic to K-flat complexes $\mathcal{F}^\bullet$ such that each component $\mathcal{F}^i_x$ belongs to $\operatorname{Add}\mathcal{P}_x$. As $\mathsf{F}'$ maps K-injective complexes to K-injective, also $\operatorname{RHom}_{\mathcal{A}_{\mathcal{H}}}(\mathsf{F}'\mathcal{M}^\bullet,\mathsf{F}'\mathcal{N}^\bullet)\simeq\operatorname{RHom}_{\mathcal{A}}(\mathcal{M}^\bullet,\mathcal{N}^\bullet)$.

Consider also the submodule $\mathcal{I}'=\left(\begin{smallmatrix}\mathcal{I}\\ \mathcal{I}\end{smallmatrix}\right)\subset\mathcal{H}_+$ and the quotient $\mathcal{Q}=\mathcal{H}_+/\mathcal{I}'=\left(\begin{smallmatrix}\mathcal{A}/\mathcal{I}\\ 0\end{smallmatrix}\right)$. Note that $\mathcal{I}'\simeq\mathsf{F}'\mathcal{I}$ and $\mathcal{I}\in\mathbf{L}_{\mathcal{H}}$, so $\mathcal{I}'$ belongs to $\mathsf{F}'(\mathbf{L}_{\mathcal{H}})$, hence to $\mathsf{lp}\,\mathcal{A}_{\mathcal{H}}$. On the other hand, $\mathsf{G}'\mathcal{Q}=(1-e)\mathcal{Q}=0$. Note also that the category $\operatorname{Ker}\mathsf{G}$ in this case is equivalent to $\mathsf{Qcoh}(\mathcal{H}/\mathcal{I}\mathcal{H})$.

**Proposition 3.6.1.** (1) $\mathscr{D}\mathcal{A}_{\mathcal{H}}=\langle\,\mathcal{Q},\mathsf{F}'(\mathbf{L}_{\mathcal{H}})\,\rangle^\omega$ *is a semi-orthogonal decomposition of* $\mathscr{D}\mathcal{A}_{\mathcal{H}}$.

(2) $\operatorname{Ext}^i_{\mathcal{A}_{\mathcal{H}}}(\mathsf{F}'\mathcal{N},\mathcal{Q})=0$ for all $i\in\mathbb{Z}$.

(3) $\operatorname{Ext}^i_{\mathcal{A}_{\mathcal{H}}}(\mathcal{Q},\mathcal{M})=0$ *for every* $\mathcal{M}\in\mathsf{Coh}\,\mathcal{A}_{\mathcal{H}}$ *and* $i>1$.

(4) *Let* $\gamma:\mathcal{N}\to\mathcal{H}om_{\mathcal{H}}(\mathcal{I},\mathcal{N})$ *be induced by the embedding* $\mathcal{I}\to\mathcal{H}$. *Then*

$$\operatorname{Hom}_{\mathcal{A}_{\mathcal{H}}}(\mathcal{Q},\mathsf{F}'\mathcal{N})\simeq\mathrm{H}^0(X,\operatorname{Ker}\gamma),$$
$$\operatorname{Ext}^1_{\mathcal{A}_{\mathcal{H}}}(\mathcal{Q},\mathsf{F}'\mathcal{N})\simeq\mathrm{H}^0(X,\operatorname{Cok}\gamma).$$

(5) $\operatorname{Ext}^i_{\mathcal{A}_{\mathcal{H}}}(\mathcal{Q},\mathcal{Q})=0$ *for* $i\neq0$.

*Proof.* (2) follows from the equality

$$\mathrm{RHom}_{\mathcal{A}_{\mathcal{H}}}(\mathsf{F}'\mathcal{N},\mathcal{Q}) \simeq \mathrm{RHom}_{\mathcal{A}}(\mathcal{N},\mathsf{G}'\mathcal{Q}) = 0.$$

(1) $\mathcal{I}' = \mathsf{F}'(\mathcal{I})$ is locally projective and belongs to $\langle\, \mathsf{F}'(\mathbf{L}_{\mathcal{H}})\,\rangle$ as well as any module $\mathcal{I}' \otimes_{\mathcal{O}_X} \mathcal{L}$, where $\mathcal{L}$ is an invertible $\mathcal{O}_X$-ideal. There is an exact sequence

$$0 \to \mathcal{I}' \otimes_{\mathcal{O}_X} \mathcal{L} \to \mathcal{H}_+^\vee \otimes_{\mathcal{O}_X} \mathcal{L} \to \mathcal{Q} \otimes_{\mathcal{O}_X} \mathcal{L} \to 0,$$

where $\mathcal{Q}\otimes_{\mathcal{O}_X}\mathcal{L} \simeq \mathcal{Q}$, since $\mathcal{Q}$ is torsion. It shows that $\mathsf{F}(\mathbf{L}_{\mathcal{A}}) \subset \langle\, \mathcal{Q}, \mathsf{F}'(\mathbf{L}_{\mathcal{H}})\,\rangle$. By Lemma 3.5.2, $\mathscr{D}\mathcal{A} = \langle\, \mathcal{Q}, \mathsf{F}'(\mathbf{L}_{\mathcal{H}})\,\rangle^\omega$.

(3-5) As $\mathcal{Q}$ is torsion, the spectral sequence

$$\mathrm{H}^p(X, \mathcal{E}xt^q_{\mathcal{A}_{\mathcal{H}}}(\mathcal{Q},\mathcal{M})) \Rightarrow \mathrm{Ext}^{p+q}_{\mathcal{A}_{\mathcal{H}}}(\mathcal{Q},\mathcal{M})$$

implies that $\mathrm{Ext}^i_{\mathcal{A}_{\mathcal{H}}}(\mathcal{Q},\mathcal{M}) \simeq \mathrm{H}^0(X, \mathcal{E}xt^i_{\mathcal{A}_{\mathcal{H}}}(\mathcal{Q},\mathcal{M}))$. The exact sequence $0 \to \mathcal{I}' \to \mathcal{H}_+^\vee \to \mathcal{Q} \to 0$, where $\mathcal{I}' \in \mathsf{lp}\,\mathcal{A}_{\mathcal{H}}$, implies that $\mathcal{E}xt^i_{\mathcal{A}_{\mathcal{H}}}(\mathcal{Q},\mathcal{M}) = 0$ for any $i > 1$ and any coherent $\mathcal{M}$, while

$$\mathcal{H}om_{\mathcal{A}_{\mathcal{H}}}(\mathcal{Q},\mathcal{M}) \simeq \mathrm{Ker}\,\beta,$$
$$\mathcal{E}xt^1_{\mathcal{A}_{\mathcal{H}}}(\mathcal{Q},\mathcal{M}) \simeq \mathrm{Cok}\,\beta,$$

where $\beta : \mathcal{H}om_{\mathcal{A}_{\mathcal{H}}}(\mathcal{H}_+^\vee,\mathcal{M}) \to \mathcal{H}om_{\mathcal{A}_{\mathcal{H}}}(\mathcal{I}',\mathcal{M})$ is induced by the embedding $\mathcal{I}' \to \mathcal{H}_+^\vee$. If $\mathcal{M} = \mathsf{F}'\mathcal{N}$, then

$$\mathcal{H}om_{\mathcal{A}_{\mathcal{H}}}(\mathcal{I}',\mathcal{M}) = \mathcal{H}om_{\mathcal{A}_{\mathcal{H}}}(\mathsf{F}'\mathcal{I},\mathcal{M}) \simeq \mathcal{H}om_{\mathcal{H}}(\mathcal{I},\mathsf{G}'\mathcal{M}) \simeq \mathcal{H}om_{\mathcal{H}}(\mathcal{I},\mathcal{N})$$

and, as $\mathcal{H}_+^\vee = \mathcal{A}_{\mathcal{H}}e$, $\mathcal{H}om_{\mathcal{A}_{\mathcal{H}}}(\mathcal{H}_+^\vee,\mathcal{M}) \simeq e\mathcal{M} \simeq \mathcal{N}$. Under these identifications $\beta$ is transformed to $\gamma$. Finally, $\mathrm{Hom}_{\mathcal{A}_{\mathcal{H}}}(\mathcal{I}',\mathcal{Q}) = 0$, which implies (5). □

## 4. Categorical resolutions for curves

In this section we suppose that $(X, \mathcal{A})$ is a non-commutative curve such that $X$ is an algebraic curve over an algebraically closed field $\Bbbk$. Without loss of generality we suppose that it is *central*, i.e. $\operatorname{cen} \mathcal{A} = \mathcal{O}_X$. We call this non-commutative curve *rational* if so are all irreducible components of $X$, i.e. $\mathcal{K} \simeq \Bbbk(t)^s$ for some $s$. Let $\mathcal{H}$ be a hereditary over-ring of $\mathcal{A}$ and $\tilde{X} = \operatorname{Spec}(\operatorname{cen} \mathcal{H})$. Then $\nu : \tilde{X} \to X$ is the normalization of $X$. In what follows we write $\mathcal{O}$ for $\mathcal{O}_X$ and $\tilde{\mathcal{O}}$ for $\nu_* \mathcal{O}_{\tilde{X}}$. If $\{ X_1, X_2, \ldots, X_s \}$ are irreducible (or, the same, connected) components of $\tilde{X}$, then $\tilde{\mathcal{O}} = \bigoplus_{i=1}^s \tilde{\mathcal{O}}_i$, where $\tilde{\mathcal{O}}_i = \nu_* \mathcal{O}_{X_i}$.

4.1. **Structure of hereditary non-commutative curves.** We recall basic facts about hereditary non-commutative curves in the algebraic case (see also [14], where Theorem 4.1.1 is given in a different, but equivalent form). Recall that the center of such a curve is always normal. Hence, without loss of generality, we can only consider central irreducible hereditary non-commutative curves.

**Theorem 4.1.1.** *Up to Morita equivalence, a central connected hereditary non-commutative curve is defined by the following data* (*called* hereditary data)*:*

- *a normal curve $X$;*
- *a finite set $S \subset X_{\mathrm{cl}}$;*
- *an integer $k(x) > 1$ for each $x \in S$.*

*The case $S = \emptyset$ correspond to normal non-commutative curves.*

*Proof.* Let $(X, \mathcal{A})$ be a a central connected hereditary non-commutative curve. It is known that the Brauer group $Br(\mathcal{K})$ is trivial [30]. Therefore, $\mathcal{K}\mathcal{H} = \operatorname{Mat}(n, \mathcal{K})$ for some $n$. Hence any normal $\mathcal{O}_x$-order in $\mathcal{K}\mathcal{H}$ is Morita equivalentto $\mathcal{O}_X$. Let $S = \{ x \in X_{\mathrm{cl}} \mid \mathcal{H}_x \text{ is not normal} \}$, $k = k(x)$ and $n_1, n_2, \ldots, n_k$ be positive integers such that $\sum_{i=1}^k n_i = n$. Consider the subring $\boldsymbol{H}_x(n_1, n_2, \ldots, n_k) \subset \operatorname{Mat}(n, \mathcal{O}_x)$ consisting of all block matrices $(A_{ij})$, where $A_{ij} \in \operatorname{Mat}(n_i \times n_j, \mathcal{O}_x)$ and $A_{ij} \equiv 0 \pmod{\mathfrak{m}_x}$. From [15, Theorem 26.28] it follows that every hereditary $\mathcal{O}_x$-order in $\operatorname{Mat}(n, \mathcal{K})$ is isomorphic to $\boldsymbol{H}(n_1, n_2, \ldots, n_k)$ for some $k$ and $n_i$. Moreover, the rings $\boldsymbol{H}_x(n_1, n_2, \ldots, n_k)$ and $\boldsymbol{H}_x(m_1, m_2, \ldots, m_l)$ are Morita equivalent if and only if $k = l$. Thus, by Theorem 3.3.1, the data $(X, S, k(x) \mid x \in S)$ define the hereditary non-commutative curve $(X, \mathcal{H})$ up to Morita equivalence. □

We denote by $\mathcal{H}(X, S, \mathbf{k}, \mathbf{n})$, where $\mathbf{k} = \{ k(x) \mid x \in S \}$,

$$\mathbf{n} = \{ n_i(x) \mid x \in S,\ 1 \le i \le k(x) \},$$

the non-commutative curve $(X, \mathcal{H})$ such that

$$S = \{ x \in X_{\mathrm{cl}} \mid \mathcal{H}_x \text{ is not normal} \}$$

and, for every $x \in S$, $\mathcal{H}_x = \boldsymbol{H}_x(n_1, n_2, \dots, n_{k(x)})$. As we have just seen, these are Morita representatives for all irreducible hereditary non-commutative curves. Note that the sums $\sum_{i=1}^{k(x)} n_i$ must have the same value $N$ for all $x \in S$. Set $N_j(x) = \sum_{i=1}^{j-1} n_i(x)$. We denote by $\mathcal{L}$ the $\mathcal{H}$-module $n\mathcal{O}$ and by $\mathcal{L}_{x,j}$ the submodule of $\mathcal{L}$ such that $(\mathcal{L}_{x,j})_y = \mathcal{L}_y$ if $y \neq x$ and $(\mathcal{L}_{x,j})_x$ consists of all columns $(a_1, a_2, \dots, a_{k(x)})$ such that $a_i \in \mathfrak{m}_x$ for $i < N_j$. Note that $\mathcal{L}_{x,1} = \mathcal{L}$ and $\mathcal{L}_{x,k(x)+1} = \mathcal{L}(-x) = \mathcal{L} \otimes_{\mathcal{O}} \mathcal{O}(-x)$. For every $x \in S$ we have the chain of submodules

$$\mathcal{L} = \mathcal{L}_{x,1} \supset \mathcal{L}_{x,2} \supset \mathcal{L}_{x,3} \dots \supset \mathcal{L}_{x,k(x)} \supset \mathcal{L}_{x,k(x)+1} = \mathcal{L}(-x).$$

Let $\mathcal{U}_{x,j} = \mathcal{L}_{x,j}/\mathcal{L}_{x,j+1}$ for $1 \le j \le k(x)$, $\mathcal{U}_y = \mathcal{L}/\mathcal{L}(-y)$ if $y \notin S$.

**Proposition 4.1.2.** (1) *The set*

$$\{\, \mathcal{U}_{x,j} \mid x \in S,\, 1 \le j \le k(x) \,\} \cup \{\, \mathcal{U}_y \mid y \notin S \,\}$$

*is a full set of representatives of isomorphism classes of simple $\mathcal{H}$-modules.*

(2) *Let*

$$\mathbf{L} = \{\, \mathcal{L}, \mathcal{L}(-x) \mid x \in X_{\mathrm{cl}} \,\} \cup \{\, \mathcal{L}_{x,j} \mid x \in S,\, 1 < j \le k(x) \,\}.$$

*Then $\langle\, \mathbf{L} \,\rangle = \mathscr{D}^c\mathcal{H}$, hence $\langle\, \mathbf{L} \,\rangle^\omega = \mathscr{D}\mathcal{H}$.*

*Proof.* (1) follows from [40, Corollary 39.18]. Hence every simple $\mathcal{H}$-module, thus every $\mathcal{H}$-module of finite length belongs to $\langle\, \mathbf{L} \,\rangle$. Let $\mathcal{M} \in \mathsf{tf}\,\mathcal{H}$, then $\mathcal{K}\mathcal{M} = m(\mathcal{K}\mathcal{L})$ for some $m$. If $\mathcal{M}' = m\mathcal{L}$, then both $\mathcal{M}/(\mathcal{M} \cap \mathcal{M}')$ and $\mathcal{M}'/(\mathcal{M}/ \cap \mathcal{M}')$ are of finite length, hence belong to $\langle\, \mathbf{L} \,\rangle$. Therefore $\mathcal{M} \in \langle\, \mathbf{L} \,\rangle$, so $\langle\, \mathbf{L} \,\rangle = \mathscr{D}^b(\mathsf{Coh}\,\mathcal{H}) = \mathscr{D}^c\mathcal{H}$. □

**Theorem 4.1.3.** *In the above notations, suppose that $X$ is rational. Let $o \in X_{\mathrm{cl}}$ be an arbitrary point.*

(1) *The set*

$$\mathbb{L}_{\mathcal{H}} = \{\, \mathcal{L}, \mathcal{L}(-o) \,\} \cup \{\, \mathcal{L}_{x,j} \mid x \in S,\, 1 < j \le k(x) \,\}$$

*generates $\mathscr{D}^c\mathcal{H}$, hence strongly generates $\mathscr{D}\mathcal{H}$.*

(2) *If $\mathcal{L}', \mathcal{L}'' \in \mathbb{L}_{\mathcal{H}}$, then $\mathrm{Ext}^i_{\mathcal{H}}(\mathcal{L}', \mathcal{L}'') = 0$ for all $i > 0$, while*

$$\dim \mathrm{Hom}_{\mathcal{H}}(\mathcal{L}', \mathcal{L}'') = \begin{cases} 1 & \text{if } \mathcal{L}' = \mathcal{L}'', \\ & \text{or } \mathcal{L}' = \mathcal{L}(-o),\ \mathcal{L}'' = \mathcal{L}_{x,j}, \\ & \text{or } \mathcal{L}' = \mathcal{L}_{x,j},\ \mathcal{L}'' = \mathcal{L}, \\ & \text{or } \mathcal{L}' = \mathcal{L}_{x,j},\ \mathcal{L}'' = \mathcal{L}_{x,k},\ j > k, \\ 2 & \text{if } \mathcal{L}' = \mathcal{L}(-o),\ \mathcal{L}'' = \mathcal{L}, \\ 0 & \text{in all other cases.} \end{cases}$$

*In particular, $\mathbb{L}_{\mathcal{H}}$ is a tilting set for the category $\mathscr{D}\mathcal{H}$.*

(3) *If $\theta_{xj}$ are generators of the spaces $\mathrm{Hom}_{\mathcal{H}}(\mathcal{L}_{x,j+1}, \mathcal{L}_{x,j})$ $(1 \le j \le k_j)$, then the products $\theta_x = \theta_{x1}\theta_{x2} \dots \theta_{x,k_j}$ are non-zero and any two of them generate $\mathrm{Hom}_{\mathcal{H}}(\mathcal{L}(-o), \mathcal{L})$.*

*Proof.* (1) If $X \simeq \mathbb{P}^1$, then all sheaves $\mathcal{O}(-x)$, hence all sheaves $\mathcal{L}(-x)$ are isomorphic, so we can apply Proposition 4.1.2.

(2) From the definition of $\mathcal{L}$ and $\mathcal{L}_{ij}$ it immediately follows that

$$
\mathcal{H}om_{\mathcal{H}}(\mathcal{L}', \mathcal{L}'') \simeq \begin{cases} \mathcal{O} & \text{if } \mathcal{L}' = \mathcal{L}'', \\ & \text{or } \mathcal{L}' = \mathcal{L}_{x,j},\ \mathcal{L}'' = \mathcal{L}, \\ & \text{or } \mathcal{L}' = \mathcal{L}_{x,j},\ \mathcal{L}'' = \mathcal{L}_{x,k},\ j > k, \\ \mathcal{O}(o - x) & \text{if } \mathcal{L}' = \mathcal{L}(-o),\ \mathcal{L}'' = \mathcal{L}_{x,j}, \\ \mathcal{O}(o) & \text{if } \mathcal{L}' = \mathcal{L}(-o),\ \mathcal{L}'' = \mathcal{L}, \\ \mathcal{O}(-o) & \text{in all other cases.} \end{cases}
$$

Since $\mathrm{Ext}^i_{\mathcal{H}}(\mathcal{L}', \mathcal{L}'') = H^i(\mathcal{H}om_{\mathcal{H}}(\mathcal{L}', \mathcal{L}''))$, it implies the statement.

(3) One easily sees that, if $x = (1 : \xi)$ as the point of $\mathbb{P}^1$, then $\theta_x$, up to a scalar, is the multiplication by $t - \xi$, where $t$ is the affine coordinate on the affine chart $\mathbb{A}^1_0$. Now the statement is obvious. $\square$

Recall that a *canonical algebra* by Ringel [41] is given by a sequence of integers $(k_1, k_2, \dots, k_r)$, where $r \geq 2$ and all $k_i \geq 2$ if $r > 2$, and a sequence $(\lambda_3, \lambda_4, \dots, \lambda_r)$ of different elements from $\Bbbk$ (if $r = 2$, this sequence is empty). Namely, this algebra, which we denote by $\mathbf{R}(k_1, k_2, \dots, k_r; \lambda_3, \dots, \lambda_r)$, is given by the quiver

(4.1.1)
$$
\begin{tikzcd}
 & \bullet \arrow[r, "\alpha_{21}"] & \bullet \cdots \bullet \arrow[r, "\alpha_{k_1-1,1}"] & \bullet \arrow[dr, "\alpha_{k_1 1}"] & \\
\bullet \arrow[ur, "\alpha_{11}"] \arrow[r, "\alpha_{12}"'] \arrow[ddr, "\alpha_{1r}"'] & \bullet \arrow[r, "\alpha_{22}"'] & \bullet \cdots \bullet \arrow[r, "\alpha_{k_2-1,2}"'] & \bullet \arrow[r, "\alpha_{k_2 2}"'] & \bullet \\
 & & \vdots & & \\
 & \bullet \arrow[r, "\alpha_{2r}"'] & \bullet \cdots \bullet \arrow[r, "\alpha_{k_r-1,r}"'] & \bullet \arrow[uur, "\alpha_{k_r r}"'] &
\end{tikzcd}
$$

with relations $\alpha_j = \alpha_1 + \lambda_j \alpha_2$ for $3 \leq j \leq r$, where $\alpha_j = \alpha_{k_j j} \dots \alpha_{2j} \alpha_{1j}$. Certainly, if $r = 2$, it is the path algebra of a quiver of type $\tilde{\mathrm{A}}_{k_1+k_2}$. In particular, if $r = 2$, $k_1 = k_2 = 1$, it is the Kronecker algebra.

**Corollary 4.1.4.** *Let $(X, \mathcal{H})$ be a rational projective hereditary non-commutative curve given by a hereditary data $\{ X, S, k(x) \mid x \in S \}$, $\mathcal{T} = \bigoplus_{\mathcal{F} \in \mathbb{L}_{\mathcal{H}}} \mathcal{F}$, $\Lambda = (\mathrm{End}_{\mathcal{H}} \mathcal{T})^{\mathrm{op}}$. If $S = \{ x_1, x_2, \dots, x_r \}$ with $r \geq 2$, we set $k_i = k(x_i)$. If $S = \{ x \}$, we set $r = 2$, $k_2 = 1$ and $k_1 = k(x)$. If $S = \emptyset$, we set $r = 2$, $k_1 = k_2 = 1$.*

1. *$\Lambda \simeq \mathbf{R}(k_1, k_2, \dots, k_r; \lambda_3, \dots, \lambda_r)$ for some $\lambda_3, \dots, \lambda_r$.*
2. *The functor $\mathrm{Hom}_{\mathcal{H}}(\mathcal{T}, \_)$ induces an equivalence $\mathscr{D}\mathcal{H} \simeq \mathscr{D}\Lambda$.*

*Proof.* (1) follows from the statement (3) of the preceding theorem. As $\mathbb{L}_{\mathcal{H}}$ strongly generates $\mathscr{D}\mathcal{H}$ and $\mathrm{Ext}^i_{\mathcal{H}}(\mathcal{L}', \mathcal{L}'') = 0$, $\mathcal{T}$ is a tilting object, whence (2) follows. $\square$

Actually, the preceding considerations show that a rational projective hereditary non-commutative curve is Morita equivalent to a *weighted projective line* by Geigle–Lenzing [20]. It can also be deduced from the description of hereditary non-commutative curves and the remark on page 271 of [20].

4.2. **König resolution.** Let $(X, \mathcal{A})$ be a non-commutative curve. We denote by $\mathcal{A}_{\rm sg}$ the subset of $X_{\rm cl}$ consisting of all points $x$ such that $\mathcal{A}_x$ is not hereditary. By $\mathcal{J} = \mathcal{J}(\mathcal{A})$ we denote the *singular ideal* of $\mathcal{A}$ defined by its local components

$$\mathcal{J}_x = \begin{cases} \mathcal{A}_x & \text{if } x \notin \mathcal{A}_{\rm sg}, \\ \operatorname{rad} \mathcal{A}_x & \text{if } x \in \mathcal{A}_{\rm sg}. \end{cases}$$

We also denote by $\mathcal{A}^\sharp$ the over-ring of $\mathcal{A}$ such that

$$\mathcal{A}^\sharp_x = \{\, \lambda \in \mathcal{K}_x \mathcal{A}_x \mid \lambda \mathcal{J}_x \subseteq \mathcal{J}_x \,\} \quad \text{for all} \quad x \in X_{\rm cl}.$$

Note that in commutative case, when $\mathcal{A} = \mathcal{O}_X$, $\mathcal{J} = \mathcal{J}_X$ is the ideal of the singular locus $X_{\rm sg}$. We use the fact which follows from [40, Theorem 39.14].

**Proposition 4.2.1.** *$\mathcal{A}$ is hereditary if and only if $\mathcal{A}^\sharp = \mathcal{A}$.*

So we construct the chain of over-rings of $\mathcal{A}$ setting $\mathcal{A}_1 = \mathcal{A}$ and $\mathcal{A}_{k+1} = \mathcal{A}_k^\sharp$. Proposition 4.2.1 shows that there is $n$ such that $\mathcal{A}_{n+1}$ is hereditary (then $\mathcal{A}_m = \mathcal{A}_{n+1}$ for all $m > n$). The smallest $n$ with this property is called the *level* of $\mathcal{A}$. Set $\mathcal{A}_\oplus = \bigoplus_{k=1}^{n+1} \mathcal{A}_k$, where $n$ is the level of $\mathcal{A}$, $\tilde{\mathcal{A}} = (\mathcal{E}nd_{\mathcal{A}}\, \mathcal{A}_\oplus)^{\rm op}$. We call $\tilde{\mathcal{A}}$ the *König resolution* of $\mathcal{A}$ and identify it with the sheaf of $(n+1) \times (n+1)$ matrices $(a_{ij})$ such that $a_{ij} \in \mathcal{A}_{ij} = \mathcal{H}om_{\mathcal{A}}(\mathcal{A}_i, \mathcal{A}_j)$, where $\mathcal{A}_i$ and $\mathcal{A}_j$ are considered as *right* $\mathcal{A}$-modules. In particular, $\mathcal{A}_{ij} = \mathcal{A}_j$ if $i \le j$ and $\mathcal{A}_{i1} = \mathcal{A}'_i = \mathcal{H}om_{\mathcal{A}}(\mathcal{A}_i, \mathcal{A})$. We denote by $\mathcal{P}_i$ the $\tilde{\mathcal{A}}$-module $\tilde{\mathcal{A}} e_{ii}$, where $e_{ij}$ denote the natural matrix units of $\tilde{\mathcal{A}}$. Note that $\mathcal{P}_1 \simeq \mathcal{H}om_{\mathcal{A}}(\mathcal{A}_\oplus, \mathcal{A})$. Let also $\mathcal{E} = (\mathcal{E}nd_{\mathcal{A}} \bigoplus_{k=2}^{n+1} \mathcal{A}_k)^{\rm op}$ identified with the sheaf of matrices $(a_{ij})$ $(2 \le i, j \le n+1)$ with $a_{ij} \in \mathcal{A}_{ij}$, and $\mathcal{I}$ be the ideal of $\mathcal{E}$ consisting of such matrices that $a_{ij} \in \mathcal{A}'_i \mathcal{A}_j$.

**Theorem 4.2.2.** (1) *$\operatorname{gl.dim} \tilde{\mathcal{A}} < \infty$.*

(2) *$\mathsf{Qcoh}\, \mathcal{A} \simeq \mathsf{Qcoh}\, \tilde{\mathcal{A}}/\mathscr{C}$, where $\mathscr{C} = \mathcal{P}_1^\perp$ is a bilocalizing subcategory of $\mathsf{Qcoh}\, \tilde{\mathcal{A}}$. So $\mathsf{Qcoh}\, \mathcal{A}$ is a bilocalization of $\mathsf{Qcoh}\, \tilde{\mathcal{A}}$. Moreover, $\mathscr{C} \simeq \mathsf{Qcoh}\, \bar{\mathcal{E}}$, where $\bar{\mathcal{E}} = \mathcal{E}/\mathcal{I}$.*

(3) *The functors $\mathsf{F} = (\mathcal{A}_\oplus)' \otimes_{\mathcal{A}} \_$ and $\mathsf{H} = \mathcal{H}om_{\mathcal{A}}(\mathcal{A}_\oplus, \_)$ define equivalences, respectively, $\mathsf{Qcoh}\, \mathcal{A} \simeq {}^\perp\mathscr{C}$ and $\mathsf{Qcoh}\, \mathcal{A} \simeq \mathscr{C}^\perp$.*

(4) *$\operatorname{Im} \mathsf{F}$ consists of such $\tilde{\mathcal{A}}$-modules $\mathcal{M}$ that for every point $x \in \mathcal{A}_{\rm sg}$ there is and exact sequence $P_1 \to P_0 \to \mathcal{M}_x$, where $P_0, P_1 \in \operatorname{Add} \mathcal{P}_{1,x}$.*

(5) *$\mathscr{D}\mathcal{A} \simeq \mathscr{D}\tilde{\mathcal{A}}/\mathscr{D}_{\mathscr{C}}\tilde{\mathcal{A}}$ and $\mathscr{D}_{\mathscr{C}}\tilde{\mathcal{A}}$ is a bilocalizing subcategory of $\mathscr{D}\tilde{\mathcal{A}}$.*

(6) *The functors $\mathsf{LF}$ and $\mathsf{RH}$ define equivalences, respectively, $\mathscr{D}\mathcal{A} \simeq {}^\perp(\mathscr{D}_{\mathscr{C}}\tilde{\mathcal{A}})$ and $\mathscr{D}\mathcal{A} \simeq (\mathscr{D}_{\mathscr{C}}\tilde{\mathcal{A}})^\perp$.*

(7) *$\operatorname{Im} \mathsf{LF}$ consists of the complexes quasi-isomorphic to K-flat complexes $\mathcal{M}^\bullet$ such that $\mathcal{F}^i_x \in \operatorname{Add} \mathcal{P}_{1,x}$ for every $i \in \mathbb{Z}$ and every $x \in \mathcal{A}_{\rm sg}$. The same is true if we replace $\mathscr{D}$ by $\mathscr{D}^-$.*

(8) *The functor* $\mathsf{H}$ *induces an equivalence of* $\mathsf{tf}\,\mathcal{A}$ *and* $\mathsf{tf}^*\,\mathcal{P}_1$*, where* $\mathsf{tf}^*\,\mathcal{P}_1$ *consists of all torsion free coherent* $\tilde{\mathcal{A}}$*-modules* $\mathcal{M}$ *such that for every* $x \in \mathcal{A}_{\mathrm{sg}}$ *there is an exact sequence* $0 \to \mathcal{M}_x \to Q \to N \to 0$*, where* $N$ *is torsion free and* $Q$ *is a multiple of* $\mathcal{A}^*_\oplus$*.*

(9) *The functor* $\mathsf{F}^\dagger$*, where* $\mathsf{F}^\dagger(\mathcal{M}) = (\mathsf{F}\mathcal{M})^{**}$*, induces an equivalence of* $\mathsf{tf}\,\mathcal{A}$ *and* $\tilde{\mathsf{F}}\mathcal{P}_1$*, where* $\tilde{\mathsf{F}}\mathcal{P}_1$ *consists of all torsion free coherent* $\tilde{\mathcal{A}}$*-modules* $\mathcal{M}$ *such that for every* $x \in \mathcal{A}_{\mathrm{sg}}$ *there is an epimorphism* $m\mathcal{P}_{1,x} \to \mathcal{M}$

(10) *If* $\mathcal{A}$ *is strongly Gorenstein, the restrictions of* $\mathsf{LF}$ *and* $\mathsf{RH}$ *onto* $\mathscr{D}^{\mathrm{c}}\mathscr{A}$ *coincide.*

*Thus the functor* $\mathsf{G} = \mathcal{H}om_{\tilde{\mathcal{A}}}((\mathcal{A}_+)', \_)$ *gives a resolution* $\mathscr{D}\tilde{\mathcal{A}} \to \mathscr{D}\mathcal{A}$ *and, if* $\mathcal{A}$ *is strongly Gorenstein, this resolution is weakly crepant.*

*Proof.* (1) follows from [26]. The other statements are partial cases of Theorems 2.3.3, 2.3.6, 3.2.1, 3.5.3 and Proposition 3.5.1, since $\mathcal{P}_1 \simeq (\mathcal{A}_\oplus)'$, $\mathcal{P}_1^\vee \simeq \mathcal{A}_\oplus$ and $(\mathcal{E}nd\,\mathcal{P}_1)^{\mathrm{op}} \simeq \mathcal{A}$. □

4.3. **Finite dimensional resolution.** Let now $(X, \mathcal{A})$ be a rational projective non-commutative curve, $(X, \mathcal{H})$ be its hereditary over-ring (for instance, a normal one). Note that $(X, \mathcal{H})$ is not necessarily central and the scheme $\tilde{X} = \mathrm{Spec}(\mathrm{cen}\,\mathcal{H})$ is not necessarily connected. Let $X_1, X_2, \dots, X_s$ be the connected (or, the same, irreducible) components of $\tilde{X}$. Then $\mathcal{H} = \prod_{\alpha=1}^s \mathcal{H}_\alpha$, where $X_\alpha = \mathrm{Spec}(\mathrm{cen}\,\mathcal{H}_\alpha)$. We use the tilting sets

$$\mathbb{L}_{\mathcal{H}_\alpha} = \{\, \mathcal{L}^\alpha, \mathcal{L}^\alpha(-o_\alpha) \,\} \cup \{\, \mathcal{L}^\alpha_{x,j} \,\}$$

from Theorem 4.1.3 and the functor $\mathsf{F}' : \mathsf{Qcoh}\,\mathcal{H} \to \mathsf{Qcoh}\,\mathcal{A}_{\mathcal{H}}$ from Section 3.5. We set $\widetilde{\mathcal{L}}^\alpha = \mathsf{F}'\mathcal{L}^\alpha$ and $\widetilde{\mathcal{L}}^\alpha_{x,j} = \mathsf{F}'\mathcal{L}^\alpha_{x,j}$. We also choose closed points $o_\alpha \in X_\alpha$ and define $\mathcal{Q} = \mathcal{H}_+/\mathcal{I}' = \begin{pmatrix} \mathcal{A}/\mathcal{I} \\ 0 \end{pmatrix}$ as on page 26.

**Theorem 4.3.1.** *The set* $\mathbf{T} = \Big\{\, \mathcal{Q}[-1], \widetilde{\mathcal{L}}^\alpha, \widetilde{\mathcal{L}}^\alpha(-o_\alpha), \widetilde{\mathcal{L}}^\alpha_{x,j} \mid 1 \le \alpha \le s \,\Big\}$ *is a tilting set for* $\mathscr{D}\mathcal{A}_{\mathcal{H}}$*. Therefore the functor* $\mathrm{Hom}_{\mathscr{D}\mathcal{A}_{\mathcal{H}}}(\mathcal{T}, \_)$*, where* $\mathcal{T} = \bigoplus_{\mathcal{C}\in\mathbf{T}} \mathcal{C}$*, induces an equivalence* $\mathscr{D}\mathcal{A}_{\mathcal{H}} \simeq \mathscr{D}\Lambda$*, where* $\Lambda = \mathrm{End}_{\mathscr{D}\mathcal{A}_{\mathcal{H}}}\mathcal{T}$*.*

Note that, since $X$ is projective, $\Lambda$ is a finite dimensional $\Bbbk$-algebra. If necessary, we denote this algebra $\Lambda(\mathcal{A}, \mathcal{H})$.

*Proof.* It follows immediately from Proposition 3.6.1 and Theorem 4.1.3. □

*Remark* 4.3.2. Let $\boldsymbol{Q} = \Gamma(X, \mathcal{A}/\mathcal{I})^{\mathrm{op}} \simeq \mathrm{End}_{\mathcal{A}_{\mathcal{H}}}\mathcal{Q}$, $\mathcal{G}_\alpha = \mathcal{L}^\alpha \oplus \mathcal{L}^\alpha(-o_\alpha) \oplus (\bigoplus_{\alpha,i,j} \mathcal{L}^\alpha_{x,j})$, $\boldsymbol{R}_\alpha = (\mathrm{End}_{\mathcal{H}_\alpha}\mathcal{G}_\alpha)^{\mathrm{op}}$, $\mathcal{G} = \bigoplus_\alpha \mathcal{G}_\alpha$ and $\boldsymbol{R} = (\mathrm{End}_{\mathcal{H}}\mathcal{G})^{\mathrm{op}}$. Then each $\boldsymbol{R}_\alpha$ is a canonical algebra (Corollary 4.1.4), $\boldsymbol{R} \simeq \prod_\alpha \boldsymbol{R}_\alpha$ and $\Lambda$ can be identified with the algebra of triangular matrices

$$\Lambda = \begin{pmatrix} \boldsymbol{Q} & \boldsymbol{E} \\ 0 & \boldsymbol{R} \end{pmatrix},$$

with $\boldsymbol{E} = \mathrm{Ext}^1_{\mathcal{A}_{\mathcal{H}}}(\mathcal{Q}, \mathsf{F}'\mathcal{G})$. Unfortunately, this algebra needs not be of finite global dimension, even if we consider "usual" (commutative) curves, as the following example shows.

**Example 4.3.3.** Let $X \subset \mathbb{P}^2$ consists of 3 lines passing through a point $x$ and $\mathcal{A} = \mathcal{O} = \mathcal{O}_X$ (a plane curve singularity of type $D_4$). Then the normalization $\tilde{X}$ of $X$ consists of 3 disjoint projective lines $X_i$ ($i \in \{1, 2, 3\}$). Let $x_i$ be the preimage of $x$ under the normalization map $\nu : \tilde{X} \to X$, $\tilde{\mathcal{O}} = \nu_* \mathcal{O}_{\tilde{X}} = \mathcal{O}_1 \oplus \mathcal{O}_2 \oplus \mathcal{O}_3$, where $\mathcal{O}_i = \nu_* \mathcal{O}_{X_i}$ and $t_i$ be a generator of the maximal ideal of $\mathcal{O}_{i,x}$. As $\tilde{\mathcal{O}}$ is a unique hereditary over-ring of $\mathcal{O}$, we must take $\mathcal{H} = \tilde{\mathcal{O}}$. Then $\mathcal{I}_x = \bigoplus_{i=1}^{3} t_i^2 \mathcal{O}_i$ and $\mathfrak{m}_x$ is generated as $\mathcal{O}_x$-module by the elements $a = (t_1, t_2, 0)$ and $b = (0, t_2, t_3)$.[8] So $\mathcal{A}/\mathcal{I} \simeq \Bbbk[\alpha, \beta]/(\alpha, \beta)^2$, where $\alpha$ is the image of $a$ and $\beta$ is the image of $b$ in $\mathfrak{m}_x/\mathcal{I}_x$. One can also verify that $\mathrm{Ext}^1_{\mathcal{A}_{\mathcal{H}}}(\mathcal{Q}, \mathcal{O}_i) \simeq \mathcal{O}_{i,x}/(t_i^2)$. Therefore, the algebra $\Lambda$ in this case is given by the quiver

$$\begin{tikzcd}
\bullet \arrow[r, bend left, "\xi_1"] \arrow[r, bend right, "\eta_1"'] & \bullet \arrow[dr, "\gamma_1"] & \\
\bullet \arrow[r, bend left, "\xi_2"] \arrow[r, bend right, "\eta_2"'] & \bullet \arrow[r, "\gamma_2"] & \bullet \arrow[loop above, "\alpha"] \arrow[loop below, "\beta"] \\
\bullet \arrow[r, bend left, "\xi_2"] \arrow[r, bend right, "\eta_1"'] & \bullet \arrow[ur, "\gamma_3"'] &
\end{tikzcd}$$

with relations

$$\begin{aligned}
&\alpha\beta = \beta\alpha = \alpha^2 = \beta^2 = 0, \\
&\gamma_i \eta_i = 0 \quad \text{for } i \in \{1, 2, 3\}, \\
&\alpha\gamma_3 = 0, \\
&\alpha\gamma_1 = \gamma_1 \xi_1, \\
&\beta\gamma_1 = 0, \\
&\beta\gamma_3 = \gamma_3 \xi_3, \\
&\alpha\gamma_2 = \beta\gamma_2 = \gamma_2 \xi_2.
\end{aligned}$$

One easily checks that $\mathrm{gl.dim}\, \Lambda = \infty$.

Nevertheless, we can obtain a "good" resolution of $\mathcal{O}_X$ using the following result.

**Theorem 4.3.4.** *Let $\Lambda$ be a finite dimensional algebra, $\mathfrak{r} = \mathrm{rad}\, \Lambda$, $n$ be the least integer such that $\mathfrak{r}^n = 0$, $M = \bigoplus_{i=1}^{n} \Lambda/\mathfrak{r}^n$, $M^\vee = \mathrm{Hom}_\Lambda(M, \Lambda)$ and $\widetilde{\Lambda} = \mathrm{End}_\Lambda M$.*

(1) $\mathrm{gl.dim}\, \widetilde{\Lambda} \le n$.
(2) *The functors $\mathsf{F} = M^\vee \otimes_\Lambda \_$ and $\mathsf{G} = \mathrm{Hom}_{\widetilde{\Lambda}}(M^\vee, \_)$ establish an equivalence between $\Lambda$-$\mathsf{Mod}$ and the full subcategory $M^\vee$-$\mathsf{Mod}$ of $\widetilde{\Lambda}$-$\mathsf{Mod}$*

[8] The curve $X$ has a singularity of type $D_4$ at the point $x$.

*consisting of such modules $N$ that there is an exact sequence $M_1 \to M_0 \to N \to 0$, where $M_0, M_1 \in \operatorname{Add} M$.*

(3) *The derived functors* $\mathsf{LF}$ *and* $\mathsf{RG}$ *establish an equivalence between $\mathscr{D}\Lambda$ and the full subcategory $\mathscr{D}M^\vee$ of $\mathscr{D}\widetilde{\Lambda}$ consisting of complexes quasi-isomorphic to K-flat complexes having all components from $\operatorname{Add} M$.*

(4) *Let $\Lambda$ be self-injective (quasi-Frobenius), $\mathsf{H} = \operatorname{Hom}_\Lambda(M, \_)$. The restrictions of* $\mathsf{LF}$ *and* $\mathsf{RH}$ *onto the subcategory $\mathscr{D}^{\mathrm{c}}\Lambda$ are isomorphic. So* $\mathsf{LF}$ *restricted onto $\mathscr{D}^{\mathrm{c}}\Lambda$ is both left and right adjoint to* $\mathsf{RG}$.

Note that $M^\vee \simeq \operatorname{Hom}_{\widetilde{\Lambda}}(M, \widetilde{\Lambda})$ is projective as $\widetilde{\Lambda}$-module, so $\mathsf{G}$ is exact and $\mathsf{LG}$ coincide with $\mathsf{RG}$ and acts as $\mathsf{G}$ applied to a complex componentwise.

*Proof.* (1) is proved in [3]. (2) and (3) are partial cases of Theorems 2.3.3 and 2.3.6.

(4) The restrictions of $\mathsf{F}$ and $\mathsf{H}$ onto the subcategory of finitely generated projective modules are isomorphic. If $C^\bullet \in \mathscr{D}^{\mathrm{c}}\Lambda$, it is quasi-isomorphic to a bounded complex $P^\bullet$ of finitely generated projective modules. Then $\mathsf{LF}P^\bullet$ is obtained by applying $\mathsf{F}$ componentwise. As $\Lambda$ is self-injective, the modules $P^i$ are also injective, so $\mathsf{RH}P^\bullet$ is obtained by applying $\mathsf{H}$ componentwise, which is the same as applying $\mathsf{F}$. □

Combining Theorems 4.3.1 and 4.3.4 with the general results on minors from Section 2.3, we see that every non-commutative curve has a resolution by a finite dimensional algebra.

**Corollary 4.3.5.** *For every rational projective non-commutative curve $(X, \mathcal{A})$ there is a resolution $\mathsf{T} : \mathscr{D}\Lambda \to \mathscr{D}\mathcal{A}$, where $\Lambda$ is a finite dimensional algebra of finite global dimension.*

Note that in this situation $\dim \mathscr{D}^b\mathcal{A} \le \dim \mathscr{D}^b\Lambda \le \operatorname{gl.dim} \Lambda$.

4.4. **Subhereditary case.** We say that a non-commutative curve $(X, \mathcal{A})$ is *subhereditary* if there is a hereditary over-ring $\mathcal{H}$ of $\mathcal{A}$ such that $\mathcal{A}/\mathcal{I}$ is semi-simple, where $\mathcal{I} = \mathcal{H}om_{\mathcal{A}}(\mathcal{H}, \mathcal{A})$. (We always identify $\mathcal{I}$ with the annihilator of the *right* $\mathcal{A}$-module $\mathcal{H}/\mathcal{A}$ and $\mathcal{A}/\mathcal{I}$ with the finite dimensional algebra $\Gamma(X, \mathcal{A}/\mathcal{I})$.) For instance, if the over-ring $\mathcal{A}^\sharp$ from Section 4.2 is hereditary, $\mathcal{A}$ is subhereditary. If $\mathcal{H}$ is normal, we say that $\mathcal{A}$ is *subnormal*. Obviously, if $\mathcal{A}$ is commutative, it is subhereditary if and only if it is subnormal.

In this subsection we suppose that $(X, \mathcal{A})$ is subhereditary, but not hereditary, and we choose $\mathcal{H}$ and $\mathcal{I}$ as in the definition above. We also use the notations from the preceding subsection.

**Theorem 4.4.1.** *If $(X, \mathcal{A})$ is subhereditary, then* $\operatorname{gl.dim} \Lambda = 2$. *Hence $\dim \mathscr{D}^b\mathcal{A} \le 2$. Namely, $\langle\, \mathcal{A}/\mathcal{I} \oplus \mathcal{G}\,\rangle_3 = \mathscr{D}^b\mathcal{A}$, where $\mathcal{G}$ is defined as in Remark 4.3.2.*

This result is a generalization of [12, Theorem 5.16].

*Proof.* In this case $\mathcal{Q}$ is semi-simple, hence so is also $\boldsymbol{Q}$. Note that every canonical algebra can be considered as the algebra of triangular matrices

$$\begin{pmatrix} \Bbbk & \boldsymbol{S} \\ 0 & \boldsymbol{H} \end{pmatrix},$$

where $\boldsymbol{H}$ is the path algebra of the quiver

$$\begin{tikzcd}
& \bullet \arrow[r, "\alpha_{21}"] & \bullet \cdots \bullet \arrow[r, "\alpha_{k_1-1,1}"] & \bullet \\
\bullet \arrow[ur, "\alpha_{11}"] \arrow[r, "\alpha_{12}"'] \arrow[ddr, "\alpha_{1r}"'] & \bullet \arrow[r, "\alpha_{22}"'] & \bullet \cdots \bullet \arrow[r, "\alpha_{k_2-1,2}"'] & \bullet \\
& & \vdots & \\
& \bullet \arrow[r, "\alpha_{2r}"'] & \bullet \cdots \bullet \arrow[r, "\alpha_{k_r-1,r}"'] & \bullet
\end{tikzcd}$$

and $\boldsymbol{S}$ is generated as right $\boldsymbol{H}$-module by the arrows $\alpha_{k_j j}$ in the notations of (4.1.1). In particular, $\boldsymbol{H}$ is hereditary.

Thus $\Lambda = \operatorname{End}_{\mathscr{D}\mathcal{A}_{\mathcal{H}}} \mathcal{T}$ can be considered as the algebra of triangular matrices

$$\Lambda = \begin{pmatrix} \boldsymbol{Q} & \boldsymbol{E}_1 & \boldsymbol{E}_2 \\ 0 & \Bbbk^s & \prod_\alpha \boldsymbol{S}_\alpha \\ 0 & 0 & \prod_\alpha \boldsymbol{H}_\alpha \end{pmatrix},$$

where each $\boldsymbol{H}_\alpha$ is hereditary. The algebra of $2 \times 2$ matrices from the upper left corner

$$\begin{pmatrix} \boldsymbol{Q} & \boldsymbol{E}_1 \\ 0 & \Bbbk^s \end{pmatrix}$$

is also hereditary. Using the result of [39], we see that $\operatorname{gl.dim} \Lambda = 2$ (obviously, $\Lambda$ is not hereditary itself).

Thus $\langle \Lambda \rangle_3 = \mathscr{D}^b \Lambda$ [42, Proposition 7.4]. As $\Lambda$ is the image of $\mathcal{T}$ under the equivalence $\mathscr{D}^b \mathcal{A}_{\mathcal{H}} \simeq \mathscr{D}^b \Lambda$, also $\langle \mathcal{T} \rangle_3 = \mathscr{D}^b \mathcal{A}_{\mathcal{H}}$. Then $\langle \mathsf{G}\mathcal{T} \rangle_3 = \mathscr{D}^b \mathcal{A}$ by Proposition 1.6.1. One easily sees that $\mathsf{G}\mathcal{T} \simeq \mathcal{A}/\mathcal{I}[-1] \oplus \mathcal{G}$, which accomplishes the proof. $\square$

**Example 4.4.2.** Fix a point $o$ in $\mathbb{P}^n$ and consider the union $X$ of $n$ lines of general position passing through this point. Then $o$ in a unique singular point of $X$, the normalization $\tilde{X}$ of $X$ is the disjoint union $\bigsqcup_{i=1}^n X_i$ of $n$ projective lines $X_i$. Let $\nu : \tilde{X} \to X$ be the normalization map, $\mathcal{O} = \mathcal{O}_X$ and $\tilde{\mathcal{O}} = \nu_* \mathcal{O}_{\tilde{X}}$. The conductor $\mathcal{I}$ of $\tilde{\mathcal{O}}$ in $\mathcal{O}$ coincides with the ideal defining the point $o$, so $\mathcal{O}/\mathcal{I} = \Bbbk(o)$. So $X$ is subnormal. $\tilde{\mathcal{O}} = \prod_{i=1}^n \tilde{\mathcal{O}}_i$, where $\tilde{\mathcal{O}}_i = \nu_* \mathcal{O}_{X_i}$. Hence the tilting set for $\tilde{\mathcal{O}}$ is $\left\{ \tilde{\mathcal{O}}_i, \tilde{\mathcal{O}}_i(-o_i) \mid 1 \le i \le n \right\}$, where $o_i$ is the preimage of $o$ on $X_i$. We suppose that the coordinates of $o_i$ on $X_i = \mathbb{P}^1$ are $(0:1)$. Then the algebra $\Lambda$ defining a categorical resolution for $X$ is given by the quiver with relations

$$\begin{tikzcd}
\bullet \arrow[r, bend left, "\xi_1"] \arrow[r, bend right, "\eta_1"'] & \bullet \arrow[dr, "\gamma_1"] & \\
\bullet \arrow[r, bend left, "\xi_2"] \arrow[r, bend right, "\eta_2"'] & \bullet \arrow[r, "\gamma_2"'] & \bullet \\
\vdots & & \\
\bullet \arrow[r, bend left, "\xi_n"] \arrow[r, bend right, "\eta_n"'] & \bullet \arrow[uur, "\gamma_n"'] &
\end{tikzcd} \qquad \gamma_i\xi_i = 0 \quad (1 \le i \le n)$$

Respectively, $\mathscr{D}^b\mathcal{O} = \langle\, \Bbbk(o) \oplus \mathcal{G}\,\rangle_3$, where $\mathcal{G} = \bigoplus_{i=1}^n (\tilde{\mathcal{O}}_i \oplus \tilde{\mathcal{O}}_i(-o_i))$.

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