VECTOR BUNDLES AND COHEN–MACAULAY MODULES

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To the memory of Sheila Brenner

CONTENTS

INTRODUCTION

The aim of this survey is to present recent results on classification of vector bundles over projective curves and Cohen–Macaulay modules over surface singularities, mainly obtained by the author in collaboration with G.-M. Greuel and I. Kashuba [**DG3**, **DGK**]. We consider this problem from the viewpoint of the representation theory, being mainly interested in the *representation type* (finite, tame or wild) and, for tame case, in the description of *all* objects. So we do not deal with stable bundles and related topics, though something can be done in this direction too (cf. Section 4). We mostly consider algebras and varieties over an algebraically closed field $\Bbbk$, though some results remain valid in a more general setting.

Recall a history of these investigations. The first general result concerning *curve singularities* was obtained by H. Jacobinski and independently by

A. Roiter and the author [**Jac**, **DR**], who gave a criterion for a curve singularity to be of *finite* Cohen–Macaulay type. We must note that there were no such *words* as 'curve singularity' or 'Cohen–Macaulay module' in these papers; they were replaced by 'commutative local ring of Krull dimension 1' and 'torsion free module.' So the first paper, where these results were indeed related to curve singularities, was that of G.-M. Greuel and H. Knörrer [**GK**], where this criterion obtained the following wonderful form (see also [**Yo**], which is a perfect survey on the Cohen–Macaulay finite type):

A curve singularity is Cohen–Macaulay finite if and only if it dominates a simple plane curve singularity (in the sense of deformation theory, cf. [**AGV**]).

This result was extended to *hypersurface singularities* in [**Kn**, **BGS**]. Namely,

A hypersurface singularity is Cohen–Macaulay finite if and only if it is simple (0-modal), i.e. of type A_n, D_n *($n \in \mathbb{N}$) or* E_n *($n = 6, 7, 8$)* (cf. [**AGV**]).

At the same time H. Ésnault and independently M. Auslander [**Esn**, **Aus**] (see also [**Yo**]) proved that

A surface singularity over an algebraically closed field $\mathbb{k}$ *of characteristic 0 is Cohen–Macaulay finite if and only if it is a quotient singularity, i.e. is isomorphic to the ring of invariants* $\mathbb{k}[[x, y]]^G$*, where* G *is a finite subgroup of* $\mathbf{GL}(2, \mathbb{k})$.

By the way, we do not know whether this result has ever been generalized to the case of positive characteristic.

The first step towards *tame* case was made by Schappert [**Sch**], who proved that a plane curve singularity has at most 1-parameter families of ideals if and only if it is *strictly unimodal* [**Wall**], or, the same, uni- or bimodal in the sense of [**AGV**]. In [**DG2**] this result was extended to all curve singularities (one only has to replace in Schappert's theorem '*it is*' by '*it dominates*'). Nevertheless, most of these singularities happened to be Cohen–Macaulay wild. Indeed, in [**DG1**] G.-M. Greuel and the author showed that

A curve singularity, which is not Cohen–Macaulay finite, is Cohen–Macaulay tame if and only if it dominates one of the singularities of type T_{pq}.

In characteristic 0 these singularities (all of them are plane) are given by the equations $x^p + y^q + \lambda x^2 y^2 = 0$ ($1/p + 1/q \leq 1/2$, $\lambda \neq 0$; if $(pq) = (36)$ or (44) some special values of λ must also be excluded). In positive characteristic the easiest way to define these singularities is by their parameterisations.

The case of surface singularities was first studied by C. Kahn [**Kahn**]. He proved that the so-called *simple elliptic* singularities are Cohen–Macaulay tame and described Cohen–Macaulay modules over these singularities. Moreover, he elaborated a very general procedure that relates Cohen–Macaulay modules over a normal surface singularity with vector bundles over the exceptional curve of its resolution. As for simple elliptic singularities the latter is an elliptic curve, he only had afterward to apply the classification of vector bundles over elliptic curves by M. Atiyah [**At**]. By the way, till

recently only vector bundles over a projective line [**Gro1**] and over elliptic curves have been classified.

To apply Kahn's technique to other surface singularities one has to know the corresponding results for vector bundles over projective curves. This investigation has been accomplished in [**DG3**] with the following output:

A projective curve is:

- *vector bundle finite if it is a configuration of projective lines of type* A_n *(just* $\mathbb{P}^1$ *if* $n = 1$*);*
- *vector bundle tame if it is either an elliptic curve or a configuration of projective lines of type* $\widetilde{A}_n$ *(if* $n = 1$ *it is a plane nodal cubic);*
- *vector bundle wild otherwise.*

In finite and tame cases a complete description of vector bundles was obtained. These results (with the corresponding definitions and outline of proofs) are presented in the first part of the paper (Sections 1–8). The technical background here is that of "*matrix problems,*" widely used before in analogous (and lots of other) questions.

The second part (Sections 9–14) consists of applications to surface and hypersurface singularities (cf. [**DGK**]). We recall the Kahn's reduction since it seems not well known to the audience. Moreover, we extend it to *families* of Cohen–Macaulay modules and vector bundles, which is necessary to deal accurately with tameness and wildness. Then we apply it, together with the results of the preceding sections, to the so-called *minimally elliptic* singularities [**Lau**]. In this case a complete answer can be obtained:

A minimally elliptic singularity is:

- *Cohen–Macaulay tame if it is either simple elliptic or a cusp singularity;*
- *Cohen–Macaulay wild otherwise.*

Here a *cusp* singularity is a such one that the exceptional curve of its minimal resolution is a configuration of projective lines of type $\widetilde{A}_n$ (the original definition by F. Hirzebruch [**Hir**] was different, though equivalent for the case $\Bbbk = \mathbb{C}$). For cusp singularities we get a complete description of Cohen–Macaulay modules. By the way, it also gives possibility to fill up a flaw in the result for curve singularities, namely to give an explicit classification of Cohen–Macaulay modules over the singularities of type T_{pq}. (Recall that in [**DG1**] there was no such classification; their tameness was proved indirectly, using considerations from the deformation theory). We also obtain a description of Cohen–Macaulay modules over the so-called *log-canonical* singularities, since they are just quotients of simple elliptic or cusp by finite groups of automorphisms [**Kaw**]. There is some strong evidence that these cases are indeed the only tame ones, all other surface singularities being Cohen–Macaulay wild.

At last, we consider the case of *hypersurface singularities.* Combining our results with the Knörrer periodicity theorem [**Kn**, **Yo**], we obtain a classification of Cohen–Macaulay modules over singularities *of type* T_{pqr}, i.e. given by equations $x^p + y^q + z^r + \lambda xyz + Q(t_1, \ldots, t_m) = 0$, where Q is a non-degenerate quadratic form, $1/p + 1/q + 1/r \leq 1$, $\lambda \neq 0$ (if $(p, q, r) = (2, 3, 6), (2, 4, 4)$ or $(3, 3, 3)$, some extra values of λ must be excluded). Again

there is an evidence that all other hypersurface singularities are Cohen–Macaulay wild.

1. An easy example: vector bundles on $\mathbb{P}^1$

A projective line $\mathbb{P}^1$ is a union of two affine lines $\mathbb{A}_i^1$ ($i = 0, 1$): if $(x_0 : x_1)$ are homogeneous coordinates in $\mathbb{P}^1$, then $\mathbb{A}_i^1 = \{ (x_0 : x_1) \,|\, x_i \neq 0 \}$. The affine coordinate on $\mathbb{A}_0^1$ is $t = x_1/x_0$ and on $\mathbb{A}_1^1$ $t^{-1} = x_0/x_1$. Thus we can identify $\mathbb{A}_0^1$ with $\operatorname{Spec} \mathbb{k}[t]$ and $\mathbb{A}_1^1$ with $\operatorname{Spec} \mathbb{k}[t^{-1}]$; their intersection is then $\operatorname{Spec} \mathbb{k}[t, t^{-1}]$. Certainly, any projective module over $\mathbb{k}[t]$ is free, i.e. all vector bundles over an affine line are trivial. Therefore to define a vector bundle over $\mathbb{P}^1$ one only has to prescribe its rank r and a *gluing matrix* $A \in \operatorname{GL}(r, \mathbb{k}[t, t^{-1}])$. Changing bases in free modules over $\mathbb{k}[t]$ and $\mathbb{k}[t^{-1}]$ corresponds to the transformations $A \mapsto TAS$, where S and T are invertible matrices of the same size, respectively over $\mathbb{k}[t]$ and over $\mathbb{k}[t^{-1}]$. Now an easy calculation, quite similar to that used in description of finitely generated modules over an euclidean ring, leads to the following

Lemma 1.1. *For any matrix $A \in \operatorname{GL}(r, \mathbb{k}[t, t^{-1}])$ there are matrices $S \in \operatorname{GL}(r, \mathbb{k}[t])$ and $T \in \operatorname{GL}(r, \mathbb{k}[t^{-1}])$ such that SAT is a diagonal matrix* $\operatorname{diag}(t^{d_1}, \ldots, t^{d_r})$.

Since 1×1 matrix (t^d) defines the line bundle $\mathcal{O}_{\mathbb{P}^1}(d)$, we get

Theorem 1.2. *Every vector bundle over a projective line uniquely decomposes into a direct sum of line bundles $\mathcal{O}_{\mathbb{P}^1}(d)$.*
(As usually, 'unique' in this context means that if $\mathcal{F} \simeq \bigoplus_{i=1}^n \mathcal{L}_i \simeq \bigoplus_{j=1}^m \mathcal{L}'_j$, then $m = n$ and there is a permutation τ of indices such that $\mathcal{L}_i \simeq \mathcal{L}'_{\tau i}$ for all i.)

This is a typical example of *finite vector bundle type*: every indecomposable is a twist of one of them, namely $\mathcal{O}_{\mathbb{P}^1}$. I think that it is an important distinction between *finite* and *discrete* type. For instance, a quiver of type A_∞ is a typical example of *discrete*, but *not finite* type: it has finitely many representation of each prescribed vector-dimension, but the dimensions of indecomposables can be arbitrary big.

2. A simple example: projective configurations of type A

A *projective configuration* is, by definition, a (singular, reduced) curve X such that

1) each irreducible component X_i ($i = 1, \ldots, s$) of X is *rational*, i.e. its normalization is isomorphic to a projective line $\mathbb{P}^1$;
2) each singular point of X is an *ordinary double point*, i.e. a transversal intersection of two components (the latter may coincide, so it may be a self-intersection of a component).

In particular, no three components pass through one point. For every projective configuration X we define its *intersection graph* (or *dual graph*) $\Delta(X)$ as follows.

- The *vertices* of $\Delta(X)$ are irreducible components of X, or rather their indices $1, 2, \ldots, s$.

- The *edges* of $\Delta(X)$ are singular points of X.
- An edge p is incident to a vertex i if $p \in X_i$; especially if p is a self-intersection point of a component X_i, it gives rise to a *loop* in the graph $\Delta(X)$.

This graph is non-oriented, but may contain loops and multiple edges between two vertices. Note that the graph $\Delta(X)$ does not, in general, define a projective configuration X up to isomorphism. For instance, if $\Delta(X)$ is a graph of type $\tilde{D}_3$, i.e.

$$
\begin{array}{c}
1 \\
| \\
2 \longrightarrow 5 \longrightarrow 4 \\
| \\
3
\end{array}
$$
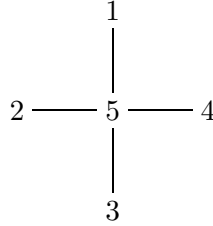
the position of 4 intersection points on the projective line X_5 corresponding to the central point depends on one parameter $\lambda \in \Bbbk \setminus \{\, 0, 1 \,\}$: their harmonic ratio. For a fixed λ these points can be chosen as $0, 1, \lambda, \infty$.

We consider now the simplest case, when $\Delta(X)$ is of type A_s, i.e.

$$
1 \longrightarrow 2 \longrightarrow 3 \longrightarrow \cdots \longrightarrow s
$$
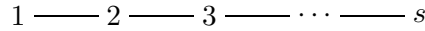
Denote by p_i ($1 \leq i < s$) the intersection point of X_i and X_{i+1}. The calculation below is typical for the case of singular curves, though is the simplest example of this sort.

The normalization $\tilde{X}$ of X is just a disjoint union $\bigsqcup_{i=1}^{s} X_i$. Each point p_i gives rise to two points on $\tilde{X}$: $p_i' \in X_i$ and $p_i'' \in X_{i+1}$. We may suppose that the isomorphisms $X_i \simeq \mathbb{P}^1$ are so chosen that $p_i' = \infty$ and $p_i'' = 0$ (in homogeneous coordinates, respectively, $(0 : 1)$ and $(1 : 0)$). As the normalization mapping $\pi : \tilde{X} \to X$ is finite and birational, it induces an embedding $\mathcal{O} = \mathcal{O}_X \to \tilde{\mathcal{O}} = \pi_* \mathcal{O}_{\tilde{X}}$, and we can (and shall) identify any vector bundle $\mathcal{F}$ over $\tilde{X}$ with its direct image $\pi_* \mathcal{F}$. We denote by $\mathcal{J}$ the *conductor* of $\tilde{\mathcal{O}}$ in $\mathcal{O}$, i.e. the biggest sheaf of $\tilde{\mathcal{O}}$-ideals contained in $\mathcal{O}$. In our example its sections are just those sections of $\tilde{\mathcal{O}}$, which have zeros at all points p_i' and p_i'' for each i. Note that $\mathcal{O}/\mathcal{J} = \bigoplus_{i=1}^{s-1} \Bbbk(p_i)$ and $\tilde{\mathcal{O}}/\mathcal{J} \simeq \bigoplus_{i=1}^{s-1} (\Bbbk(p_i') \times \Bbbk(p_i''))$.

Let $\mathcal{G}$ be a vector bundle over X of rank r, $\tilde{\mathcal{G}} = \tilde{\mathcal{O}} \otimes_{\mathcal{O}} \mathcal{G}$. Then $\tilde{\mathcal{G}} \supset \mathcal{G} \supset \mathcal{J}\mathcal{G} = \mathcal{J}\tilde{\mathcal{G}}$. We already know that $\tilde{\mathcal{G}} \simeq \bigoplus_{i=1}^{s}(\bigoplus_{j=1}^{r} \mathcal{O}_i(d_{ij}))$ for some integers d_{ij}, where $\mathcal{O}_i = \mathcal{O}_{X_i}$. For every d

$$
\mathcal{O}_i(d)/\mathcal{J}\mathcal{O}_i(d) \simeq \mathcal{O}_i/\mathcal{J}\mathcal{O}_i \simeq
\begin{cases}
\Bbbk(p_{i-1}'') \oplus \Bbbk(p_i'), & \text{if } 1 < i < s, \\
\Bbbk(p_1'), & \text{if } i = 1, \\
\Bbbk(p_{s-1}''), & \text{if } i = s.
\end{cases}
$$

Therefore the factor $\tilde{\mathcal{G}}/\mathcal{J}\tilde{\mathcal{G}}$ is isomorphic to $\bigoplus_{i=1}^{s-1} r(\Bbbk(p_i') \oplus \Bbbk(p_i''))$. The factor $\mathcal{G}/\mathcal{J}\mathcal{G}$ is isomorphic to $\bigoplus_{i=1}^{s} r\Bbbk(p_i)$, where each $r\Bbbk(p_i)$ is embedded into $r(\Bbbk(p_i') \oplus \Bbbk(p_i''))$. Moreover, the projections of $r\Bbbk(p_i)$ onto both $r\Bbbk(p_i')$ and $r\Bbbk(p_i'')$ are isomorphisms. On the contrary, given r-dimensional subspaces $V_i \in r(\Bbbk(p_i') \oplus \Bbbk(p_i''))$ for each i such that their projections onto both $r\Bbbk(p_i')$

and $r\Bbbk(p_i'')$ are isomorphisms, we can construct a vector bundle $\mathcal{G}$ over X taking the preimage of $\bigoplus_{i=1}^{s-1} V_i$ in $\tilde{\mathcal{G}}/\mathcal{J}\tilde{\mathcal{G}}$. Hence $\mathcal{G}$ can be defined by a set of invertible $r \times r$ matrices $\{\, M_i', M_i'' \,|\, i = 1, \ldots, s-1 \,\}$ describing the projections of V_i respectively onto $r\Bbbk(p_i')$ and $r\Bbbk(p_i'')$. It is important that every row of these matrices has a *weight* d_{ij}: the degree of the corresponding vector bundle $\mathcal{O}_i(d_{ij})$ (this weight is common to the j-th rows of M_i' and of M_{i-1}'').

Certainly, we can change these matrices using automorphisms of $\tilde{\mathcal{G}}$ and of V_i. Recall that

$$\mathrm{Hom}_{\mathcal{O}_i}(\mathcal{O}_i(d), \mathcal{O}_i(d')) = \begin{cases} 0, & \text{if } d > d', \\ \Bbbk, & \text{if } d = d', \\ \mathrm{Poly}(d'-d), & \text{if } d < d', \end{cases}$$

where $\mathrm{Poly}(m)$ is the space of homogeneous polynomials of degree m over $\Bbbk$. Namely, if s is a section of $\mathcal{O}_i(d)$ and $f \in \mathrm{Poly}(d'-d)$, then $(fs)(\xi_0 : \xi_1) = f(\xi_0, \xi_1)s(\xi_0 : \xi_1)$. In particular, for any scalars $\lambda, \mu \in \Bbbk$ one can choose $f \in \mathrm{Poly}(d'-d)$ such that $(fs)(0) = \lambda s(0)$ and $(fs)(\infty) = \mu s(\infty)$. Therefore two sets of matrices $\mathfrak{M} = \{\, M_i', M_i'' \,\}$ and $\mathfrak{N} = \{\, N_i', N_i'' \,\}$ define isomorphic vector bundles over X if and only if $\mathfrak{N}$ can be obtained from $\mathfrak{M}$ by a sequence of transformations of the following sorts:

1) $M_i' \mapsto M_i'S$ and $M_i'' \mapsto M_i''S$ for some i and some invertible matrix S;
2) $M_i' \mapsto T'M_i'$ and $M_{i-1}'' \mapsto T''M_{i-1}''$ $(1 < i < s-1)$, where $T' = (t_{jk}')$ and $T'' = (t_{jk}'')$ are invertible matrices such that
   (a) $t_{jk}' = t_{jk}''$ if $d_{ij} = d_{ik}$;
   (b) $t_{jk}' = t_{jk}'' = 0$ if $d_{ij} < d_{ik}$;
3) $M_1' \mapsto TM_1'$ for some invertible matrix T;
4) $M_{s-1}'' \mapsto TM_{s-1}''$ for some invertible matrix T.

The following result is a rather simple exercise in matrix calculation.

Proposition 2.1. *Using transformations* (1–4) *from above one can transform any set* $\mathfrak{M} = \{\, M_i', M_i'' \,\}$ *to the set* $\mathfrak{I}$ *only consisting of unit matrices.*

(Actually, one has to start from M_1', make it unit, then consider transformations of M_1'' that do not change this form of M_1', etc.)

Evidently, it can be reformulated as a description of all vector bundles over X.

Theorem 2.2. *Let X be a projective configuration of type* A_s.

1. *Every vector bundle over X uniquely decomposes into a direct sum of line bundles.*
2. *A line bundle $\mathcal{L}$ over X is uniquely determined by its* vector-degree, *i.e. the sequence* $\mathbf{d} = (d_1, d_2, \ldots, d_s)$, *where* $d_i = \deg_{X_i} \mathcal{L}$.

Especially every line bundle is a twist of the trivial line bundle $\mathcal{O}$. Thus a projective configuration of type A_s, as well as projective line, is of finite vector bundle type. Further we shall see that there are no more such curves.

Note that just the same calculation also gives a description of all *torsion free* coherent sheaves $\mathcal{F}$ over X. The distinctions are the following:

- The ranks r_j of the restrictions $\mathcal{F}|X_j$ need not coincide, as well as the dimensions m_i of the fibres $\mathcal{F}(p_i)$.
- The projections $m_i \mathbb{k}(p_i) \to r_i \mathbb{k}(p'_i)$ and $m_i \mathbb{k}(p_i) \to r_{i+1} \mathbb{k}(p''_i)$ must be surjective (but not necessarily bijective).
- The mappings $m_i \mathbb{k}(p_i) \to r_i \mathbb{k}(p'_i) \oplus r_{i+1} \mathbb{k}(p''_i)$ must be injective.

It means that the matrices M'_i (of size $r_i \times m_i$) and M''_i (of size $r_{i+1} \times m_i$) are not necessarily square, but have the ranks, respectively, r_i and r_{i+1}, while the "big" matrix

$$\begin{pmatrix} M'_i \\ M''_i \end{pmatrix}$$

has rank m_i. In any case, using transformations (1–4) from above these matrices can be transformed to diagonal forms with 1 and 0 on the diagonals. Thus an indecomposable torsion free sheaf is actually a vector bundle on a connected part of our configuration, i.e. on a curve consisting of components $X_k, X_{k+1}, \ldots, X_l$ for some $1 \le k < l \le s$. Again we only get, up to twist, a finite number of indecomposables, so with this respect nothing changes.

3. ELLIPTIC CURVES ARE VECTOR BUNDLE TAME

Another known case is that of *elliptic curves*, i.e. smooth projective curves of genus 1. Such a curve can always be represented as a 2-fold covering of a projective line with 4 ramification points of degree 2, which can be chosen as $0, 1, \lambda, \infty$ ($\lambda \in \mathbb{k} \setminus \{0, 1\}$). If char $\mathbb{k} \ne 2$ it can also be considered as a smooth cubic curve in $\mathbb{P}^2$ given in one of its affine parts by the equation $y^2 = x(x-1)(x-\lambda)$. Recall [**Ha**, Section IV.4] that in this case the line bundles of a prescribed degree d are in one-to-one correspondence with the points of the curve X. Namely, if we fix one point o, such a bundle is isomorphic to $\mathcal{O}_X(x+(d-1)o)$ for a uniquely determined point x. Moreover, there is a line bundle $\mathcal{P}$ on $X \times X$ (*Poincaré bundle*) such that, for every $x \in X$,

$$\mathcal{O}_X(x + (d-1)o) \simeq \mathcal{O}_X(do) \otimes_{\mathcal{O}_X} i_x^* \mathcal{P} \simeq i_x^* \mathcal{P}(d(o \times X)),$$

where i_x is the embedding $X \simeq X \times x \to X \times X$. Thus the line bundles of degree d form a 1-parameter family (parameterised by X).

It so happens that the description of indecomposable vector bundles of an arbitrary rank and degree is quite similar. Nearby we present the results of Atiyah [**At**] (with the modifications of Oda [**Oda**], who has shown that the Atiyah's classification can be formulated in terms of families). We denote by nx the closed subscheme of X defined by the sheaf of ideals $\mathcal{O}_X(-nx)$ and by i_{nx} the embedding $X \times nx \to X \times X$.

Theorem 3.1. *For every pair of coprime integers (r, d) with $r > 0$ there is a vector bundle $\mathcal{P}_{r,d}$ over $X \times X$ such that every indecomposable vector bundle over X of rank nr and degree nd, where n is a positive integer, is isomorphic to $p_{1*} i_{nx}^* \mathcal{P}_{r,d}$ for a uniquely determined point $x \in X$. Moreover, $\mathcal{P}_{r,d+mr} \simeq \mathcal{P}_{r,d}(m(o \times X))$ and $\mathcal{P}_{1,0} \simeq \mathcal{P}$.*

The proof of this theorem uses rather sophisticated considerations specific to elliptic curves, and we omit it referring to [**At**, **Oda**].

Theorem 3.1 shows that any elliptic curve is vector bundle *tame*: there are 1-paramenter families, at most one for any prescribed rank and degree, such that every indecomposable vector bundle over X can be obtained from this family by specialization. The latter may include "blowing," which means that we consider the values not only at points, but also at subschemes of the sort nx. (We shall give precise definitions in Section 5.)

4. Curves of genus $g > 1$ are vector bundle wild

Suppose now that X is a smooth projective curve of genus $g > 1$, $\mathcal{O} = \mathcal{O}_X$. Then for any two points $x \neq y$ from X the Riemann–Roch Theorem implies that $\operatorname{Hom}_{\mathcal{O}}(\mathcal{O}(x), \mathcal{O}(y)) \simeq \operatorname{H}^0(X, \mathcal{O}(y - x)) = 0$ and $\operatorname{Ext}^1_{\mathcal{O}}(\mathcal{O}(x), \mathcal{O}(y)) \simeq \operatorname{H}^1(X, \mathcal{O}(y - x)) \neq 0$. Fix 5 different points $x_1, \ldots, x_5$ of the curve X, choose non-zero elements $\xi_{ij} \in \operatorname{Ext}^1(\mathcal{O}(x_j), \mathcal{O}(x_i))$ for $i \neq j$ and consider vector bundles $\mathcal{F}(A, B)$, where $A, B \in \operatorname{Mat}(n \times n, \Bbbk)$, and $\mathcal{F}(A, B)$ is given as an extension

$$0 \longrightarrow \underbrace{n(\mathcal{O}(x_1) \oplus \mathcal{O}(x_2))}_{\mathcal{B}} \longrightarrow \mathcal{F}(A, B) \longrightarrow \underbrace{n(\mathcal{O}(x_3) \oplus \mathcal{O}(x_4) \oplus \mathcal{O}(x_5))}_{\mathcal{A}} \longrightarrow 0$$

corresponding to the element $\xi(A, B)$ of $\operatorname{Ext}^1(\mathcal{A}, \mathcal{B})$ presented by the matrix

$$\begin{pmatrix} \xi_{13}I & \xi_{14}I & \xi_{15}I \\ \xi_{23}I & \xi_{24}A & \xi_{25}B \end{pmatrix}$$

(I denotes the unit $n \times n$ matrix). If (A', B') is another pair of matrices, any homomorphism $\mathcal{F}(A, B) \to \mathcal{F}(A', B')$ maps $\mathcal{O}(x_i)$ to $\mathcal{O}(x_i)$. It means that there are homomorphisms $\phi : \mathcal{A} \to \mathcal{A}'$ and $\psi : \mathcal{B} \to \mathcal{B}'$ such that $\psi\xi(A, B) = \xi(A', B')\phi$ (this is the Yoneda multiplication). Note that both ϕ and ψ also map $\mathcal{O}(x_i)$ to $\mathcal{O}(x_i)$ for each i. Now one can easily deduce that $\phi = \operatorname{diag}(S, S, S)$ and $\psi = \operatorname{diag}(S, S)$ for some matrix $S \in \operatorname{Mat}(n' \times n, \Bbbk)$ such that $SA = A'S$ and $SB = B'S$.

If we consider a pair (A, B) as a representation of the free algebra Σ_2 in 2 generators, the correspondence $(A, B) \mapsto \mathcal{F}(A, B)$ becomes a *full, faithful, exact* functor $\Sigma_2\text{-mod} \to \operatorname{VB}(X)$. In particular, it maps non-isomorphic modules to non-isomorphic vector bundles and indecomposable modules to indecomposable vector bundles. Using terminology of the representation theory of algebras, we say that the curve X is *vector bundle wild*. Again we give a precise definition in the next section.

Recall that the algebra Σ_2 here can be replaced by *any* finitely generated algebra $\Lambda = \Bbbk\langle a_1, a_2, \ldots, a_m \rangle$. Indeed, any Λ-module M such that $\dim_{\Bbbk} M = n$ is given by a set of matrices $\{A_1, A_2, \ldots, A_m\}$ of size $n \times n$. One gets a full, faithful, exact functor $\Lambda\text{-mod} \to \Sigma_2\text{-mod}$ mapping the module M to the Σ_2-module of dimension mn defined by the pair of matrices

$$\begin{pmatrix} \lambda_1 I & 0 & \ldots & 0 \\ 0 & \lambda_2 I & \ldots & 0 \\ \multicolumn{4}{c}{\ldots\ldots\ldots\ldots\ldots\ldots\ldots} \\ 0 & 0 & \ldots & \lambda_n I \end{pmatrix}, \quad \begin{pmatrix} A_1 & I & 0 & \ldots & 0 \\ 0 & A_2 & I & \ldots & 0 \\ \multicolumn{5}{c}{\ldots\ldots\ldots\ldots\ldots\ldots} \\ 0 & 0 & 0 & \ldots & A_n \end{pmatrix},$$

where $\lambda_1, \lambda_2, \ldots, \lambda_m$ are different elements from the field $\Bbbk$. Thus a classification of vector bundles over X would imply a classifications of *all* representations of *all* finitely generated algebras, the goal that perhaps nobody considers as achievable.

As the first result of our investigation, we may formulate a theorem that describes vector bundle types of smooth projective curves.[1]

Theorem 4.1. *A smooth projective curve X is*

- *vector bundle finite if $X \simeq \mathbb{P}^1$;*
- *vector bundle tame if it is an elliptic curve (i.e. of genus 1);*
- *vector bundle wild otherwise.*

Remark 4.2. As $\mathcal{F}(M)$ is an iterated extension of line bundles of degree 1, it is *semi-stable* in the usual sense [**Ses**]. Thus even the classification of semi-stable vector bundles is wild in this case. The same can be shown in other cases too, though we do not bother to present explicit explanations.

5. Vector bundle types: definitions

The aim of this section is to precise the definitions concerning vector bundle types, especially make them available for non-smooth and even reducible curves. First we fix some notations.

Let X be a projective curve (connected, reduced, but maybe reducible) over an algebraically closed field $\Bbbk$. We denote by $X_1, X_2, \ldots, X_s$ its irreducible components, by $\pi : \tilde{X} \to X$ its normalization, by $\mathrm{Sing}\, X$ the set of singular points, and by $\widetilde{\mathrm{Sing}}\, X = \pi^{-1}(\mathrm{Sing}\, X)$ its preimage on $\tilde{X}$. Note that $\tilde{X} = \bigsqcup_{i=1}^s \tilde{X}_i$, where $\tilde{X}_i$ is the normalization of X_i, so if $s > 1$ it is not connected. We often write $\mathcal{O}$ and $\tilde{\mathcal{O}}$ instead of, respectively, $\mathcal{O}_X$ and $\mathcal{O}_{\tilde{X}}$.

Denote by $\mathrm{VB}(X)$ the category of vector bundles over X. We always identify *vector bundles* over X with their sheaves of sections, thus with *locally free coherent sheaves* of $\mathcal{O}$-modules. If $\mathcal{F}$ is such a sheaf, we set $\tilde{\mathcal{F}} = \mathcal{F} \otimes_{\mathcal{O}} \tilde{\mathcal{O}}$; it is a vector bundle over $\tilde{X}$. As π is finite and birational, the direct image functor π_* is a full embedding on the category of vector bundles, so we usually identify a vector bundle $\mathcal{G}$ over $\tilde{X}$ with $\pi_* \mathcal{G}$, which is a coherent sheaf on X (but not a vector bundle over X). In particular, we usually identify $\tilde{\mathcal{O}}$ with $\pi_* \tilde{\mathcal{O}}$.

As X is connected, every vector bundle $\mathcal{F}$ has a constant rank $\mathrm{rk}\,\mathcal{F} = \dim_{\Bbbk} \mathcal{F}(x)$, where x is an arbitrary closed point of X and $\mathcal{F}(x) = \mathcal{F}_x / \mathfrak{m}_x \mathcal{F}_x$. On the other hand, if $s > 1$ we must consider the *degree* of a vector bundle as a *vector* $\deg \mathcal{F} = (d_1, d_2, \ldots, d_s)$, where d_i denotes the degree of the restriction $\mathcal{F}|X_i$. The degree defines an epimorphism $\deg : \mathrm{Pic}\, X \to \mathbb{Z}^s$. We denote by $\mathrm{Pic}^\circ X$ its kernel. We fix a section $s : \mathbb{Z}^s \to \mathrm{Pic}\, X$ of this epimorphism and denote by $\mathcal{O}(\mathbf{d})$ the line bundle $s(\mathbf{d})$ ($\mathbf{d} \in \mathbb{Z}^s$). Setting $\mathcal{F}(\mathbf{d}) = \mathcal{F} \otimes_{\mathcal{O}} \mathcal{O}(\mathbf{d})$ we define an action of the group $\mathbb{Z}^s$ on vector bundles.

To define vector bundle tame and wild curves we need not individual sheaves, but their families, moreover, those with non-commutative bases.

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[1]The same results has been obtained by W. Scharlau (preprint of the Münster University). Moreover, he has also shown, almost in the same way, that every algebraic variety of dimension greater than 1 is vector bundle wild.

We provide the necessary definitions. Note that symbols like $\otimes$, Hom, etc. always denote $\otimes_\Bbbk$, $\mathrm{Hom}_\Bbbk$, etc.

Definition 5.1. Let X be a projective curve, Λ be a $\Bbbk$-algebra.

1. A *family of vector bundles over X based on Λ* is a flat coherent sheaf of $\mathcal{O} \otimes \Lambda^{\mathrm{op}}$-modules $\mathcal{F}$ on X (it is convenient to suppose that Λ acts on the right). We denote the category of such sheaves by $\mathrm{VB}(X, \Lambda)$.

2. Given such a family and any finite dimensional (over $\Bbbk$) Λ-module M, we set $\mathcal{F}(M) = \mathcal{F} \otimes_\Lambda M$; it is a vector bundle over X; moreover, for each vector $\mathbf{d} \in \mathbb{Z}^s$ we set $\mathcal{F}(\mathbf{d}, M) = \mathcal{F}(\mathbf{d}) \otimes_\Lambda M$.
   If Λ is commutative and $M = \Bbbk(x) = \Lambda/\mathfrak{m}$, where x is the closed point of $S = \mathrm{Spec}\,\Lambda$ corresponding to a maximal ideal $\mathfrak{m} \subset \Lambda$, then $\mathcal{F}$ can be considered as a family of vector bundles with the base S, and $\mathcal{F}(M) = \mathcal{F}(x)$ is the fibre of this family at the point x. If S is connected (i.e. Λ is indecomposable), the rank $\mathrm{rk}\,\mathcal{F}(x)$ and the degree $\deg \mathcal{F}(x)$ are constant on S; we call them the *rank* and the *degree of the family $\mathcal{F}$*. If M is an indecomposable, but not simple Λ-module, $\mathcal{F}(M)$ can be considered as a "generalized" fibre. For instance, if $M = \Lambda/I$ for some ideal I, we consider $\mathcal{F}(M)$ as the value of $\mathcal{F}$ on the closed subscheme of S defined by the ideal I, just as we have done in Section 3. Note that we can consider families over arbitrary schemes, not only affine. The corresponding obvious changes in the definitions are left to the reader (cf. also [**DG3**]).

3. A family $\mathcal{F}$ of vector bundles over an algebra Λ is called *strict* if, for every finite dimensional Λ-modules M, M',
   (a) $\mathcal{F}(M) \simeq \mathcal{F}(M')$ if and only if $M \simeq M'$;
   (b) $\mathcal{F}(M)$ is indecomposable if and only if so is M.

4. We call a curve X
   - *vector bundle finite* if it has finitely many non-isomorphic indecomposable vector bundles *up to twist*, i.e. there is a finite set of vector bundles $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_n$ such that every indecomposable vector bundle over X is isomorphic to $\mathcal{F}_k(\mathbf{d})$ for some $k \in \{1, \ldots, n\}$ and some $\mathbf{d} \in \mathbb{Z}^s$;
   - *vector bundle tame* if there is a set $\mathcal{S}$ of families of vector bundles over X satisfying the following conditions:
     (a) every $\mathcal{F} \in \mathcal{S}$ is a strict family over a smooth connected curve $S_\mathcal{F}$;
     (b) the set $\mathcal{S}(r, \mathbf{d}) = \{\mathcal{F} \in \mathcal{S} \mid \mathrm{rk}\,\mathcal{F} = r,\ \deg \mathcal{F} = \mathbf{d}\}$ is finite for each r and $\mathbf{d}$;
     (c) for each r and $\mathbf{d}$ almost all (i.e. all but a finite number) indecomposable vector bundles over X of rank r and degree $\mathbf{d}$ are isomorphic to $\mathcal{F}(\mathbf{c}, M)$ for some $\mathcal{F} \in \mathcal{S}$, some vector $\mathbf{c} \in \mathbb{Z}^s$, and some sky-scraper sheaf M, or, the same, a finite dimensional $\mathcal{O}_{S_\mathcal{F}, x}$-module for some point $x \in S_\mathcal{F}$
     (such a set $\mathcal{S}$ is called a *parameterising set* for vector bundles over X);
   - *vector bundle wild* if it possesses a strict family over every finitely generated $\Bbbk$-algebra Λ.

5. For a vector bundle tame curve X and a parameterising set $\mathcal{S}$ denote by $\nu(r, \mathbf{d}, \mathcal{S})$ the cardinality of $\mathcal{S}(r, \mathbf{d})$, and by $\nu(r, \mathbf{d}, X)$ the smallest value of $\nu(r, \mathbf{d}, \mathcal{S})$, when $\mathcal{S}$ runs through all parameterising families. The curve X is called
- *tame bounded* if there is a polynomial $N(r, \mathbf{d})$ such that $\nu(r, \mathbf{d}, X) \leq N(r, \mathbf{d})$ for all r and $\mathbf{d}$;
- *tame unbounded* otherwise.

For instance, elliptic curves are vector bundle tame bounded (actually in this case $\nu(r, d, X) = 1$ for all r and d).

Remark 5.2.   1. The use of twists in the definitions of finite and tame is indeed indispensable. On the other hand, the parameterising families that we shall construct later actually cover *all* vector bundles (up to twist), so the words 'almost all' could be replaced by 'all' in this context. Nevertheless, we have included them in order that our definition fits the usual pattern.

2. One can be interested not only in vector bundles, but also in *torsion free* sheaves, or even in all coherent sheaves. Certainly, all definitions of 'finite,' 'tame' or 'wild' can be almost literally reproduced for these cases too. Moreover, the same calculations as for vector bundles show that nothing will change if we consider torsion free sheaves (though this time 'almost all' is indispensable). We shall comment their structure at the corresponding places. The things become more complicated if we are interested in *all* coherent sheaves over *singular* curves, because they do not split into direct sums of torsion and torsion free ones. Nevertheless, I. Burban and the author have shown that even the *derived category* of coherent sheaves remains tame for all vector bundle tame curves (cf. the talk of I. Burban presented at this workshop).

3. It is well-known that to prove that a curve X is vector bundle wild it is enough to construct a strict family from $\mathrm{VB}(X, \Lambda)$, where Λ is either the free algebra $\Bbbk\langle z_1, z_2 \rangle$ in two generators, or the polynomial algebra $\Bbbk[z_1, z_2]$, or the power series algebra $\Bbbk[[z_1, z_2]]$. Moreover, we can even replace them by a finite dimensional algebra, for instance, $\Bbbk[z_1, z_2]/(z_1^2, z_2^3, z_1 z_2^2)$ or the path algebra of a wild quiver without cycles (the latter is even *hereditary* that is sometimes convenient). In what follows we constantly use this observation.

6. Vector bundles and matrix problems

We have already established the vector bundle types of smooth curves. Suppose now that X is a singular curve. We use a procedure similar to that of Section 2. Namely, let $\mathcal{J}$ be the *conductor* of $\tilde{\mathcal{O}}$ in $\mathcal{O}$, i.e. $\mathrm{Ann}_{\mathcal{O}}(\tilde{\mathcal{O}}/\mathcal{O})$. Then $\mathcal{J}\tilde{\mathcal{F}} = \mathcal{J}\mathcal{F} \subset \mathcal{F} \subset \tilde{\mathcal{F}}$ for every vector bundle $\mathcal{F}$. Denote by $\mathbf{F} = \mathcal{O}/\mathcal{J}$ and by $\tilde{\mathbf{F}} = \tilde{\mathcal{O}}/\mathcal{J}$. Both $\mathbf{F}$ and $\tilde{\mathbf{F}}$ have 0-dimensional support (respectively $\mathrm{Sing}\, X$ and $\widetilde{\mathrm{Sing}\, X}$). Thus we may (and shall) identify them with the finite dimensional algebras of their sections $\Gamma(X, \mathbf{F})$ and $\Gamma(\tilde{X}, \tilde{\mathbf{F}})$. Evidently $\mathbf{F}$ is a subalgebra in $\tilde{\mathbf{F}}$.

We define the category $\mathrm{Fsm}(X, \Lambda)$, where Λ is a $\Bbbk$-algebra, as follows:

- Its *objects* are the pairs $(\mathcal{A}, M)$, where $\mathcal{A}$ is an object from $\mathrm{VB}(\tilde{X}, \Lambda)$, i.e. a coherent flat sheaf of $\tilde{\mathcal{O}} \otimes \Lambda$-modules, and M is a projective $\mathbf{F} \otimes \Lambda$-submodule in $\mathcal{A}/\mathcal{J}\mathcal{A}$ such that the natural homomorphism $\tilde{\mathbf{F}} \otimes_{\mathbf{F}} M \to \mathcal{A}/\mathcal{J}\mathcal{A}$ is an isomorphism.
- A *morphism* from $(\mathcal{A}, M)$ to $(\mathcal{A}', M')$ is a homomorphism $\phi : \mathcal{A} \to \mathcal{A}'$ such that $\bar{\phi}(M) \subseteq M'$, where $\bar{\phi}$ is the induced homomorphism $\mathcal{A}/\mathcal{J}\mathcal{A} \to \mathcal{A}'/\mathcal{J}\mathcal{A}'$.

We write $\mathrm{Fsm}(X)$ instead of $\mathrm{Fsm}(X, \Bbbk)$. If L is a finite dimensional Λ-module and $\mathcal{P} = (\mathcal{A}, M) \in \mathrm{Fsm}(X, \Lambda)$, set $\mathcal{P}(L) = (\mathcal{A} \otimes_\Lambda L, M \otimes_\Lambda L) \in \mathrm{Fsm}(X)$. Note also that the group $\mathbb{Z}^s$ acts on $\mathrm{Fsm}(X)$. Indeed, the factors $\mathcal{A}/\mathcal{J}\mathcal{A}$ and $\mathcal{A}(\mathbf{d})/\mathcal{J}\mathcal{A}(\mathbf{d})$ are naturally isomorphic, so we can just set $\mathcal{P}(\mathbf{d}) = (\mathcal{A}(\mathbf{d}), M)$. It allows to transfer the definitions of finite, tame and wild types to the categories $\mathrm{Fsm}(X)$ (we leave it to the reader).

Consider the functor $\mathbf{P} : \mathrm{VB}(X, \Lambda) \to \mathrm{Fsm}(X, \Lambda)$ that maps a sheaf $\mathcal{F}$ to the pair $(\tilde{\mathcal{F}}, \mathcal{F}/\mathcal{J}\mathcal{F})$. Moreover, for each pair $\mathcal{P} = (\mathcal{A}, M) \in \mathrm{Fsm}(X, \Lambda)$ denote by $\mathbf{V}(\mathcal{P})$ the preimage of M in $\mathcal{A}$; it is a sheaf of $\mathcal{O} \otimes \Lambda$-modules. Obviously, any morphism $\phi : \mathcal{P} \to \mathcal{P}'$ induces a homomorphism $\mathbf{V}(\mathcal{P}) \to \mathbf{V}(\mathcal{P}')$ and one easily verifies the main property of these constructions:

Proposition 6.1. *For every pair $\mathcal{P} \in \mathrm{Fsm}(X, \Lambda)$ the sheaf $\mathbf{V}(\mathcal{P})$ belongs to $\mathrm{VB}(X, \Lambda)$, and the functors $\mathbf{P}, \mathbf{V}$ define an equivalence of the categories $\mathrm{VB}(X, \Lambda)$ and $\mathrm{Fsm}(X, \Lambda)$. Moreover, $\mathbf{P}(\mathcal{P}(L)) \simeq \mathbf{P}(\mathcal{P})(L)$ and $\mathbf{V}(\mathcal{F}(L)) \simeq \mathbf{V}(\mathcal{F})(L)$ for every finite dimensional Λ-module L.*

Corollary 6.2. *A curve X is vector bundle finite (tame, wild) if and only if so is the category $\mathrm{Fsm}(X)$.*

It is easier to deal with the category $\mathrm{Fsm}(X)$, because it can be identified with a bimodule category, so it carries us to the better explored world of "matrix problems." (We refer to [**GR**] and [**DG3**] for the corresponding definitions and note that in [**DG3**, Section 3] the *shifted bimodules* have been introduced, which are necessary for the application to vector bundles.) Namely, let $\mathsf{A} = \mathrm{VB}(\tilde{X})$, $\mathsf{B} = \mathbf{F}$-pro, the category of finitely generated projective right $\mathbf{F}$-modules. Consider the A-B-bimodule U such that $\mathsf{U}(P, \mathcal{A}) = \mathrm{Hom}_{\mathbf{F}}(P, \mathcal{A}/\mathcal{J}\mathcal{A})$, where $\mathcal{A} \in \mathsf{A}$, $P \in \mathsf{B}$. In the category $\mathrm{El}(\mathsf{U})$ of elements of the bimodule U (or matrices over U) consider the full subcategory $\mathrm{El}_c(\mathsf{U})$ consisting of all homomorphisms $\alpha : P \to \mathcal{A}$ such that the induced homomorphism $\tilde{\mathbf{F}} \otimes_{\mathbf{F}} P \to \mathcal{A}$ is an isomorphism. We call elements from El_c *correct*.

Proposition 6.3. *The categories $\mathrm{Fsm}(X)$ and $\mathrm{El}_c(\mathsf{U})$ are equivalent.*

Moreover, one can obviously define twists by $\mathbf{d} \in \mathbb{Z}^s$ on $\mathrm{El}(\mathsf{U})$ so that this equivalence is compatible with twists.

Corollary 6.4. *A curve X is vector bundle finite (tame, wild) if and only if so is the category $\mathrm{El}_c(\mathsf{U})$.*

7. Vector bundle types: results

Now we use the reduction to matrix problems to find the vector bundle types of singular curves. First we establish several *wild* cases. We keep

the notations and definitions of the preceding section and suppose that X is indeed *singular*. Since all proofs consist in more or less standard matrix calculations, we only write down the matrices that describe strict families, leaving the verification of strictness (always straightforward, though sometimes rather cumbersome) to the reader, who can also look into [**DG3**].

Step 1. *If one of the components $\tilde{X}_i$ is not rational, the curve X is vector bundle wild.*

Proof. Suppose that $\tilde{X}_1$ is of genus $g \geq 1$. As X is connected, there is a singular point p that belongs to X_1. We suppose that p has at least 2 preimages on $\tilde{X}$. If it only has one, the algebra $\tilde{\mathbf{F}}_p$ is not semi-simple, which simplifies the calculations. Let $\{ p_1, p_2, \ldots, p_t \}$ be all preimages of p, with $p_1 \in \tilde{X}_1$, and let Y be the component that contains p_2 (we allow $Y = \tilde{X}_1$). Let $\{ p_{t+1}, \ldots, p_l \}$ be all other points from $\widetilde{\mathrm{Sing}}\, X$. Choose 4 different regular points x_i $(i = 1, \ldots, 4)$ on $\tilde{X}_1$ and another regular point y on Y, and consider the family $\mathbf{u}$ of elements of $\mathrm{El}_c(\mathsf{U})$ over the free algebra $\Lambda = \Bbbk\langle z_1, z_2 \rangle$ that belongs to $\mathsf{U}(\mathcal{A} \otimes \Lambda, 4\mathbf{F} \otimes \Lambda)$, where $\mathcal{A} = \bigoplus_{k=1}^4 \tilde{\mathcal{O}}(x_k + ky)$, all components of $\mathbf{u}$ in $\mathrm{Hom}_{\mathbf{F}}(4\mathbf{F}, \mathcal{A}_{p_i}/\mathcal{J}\mathcal{A}_{p_i}) \otimes \Lambda$ $(1 < i \leq l)$ are unit matrices, and its component in $\mathrm{Hom}_{\mathbf{F}}(4\mathbf{F}, \mathcal{A}_{p_1}/\mathcal{J}\mathcal{A}_{p_1}) \otimes \Lambda$ equals

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & z_1 \\ 1 & 0 & 1 & z_2 \end{pmatrix}.$$

Since $\mathrm{Hom}_{\mathcal{O}_{X_1}}(\mathcal{O}_{X_1}(x_k), \mathcal{O}_{X_1}(x_j)) = 0$ if $k \neq j$ and $\mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{O}_Y(ky), \mathcal{O}_Y(jy)) = 0$ if $k > j$, one can check that $\mathbf{u}$ is actually a strict family of correct elements. Hence El_c is wild and thus X is vector bundle wild. $\qquad\square$

From now on we suppose that $\tilde{X}_i \simeq \mathbb{P}^1$ for every $i = 1, \ldots, s$.

Step 2. *If the algebra $\tilde{\mathbf{F}}$ is not semi-simple, the curve X is wild.*

(Note that $\tilde{\mathbf{F}}$ is semi-simple if and only if all singular points of X are *ordinary multiple points*, i.e. such that at each of them the number of linearly independent tangent directions to X equals the multiplicity of this point.)

Proof. Choose a point $p \in \widetilde{\mathrm{Sing}}\, X$ such that $\tilde{\mathbf{F}}_p$ is not semi-simple, and a non-zero element $\alpha \in \tilde{\mathbf{F}}_p$ with $\mathrm{Ann}_{\tilde{\mathbf{F}}_p} \alpha = \mathrm{rad}\,\tilde{\mathbf{F}}_p$. Let Λ be the path algebra of the graph

$$5 \xrightarrow{\ z_5\ } 4 \xrightarrow{\ z_4\ } 3$$
$$\Big\downarrow{\scriptstyle z_3} \qquad\qquad \Big\downarrow{\scriptstyle z_2}$$
$$2 \xrightarrow[\ z_1\ ]{} 1$$

It is known to be wild [**DF**, **Na1**], so we only have to construct a strict family of correct elements over Λ. Denote by P_j the indecomposable projective Λ-module corresponding to the vertex j of the graph. We identify an arrow $z : j \to k$ with the corresponding homomorphism $P_j \to P_k$ (left multiplication

by z). Let $p \in \tilde{X}_1$. Choose a regular point x on $\tilde{X}_1$. Set

$$\mathcal{A} = \tilde{\mathcal{O}} \otimes (P_3 \oplus P_1) \oplus \tilde{\mathcal{O}}(x) \otimes (P_1 \oplus P_2) \oplus \tilde{\mathcal{O}}(2x) \otimes (P_3 \oplus P_4) \oplus \tilde{\mathcal{O}}(3x) \otimes P_5,$$
$$B = \mathbf{F} \otimes (P_3 \oplus P_1 \oplus P_1 \oplus P_2 \oplus P_3 \oplus P_4 \oplus P_5)$$

and consider the family $\mathbf{u}$ of elements from El_c over Λ such that all components of $\mathbf{u}$ except that in $\mathrm{Hom}_{\mathbf{F}}(B, \mathcal{A}_p/\mathcal{J}\mathcal{A}_p)$ equal unit matrices, while the last is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \alpha \otimes 1 & 0 & 0 \\ 0 & 1 & \alpha \otimes 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \alpha \otimes z_1 & \alpha \otimes z_2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \alpha \otimes z_3 & 0 \\ 0 & 0 & 0 & 0 & 1 & \alpha \otimes z_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \alpha \otimes z_5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Again a straightforward calculation shows that this family is strict, hence X is vector bundle wild. □

Step 3. *If the curve X has a singular point of multiplicity $m \geq 3$, it is vector bundle wild.*

Proof. Let p be a point of multiplicity $l \geq 3$, which we suppose an ordinary multiple point, $p_1, p_2, \ldots, p_l$ be its preimages on $\tilde{X}$. Denote by Y_i the component of $\tilde{X}$ containing p_i (some of them may coincide). Choose regular points $y_i \in Y_i$ and set $\mathcal{A} = \bigoplus_{k=1}^4 \tilde{\mathcal{O}}(ky_1 + ky_2 + ky_3)$. Consider the family of elements from $\mathrm{El}_c(\mathsf{U})$ over $\Lambda = \Bbbk\langle z_1, z_2 \rangle$ given by the element $\mathbf{u} \in \mathrm{Hom}_{\mathbf{F}}(4\mathbf{F}, \mathcal{A}/\mathcal{J}\mathcal{A})$ that has unit matrices as all its components except those at the points p_1 and p_2, the last two being respectively

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 & z_1 & z_2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

One can check that $\mathbf{u}$ is a strict family, so X is vector bundle wild. □

Thus from now on we only consider the case when all singular points of X are *ordinary double points* (or *nodes*). It means that X is a *projective configuration* in the sense of Section 2, so its *dual graph* $\Delta(X)$ is defined.

Step 4. *If a vertex of $\Delta(X)$ is incident to three edges or to a loop and another edge, the curve X is vector bundle wild.*

Proof. We consider the case when the graph $\Delta(X)$ is
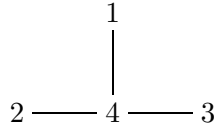
(In other cases the calculations are even easier.) It means that the component X_4 intersects transversally the components X_i ($i = 1, 2, 3$) at the points p_i ($i = 1, 2, 3$). We denote by p_{ij} the preimage of p_i on the component X_j ($j = i$ or $j = 4$). If $\mathbf{u} \in \mathrm{Hom}_{\mathbf{F}}(P, \mathcal{A}/\mathcal{J}\mathcal{A})$ is an element of U,

we denote by $\mathbf{u}_{ij}$ the component of $\mathbf{u}$ in $\mathrm{Hom}_{\mathbf{F}}(P, \mathcal{A}_{p_{ij}}/\mathcal{J}\mathcal{A}_{p_{ij}})$. We can choose homogeneous coordinates on $X_4 \simeq \mathbb{P}^1$ so that $p_1 = (1 : 0)$, $p_2 = (0 : 1)$, $p_3 = (1 : 1)$. Fix regular points $x_i \in X_i$ and consider a family $\mathbf{u} \in \mathrm{Hom}_{\mathbf{F}}(14\mathbf{F}, \mathcal{A}) \otimes \Bbbk\langle z_1, z_2 \rangle$ of elements of U such that

$$\mathcal{A} = \bigoplus_{j=1}^{14} \tilde{\mathcal{O}}(k_j(x_1 + x_2 + x_4) + l_j x_3)$$

where

$$k_j = \begin{cases} 1 & \text{if } 1 \le j \le 3, \\ 2 & \text{if } 4 \le j \le 7, \\ 3 & \text{if } 8 \le j \le 11, \\ 4 & \text{if } 12 \le j \le 14, \end{cases}$$

$l_j = [(j+1)/2]$, $\mathbf{u}_{ij}$ is a unit matrix if $j \ne 4$ or $i = 1$,

$$\mathbf{u}_{24} = \begin{pmatrix} 0 & \dots & 0 & 1 \\ 0 & \dots & 1 & 0 \\ \hdotsfor{4} \\ 1 & \dots & 0 & 0 \end{pmatrix},$$

while $\mathbf{u}_{34} = (t_{pq})$ $(p, q = 1, \dots, 14)$, where $t_{13,11} = z_1$, $t_{14,11} = z_2$, $t_{pq} = 1$ if $p = q$ or (p, q) is from the set

$$\{ (5, 3), (6, 1), (7, 2), (9, 4), (10, 6), (10, 9), (11, 5), (11, 7), (11, 10),$$
$$(12, 8), (13, 9), (13, 12), (14, 10), (14, 12) \},$$

and $t_{pq} = 0$ otherwise. Again a straightforward though cumbersome calculation shows that it is a strict family, thus X is vector bundle wild. $\square$

Remark. Actually under the given shape of $\mathbf{u}_{ij}$ for $(ij) \ne (34)$ the matrix $\mathbf{u}_{34}$ splits into blocks $\mathbf{v}_{kl}$ $(1 \le k \le 4, 1 \le l \le 7)$ (corresponding to the values $k_j = k$, $l_j = l$), all of them with 2 columns, the number of rows is 3 for $k = 1, 4$ and 4 for $k = 2, 3$. With respect to the transformations that do not change other matrices these blocks form a representation of the pair of posets (N, L), where L is a chain with 7 elements and N is
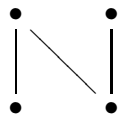
The matrix $\mathbf{u}_{34}$ described above just presents a strict family of representations of this pair over $\Bbbk\langle z_1, z_2 \rangle$. Certainly, this matrix problem is known to be wild [**Na2**], but we had to ensure the matrix $\mathbf{u}_{34}$ to be invertible. That is why we had to take L with 7 elements, though for the wildness of the pair of posets 6 elements would suffice.

Step 4 shows that projective configurations that are not wild can only have the following dual graphs:

$$\text{chain } \mathrm{A}_s \qquad\qquad 1 \ \rule[0.5ex]{2em}{0.4pt}\ 2 \ \rule[0.5ex]{2em}{0.4pt}\ 3 \ \rule[0.5ex]{1.5em}{0.4pt}\ \cdots \ \rule[0.5ex]{1.5em}{0.4pt}\ s$$

or

cycle $\tilde{A}_s$
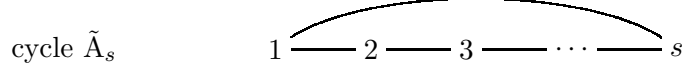
Here s denotes the number of vertices. If $s = 1$, there are no edges in the configuration of type A_1; thus the corresponding curve is just a projective line. The configuration $\tilde{A}_1$ corresponds to a rational curve with one ordinary double point (a nodal plane cubic, an affine part of which can be given by the equation $y^2 = x^3 + x^2$).

We already know that the projective configurations of type A_s are vector bundle finite. In the next section we show that all projective configurations of type $\tilde{A}_s$ are vector bundle tame unbounded, thus accomplishing the proof of the following theorem announced in the Introduction.

Theorem 7.1. *A projective curve is*

- *vector bundle finite if and only if it is a projective configuration of type A_s,*
- *vector bundle tame bounded if and only if it is a elliptic curve,*
- *vector bundle tame unbounded if and only if it is a projective configuration of type $\tilde{A}_s$,*
- *vector bundle wild otherwise.*

Moreover, we present there a description of indecomposable vector bundles over the projective configurations of types $\tilde{A}_s$.

8. Vector bundles over projective configurations of type $\widetilde{A}$

Now we consider the projective configurations of type $\tilde{A}_s$. We follow the way of Section 2 with evident changes. There are s irreducible components $\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_s$ of $\tilde{X}$ and s singular points $p_1, p_2, \ldots, p_s$ on X, each of them having two preimages p_i' and p_i'' on $\tilde{X}$. We can arrange the numeration and coordinates on $\tilde{X}_i \simeq \mathbb{P}^1$ so that $p_i' = (1 : 0) \in \tilde{X}_i$, $p_i'' = (0 : 1) \in \tilde{X}_{i+1}$ (we use the cyclic numeration modulo s, so $X_{s+1} = X_1$, etc.). Then $\mathbf{F} = \prod_{i=1}^s \Bbbk_i$ and $\tilde{\mathbf{F}} = \prod_{i=1}^s (\Bbbk_i' \times \Bbbk_i'')$, where $\Bbbk_i = \Bbbk(p_i)$, $\Bbbk_i' = \Bbbk(p_i')$, $\Bbbk_i'' = \Bbbk(p_i'')$. All these fields coincide with $\Bbbk$ and the embedding $\mathbf{F} \to \tilde{\mathbf{F}}$ maps each $\Bbbk_i$ into $\Bbbk_i' \times \Bbbk_i''$ diagonally.

Let $\mathbf{u} \in \mathrm{El}_c(P, \mathcal{A})$. Then

$$P = r\mathbf{F}, \quad \mathcal{A} = \bigoplus_{i=1}^s \bigoplus_{k=1}^r \mathcal{O}_i(d_{ik}),$$

where $\mathcal{O}_i = \mathcal{O}_{\tilde{X}_i}$ and $d_{ik} \in \mathbb{Z}$ are the degrees of direct summands; r is the rank of $\mathcal{A}$ (it must be constant). Note that $\mathcal{O}_i(d)/\mathcal{J}\mathcal{O}_i(d) \simeq \Bbbk_i' \oplus \Bbbk_{i-1}''$ (again $\Bbbk_0 = \Bbbk_s$). Choosing bases in each summand $\Bbbk_i'$ and $\Bbbk_i''$ as well as in each component $r\Bbbk_i$ of M, we present $\mathbf{u}$ as a set of $r \times r$ invertible matrices $\{ M_i', M_i'' \,|\, i = 1, \ldots, s \}$. Moreover, the rows of these matrices are endowed with *weights* d_{ik} (it is the common weight of the k-th row of M_i' and of M_{i-1}''). Taking into account the description of homomorphisms of vector bundles over $\mathbb{P}^1$ from Section 2, we see that the automorphisms of $\mathcal{A}$ and P give rise to the following transformations of these matrices:

1) $M_i' \mapsto M_i'S$ and $M_i'' \mapsto M_i''S$ for some i and some invertible matrix S;
2) $M_i' \mapsto T'M_i'$ and $M_{i-1}'' \mapsto T''M_{i-1}''$ $(1 \le i \le s)$, where $T' = (t_{jk}')$ and $T'' = (t_{jk}'')$ are invertible matrices such that
   (a) $t_{jk}' = t_{jk}''$ if $d_{ij} = d_{ik}$;
   (b) $t_{jk}' = t_{jk}'' = 0$ if $d_{ij} < d_{ik}$.

Two sets of matrices correspond to isomorphic vector bundles if and only if one of them can be converted to the other by a sequence of such transformations.

Note that this problem is much more complicated than that of Section 2. Even if $s = 1$ and all rows have the same weight, it becomes the *Kronecker problem*, or that of *pencils of matrices*, which is of infinite type. Fortunately, one can recognize the arising matrix problem as belonging to the so-called "representations of *bunches of chains*," or "Gelfand problems," or "clans" (cf. [**CB**, **Bon**] or [**DG3**, Appendix B]; we refer to the last paper, because the presentation there is more convenient to our purpose). Namely, in our case we have the pairs of chains

$$\mathfrak{C}_i' = \left\{ E_{id}' \,|\, i = 1, \dots, s; \; d \in \mathbb{Z} \right\}, \quad \mathfrak{F}_i' = \left\{ f_i' \right\},$$
$$\mathfrak{C}_i'' = \left\{ E_{id}'' \,|\, i = 1, \dots, s; \; d \in \mathbb{Z} \right\}, \quad \mathfrak{F}_i'' = \left\{ f_i'' \right\}$$

with the natural ordering in each $\mathfrak{C}_i'$, $\mathfrak{C}_i''$ (according to the index d), while the equivalence relation $\sim$ is given by the rules:

$$E_{id}' \sim E_{i-1,d}'', \quad f_i' \sim f_i''.$$

Slightly rearranging the list of indecomposable objects, we obtain the following result.

Theorem 8.1. *Indecomposable vector bundles over a projective configuration of type $\tilde{A}_s$ are described by "band data," which are triples $(\mathbf{d}, m, \lambda)$, where $m \in \mathbb{N}$, $\lambda \in \Bbbk \setminus \{0\}$ and $\mathbf{d}$ is a sequence from $\mathbb{Z}^{rs}$ for some r that is* non-s-periodic, *i.e. cannot be presented as a repetition of a shorter sequence $\mathbf{c} \in \mathbb{Z}^{ls}$ $(l < r)$. Two such triples $(\mathbf{d}, m, \lambda)$ and $(\mathbf{d}', m', \lambda')$ correspond to isomorphic vector bundles if and only if $m = m'$, $\lambda = \lambda'$ and $\mathbf{d}'$ can be obtained from $\mathbf{d}$ by an s-shift, i.e. $\mathbf{d} = (d_1, d_2, \dots, d_{rs})$, while $\mathbf{d}' = (d_{ls+1}, d_{ls+2} \dots, d_{rs}, d_1, \dots, d_{ls})$ for some $l \le r$.*

Note that neither "string data" gives a set of *invertible* matrices, though most of them can be interpreted as corresponding to *torsion free*, but not locally free sheaves (cf. [**DG3**] or Remark 8.3 below).

Moreover, from the explicit description of indecomposable representations of a bunch of chains one can deduce an explicit description of vector bundles over such a configuration. Namely, the vector bundle $\mathcal{V} = \mathcal{V}(\mathbf{d}, m, \lambda)$, where $\mathbf{d} = (d_1, d_2, \dots, d_{rs})$, is a subsheaf of $\tilde{\mathcal{V}} = \bigoplus_{j=1}^{rs} m\mathcal{O}_j(d_j)$ defined as follows:

- If $x \notin \mathrm{Sing}\, X$, $\mathcal{V}_x = \tilde{\mathcal{V}}_x$.
- Choose bases $\mathbf{e}_{jk}'$ and $\mathbf{e}_{j,k}''$ $(j = 1, \dots, rs, \; k = 1, \dots, m)$ in each vector space $m(\mathcal{O}_j(d_j)/\mathcal{J}\mathcal{O}_j(d_j))_{p_j'}$ and $m(\mathcal{O}_{j+1}(d_{j+1})/\mathcal{J}\mathcal{O}_{j+1}(d_{j+1}))_{p_j''}$

respectively, and set

$$\mathbf{e}_{jk} = \begin{cases} \mathbf{e}'_{jk} + \mathbf{e}''_{jk} & \text{if } 1 \le j < rs, \\ \mathbf{e}'_{rs,1} + \lambda\mathbf{e}''_{rs,1} & \text{if } j = rs, \, k = 1, \\ \mathbf{e}'_{rs,k} + \lambda\mathbf{e}''_{rs,k} + \mathbf{e}''_{rs,k-1} & \text{if } j = rs, \, k > 1. \end{cases}$$

- For each singular point p_i the stalk $\mathcal{V}_{p_i}$ is generated by the preimages of $\mathbf{e}_{jk}$ with $j \equiv i \pmod{s}$.

Especially $\operatorname{rk}\mathcal{V}(\mathbf{d}, m, \lambda) = mr$ and $\deg\mathcal{V}(\mathbf{d}, m, \lambda) = (\delta_1, \delta_2, \ldots, \delta_s)$, where $\delta_i = \sum_{j \equiv i \pmod{s}} d_j$. For instance, if $s = 1$ (the case of nodal cubic) and $m = 1$, one can present this vector bundle as the following gluing of line bundles over $\tilde{X} = \mathbb{P}^1$:
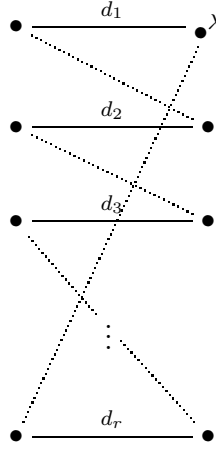
Here horizontal lines symbolize line bundles over $\tilde{X}$ of the superscripted degrees, their left (right) ends are basic elements of these bundles at the point $0 = (1 : 0)$ (respectively $\infty = (0 : 1)$), and the dotted lines show which of them must be glued. All gluings are trivial, except that going from the uppermost right point to the lowermost left one, where we glue one vector to another multiplied by λ. If $m > 1$, one has to take m copies of each vector bundle from this picture, make again trivial all gluings except the last one, where identifications must be made using the Jordan $m \times m$ cell with eigenvalue λ. The necessary changes for $s > 1$ are quite obvious.

Corollary 8.2. *Projective configurations of type $\tilde{A}_s$ are vector bundle tame unbounded.*

Proof. Let $\Lambda = \Bbbk[x, x^{-1}]$. For each vector $\mathbf{d} \in \mathbb{Z}^{rs}$ we define a family $\mathcal{V}_\mathbf{d}$ of vector bundles of rank r with base $\operatorname{Spec}\Lambda = \mathbb{A}^1 \setminus \{0\}$ just as we have defined the vector bundles $\mathcal{V}(\mathbf{d}, 1, \lambda)$, but replacing $\mathcal{O}_j$ by $\mathcal{O}_j \otimes \Bbbk[x, x^{-1}]$ and λ by x. Obviously, then $\mathcal{V}(\mathbf{d}, m, \lambda) \simeq \mathcal{V}_\mathbf{d}(L)$, where $L = \Lambda/(x - \lambda)^m$, hence X is vector bundle tame. As we allow twists, it is enough to consider sequences $\mathbf{d}$ of non-negative integers (even those containing 0). Evidently, for every fixed degree $\delta = (\delta_1, \delta_2, \ldots, \delta_s)$ and every fixed rank r there are finitely many such $\mathbf{d}$ with $\sum_{j \equiv i \pmod{s}} d_j = \delta_i$. On the other hand, the number of such sequences $\mathbf{d}$ grows exponentially when $r \to \infty$. Thus X is indeed tame unbounded. $\qquad\square$

Remark 8.3. Just in the same way we can describe *torsion free* sheaves over any configuration of type $\tilde{A}_s$. The difference is that it is allowed not to glue all basic vectors at 0 and ∞, leaving the first and the last ones "free" (for instance, in the picture above we can exclude one of the gluings). Note that in this case we can make all gluings trivial. It means that there are no families of torsion free sheaves that are not vector bundles; all of them stand apart. See [**DG3**] for details.

9. Cohen–Macaulay modules: generalities

From now on we consider Cohen–Macaulay modules over *normal surface singularities*. For our purpose, it means a complete noetherian algebra $\mathbf{A}$ over an algebraically closed field $\Bbbk$ such that

- $\mathbf{A}$ is *normal*, i.e. integral and integrally closed;
- $\mathrm{Kr.dim}\,\mathbf{A} = 2$;
- $\mathbf{A}/\mathfrak{m} \simeq \Bbbk$, where $\mathfrak{m}$ denotes the maximal ideal of $\mathbf{A}$.

Note that for 2-dimensional integral rings '*Cohen–Macaulay*' means that $\mathbf{A} = \bigcap_{\mathrm{ht}\,\mathfrak{p}=1} \mathbf{A}_{\mathfrak{p}}$, while '*normal*' means that $\mathbf{A}_{\mathfrak{p}}$ is a discrete valuation ring for each prime ideal $\mathfrak{p}$ of height 1. Moreover, a finitely generated module M over such a ring is Cohen–Macaulay if and only if it is torsion free and $M = \bigcap_{\mathrm{ht}\,\mathfrak{p}=1} M_{\mathfrak{p}}$. We denote $M^{\vee} = \mathrm{Hom}_{\mathbf{A}}(M, \mathbf{A})$; it is always a Cohen–Macaulay $\mathbf{A}$-module. For Cohen–Macaulay $\mathbf{A}$-modules M, N we denote by $M \boxtimes_{\mathbf{A}} N$ their "reflexive product" $(M \otimes_{\mathbf{A}} N)^{\vee\vee}$.

A *family of Cohen–Macaulay $\mathbf{A}$-modules* based on an algebra Λ is defined as a finitely generated $\mathbf{A}$-Λ-bimodule $\mathcal{M}$ such that

- $\mathcal{M}$ is flat as Λ-module;
- for every finite dimensional Λ-module L the $\mathbf{A}$-module $\mathcal{M}(L) = \mathcal{M} \otimes_{\Lambda} L$ is Cohen–Macaulay.

Obviously, the latter condition is only to be checked for simple modules L. Having this notion, one can define *strict families, Cohen–Macaulay tame and wild singularities* just as it has been done in Section 5 for vector bundles, so we omit the details of these definitions. We also leave to the reader the obvious changes in these definitions, when families based on arbitrary $\Bbbk$-schemes are considered. We call modules $\mathcal{M}(L)$ (generalized) fibres of $\mathcal{M}$.

We denote by $S = \mathrm{Spec}\,\mathbf{A}$ and by p the unique closed point of S (corresponding to the ideal $\mathfrak{m}$). Recall that there always is a *resolution* of such a singularity, i.e. a projective birational morphism of schemes $\pi : X \to S$, where X is regular, such that the restriction of π onto $\check{X} = X \setminus \pi^{-1}(p)$ is an isomorphism $\check{X} \to \check{S} = S \setminus \{\,p\,\}$, cf. e.g. [**Lip**]. The reduced preimage $E = \pi^{-1}(p)_{\mathrm{red}}$ is called the *exceptional curve* of this resolution. It is indeed a projective curve, possibly singular and even reducible. We denote by $E_1, E_2, \ldots, E_s$ its irreducible components. We always identify $\check{X}$ and $\check{S}$ so that the diagram

$$
\begin{array}{ccc}
\check{X} & \xrightarrow{\ i\ } & X \\
\Big\| & & \Big\downarrow \pi \\
\check{S} & \xrightarrow[\ j\ ]{} & S
\end{array}
$$

where i, j are embeddings, commutes.

To obtain a criterion of Cohen–Macaulay finiteness, as well as for some other results, the following considerations are important. Suppose that a finite group G acts on a normal surface singularity $\mathbf{B}$ and $\mathbf{A} = \mathbf{B}^G$ is the ring of invariants. It is again a normal surface singularity. A Cohen–Macaulay $\mathbf{B}$-module N is called *induced from* $\mathbf{A}$ if it is isomorphic to $\mathbf{B} \boxtimes_{\mathbf{A}} M$ for some Cohen–Macaulay $\mathbf{A}$-module M. We say that $\mathbf{B}$ is *unramified in codimension* 1 if, for every prime ideal $\mathfrak{p} \subset \mathbf{A}$ of height 1, $\mathbf{B}_{\mathfrak{p}}/\mathfrak{p}\mathbf{B}_{\mathfrak{p}}$ is a separable algebra over the field $\mathbf{A}_{\mathfrak{p}}/\mathfrak{p}\mathbf{A}_{\mathfrak{p}}$. Equivalently, the natural epimorphism of $\mathbf{B}$-bimodules $\mathbf{B}_{\mathfrak{p}} \otimes_{\mathbf{A}_{\mathfrak{p}}} \mathbf{B}_{\mathfrak{p}} \to \mathbf{B}_{\mathfrak{p}}$ splits. For instance it is so if G acts freely on the set of prime ideals of $\mathbf{B}$ of height 1.

Proposition 9.1 (cf. [**Her**]). *If the order $g = |G|$ is invertible in $\Bbbk$, every Cohen–Macaulay $\mathbf{A}$-module is a direct summand of a Cohen–Macaulay $\mathbf{B}$-module considered as $\mathbf{A}$-module. If, moreover, $\mathbf{B}$ is unramified in codimension 1, every Cohen–Macaulay $\mathbf{B}$-module is a direct summand of a module induced from $\mathbf{A}$.*

Proof. Let M be any Cohen–Macaulay $\mathbf{A}$-module. We identify it with the $\mathbf{A}$-submodule $1 \otimes M$ of $\mathbf{B} \otimes_{\mathbf{A}} M$. There is a retraction of $\mathbf{A}$-modules $\mathbf{B} \otimes_{\mathbf{A}} M \to M$ mapping $b \otimes v$ to $g^{-1} \sum_{\sigma \in G} \sigma(b) \otimes v$. Thus $\mathbf{B} \otimes_{\mathbf{A}} M \simeq M \oplus M'$ for some M'. Taking second duals, we get $\mathbf{B} \boxtimes_{\mathbf{A}} M \simeq M \oplus (M')^{\vee\vee}$.

Suppose now that $\mathbf{B}$ is unramified in codimension 1. Denote by $\mathbf{K}$ and $\mathbf{L}$ respectively the fields of fractions of $\mathbf{A}$ and $\mathbf{B}$. Then $\mathbf{L}$ is a Galois extension of $\mathbf{K}$ with Galois group G. Hence $\mathbf{L}^e = \mathbf{L} \otimes_{\mathbf{K}} \mathbf{L} = \bigoplus_{\sigma \in G} \mathbf{L}^{\sigma}$ as $\mathbf{L}$-bimodule, where $\mathbf{L}^{\sigma} = \{\lambda \in \mathbf{L} \mid \forall \alpha \in \mathbf{L}\ \alpha\lambda = \lambda\sigma(\alpha)\}$ [**DK**]. Especially there is a unique element $\varepsilon \in \mathbf{L}^e$ such that $\lambda\varepsilon = \varepsilon\lambda$ and $\phi(\varepsilon) = 1$, where ϕ denotes the natural epimorphism $\mathbf{L}^e \to \mathbf{L}$. Since the restriction of ϕ onto $\mathbf{B}_{\mathfrak{q}}^e$ splits, $\varepsilon \in \mathbf{B}_{\mathfrak{q}}^e$. Therefore $\varepsilon \in \bigcap_{\mathrm{ht}\,\mathfrak{q}=1} \mathbf{B}_{\mathfrak{q}}^e = \mathbf{B} \boxtimes_{\mathbf{A}} \mathbf{B}$, so the natural epimorphism $\mathbf{B} \boxtimes_{\mathbf{A}} \mathbf{B} \to \mathbf{B}$ splits too. Let N be any Cohen–Macaulay $\mathbf{B}$-module. Then $\mathbf{B} \otimes_{\mathbf{A}} N \simeq (\mathbf{B} \otimes_{\mathbf{A}} \mathbf{B}) \otimes_{\mathbf{A}} N$, hence $\mathbf{B} \boxtimes_{\mathbf{A}} N \simeq (\mathbf{B} \boxtimes_{\mathbf{A}} \mathbf{B}) \boxtimes_{\mathbf{B}} N$, which gives a natural epimorphism $\mathbf{B} \otimes_{\mathbf{A}} N \to N$. It arises from $\mathbf{B} \boxtimes_{\mathbf{A}} \mathbf{B} \to \mathbf{B}$, thus splits, and N is a direct summand of $\mathbf{B} \boxtimes_{\mathbf{A}} N$. $\qquad\square$

Obviously, in the situation of Proposition 9.1 an indecomposable Cohen–Macaulay $\mathbf{A}$-module (respectively $\mathbf{B}$-module) is isomorphic to a direct summand of an indecomposable Cohen–Macaulay $\mathbf{B}$-module (respectively $\mathbf{A}$-module).

Corollary 9.2. *If the order $|G|$ is invertible in $\Bbbk$ and $\mathbf{B}$ is Cohen–Macaulay finite, tame or wild, then so is $\mathbf{A}$. If, moreover, $\mathbf{B}$ is unramified in codimension 1 and $\mathbf{A}$ is Cohen–Macaulay finite, tame or wild, then so is $\mathbf{B}$.*

It implies immediately the Ésnault–Auslander criterion. Namely call $\mathbf{A}$ a *quotient singularity* if it is isomorphic to a ring of invariants $\mathbf{R}^G$, where $\mathbf{R} = \Bbbk[[x,y]]$ and G is a finite group of automorphisms of $\mathbf{R}$ (it is well-known that in this case we may always suppose that G acts linearly, so it is a finite subgroup of $\mathrm{GL}\,(2,\Bbbk)$).

Theorem 9.3 (Ésnault–Auslander). *Suppose that $\mathrm{char}\,\Bbbk = 0$. Then $\mathbf{A}$ is Cohen–Macaulay finite if and only if it is a quotient singularity.*

Proof. Necessity. If $\mathbf{A}$ is Cohen–Macaulay finite, it has in particular finitely many rank 1 Cohen–Macaulay modules, or, the same, divisorial ideals (up to isomorphism). Thus its Picard group (group of classes of divisors) is finite. It is known (cf. e.g. [**Lip**]) that such a singularity is *rational*, which means that $\mathrm{H}^1(X, \mathcal{O}_X) = 0$. Consider the *canonical divisor* $K = K_S$, i.e. the class of a (rational) differential 2-form. It is of finite order in the Picard group, i.e. nK is a principal divisor for some n. Choose an ideal J of class K; then $J^{-n} = \alpha\mathbf{A}$ and one can consider the ring $\mathbf{B} = \bigoplus_{i=0}^{n-1} J^{-i} t^i$ with $t^n = \alpha^{-1}$. It is Gorenstein and $\mathbf{A} \simeq \mathbf{B}^H$, where H is the cyclic group of order n acting naturally on $\mathbf{B}$, namely leaving elements of $\mathbf{A}$ intact and mapping t to εt, where ε is a primitive root of 1. For every prime $\mathfrak{q} \subset \mathbf{A}$ of height 1 $\mathbf{A}_\mathfrak{q}$ is a discrete valuation ring, hence $J_\mathfrak{q} = \gamma\mathbf{A}$ for some γ such that $\alpha = \mu\gamma^{-n}$ with an invertible μ. Moreover, since $\mathrm{char}\,\Bbbk = 0$, μ is an n-th power of some element, so we may suppose that $\alpha = \gamma^{-n}$ and $\mathbf{B}_\mathfrak{q} = \mathbf{A}_\mathfrak{q}[\gamma^{-1}t] \simeq \mathbf{A}_\mathfrak{q}[x]/(x^n - 1) \simeq \mathbf{A}_\mathfrak{q}^n$. Therefore $\mathbf{B}$ is unramified in codimension 1, so it is also Cohen–Macaulay finite by Proposition 9.1, thus rational. But all rational Gorenstein singularities are well-known [**Lau**]. They are *rational double points*, or *du Val singularities*. All of them are quotient singularities. Therefore $\mathbf{A}$ is quotient too.

Sufficiency follows directly from Proposition 9.1, since any regular local ring is Cohen–Macaulay finite: all Cohen–Macaulay modules are free. $\square$

Remark 9.4. As far as I know, the finiteness criterion is still unknown if $\mathrm{char}\,\Bbbk > 0$, though it seems very plausible that the answer must be the same (maybe modulo some minor changes of definitions).

10. Kahn's reduction

In this section we recall the main results of the Kahn's paper [**Kahn**] and extend them to families of Cohen–Macaulay modules and vector bundles.

An *(exceptional) cycle* on X is a divisor $Z = \sum_{i=1}^{s} z_i E_i$. If $z_i \geq 0$ it is called *effective*. We treat an effective cycle as a projective curve (non-reduced if $z_i > 1$ for some i), namely the subvariety of X defined by the sheaf of ideals $\mathcal{O}_X(-Z)$. We also denote by ω_X the *dualizing sheaf* of X and by $\omega_Z = \omega_X(Z) \otimes_{\mathcal{O}_X} \mathcal{O}_Z$ the dualizing sheaf of Z. The latter defines the *Serre's duality*

$$\mathrm{H}^i(E, \mathcal{F}) \simeq \mathrm{DH}^{1-i}(E, \mathcal{F}^\vee \otimes_{\mathcal{O}_Z} \omega_Z) \qquad (i = 0, 1)$$

for every vector bundle $\mathcal{F}$ on Z, where $\mathrm{D}V = \mathrm{Hom}(V, \Bbbk)$, the dual vector space to V, and $\mathcal{F}^\vee = \mathcal{H}om_{\mathcal{O}_Z}(\mathcal{F}, \mathcal{O}_Z)$.

For any coherent sheaf $\mathcal{F}$ on X denote by $\mathcal{F}^g$ the image of the evaluation mapping $\Gamma(X, \mathcal{F}) \otimes \mathcal{O}_X \to \mathcal{F}$. If $\mathcal{F}^g = \mathcal{F}$, we say that $\mathcal{F}$ is *generated by global sections*, or *globally spanned*. If the support of the factor $\mathcal{F}/\mathcal{F}^g$ is 0-dimensional (i.e. a finite set of closed points), we say that $\mathcal{F}$ is *generically generated* by global sections, or *generically spanned*.

The main notion of the Kahn's theory is the following.

Definition 10.1. An effective cycle Z is called a *weak reduction cycle* if

    1) the sheaf $\mathcal{O}_Z(-Z)$ is generically spanned;

2) $\mathrm{H}^1(E, \mathcal{O}_Z(-Z)) = 0$.

It is called a *reduction cycle* if, moreover,

3) the sheaf $\omega_Z^\vee$ is generically spanned.

A reduction cycle always exists: it easily follows from the fact that the *intersection form* $(A.B)$ is negative definite on the group of all exceptional cycles [**Lip**].

We identify **A**-modules with quasi-coherent sheaves over S (their sheafifications). In particular, we consider the inverse image functor $\pi^* : \mathbf{A}\text{-mod} \to \mathrm{Coh}\,X$. Even if $M \in \mathrm{CM}(\mathbf{A})$, usually π^*M can have torsion. So we define the functor $\pi^\sharp : \mathrm{CM}(\mathbf{A}) \to \mathrm{VB}(X)$ setting $\pi^\sharp(M) = (\pi^*M)^{\vee\vee}$, where $\mathcal{F}^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$. As π is isomorphism outside E and any Cohen–Macaulay module M is completely defined by its stalks outside p, $M \simeq \pi_*\pi^\sharp M$, so this functor is full and faithful. The following theorem describes its image.

Theorem 10.2 ([**Kahn**]). *A vector bundle $\mathcal{F}$ over X is isomorphic to $\pi^\sharp M$ for some Cohen–Macaulay $\mathbf{A}$-module M if and only if*

1) $\mathcal{F}$ *is generically spanned;*
2) *the restriction $\Gamma(X, \mathcal{F}) \to \Gamma(\check{X}, \mathcal{F})$ is surjective.*

We call such vector bundles *full* and denote by $\mathrm{VB}^f(X)$ the subcategory of full vector bundles.

If Z is any effective cycle, we can consider the functor "restriction on Z," $\mathrm{res}_Z : \mathrm{VB}(X) \to \mathrm{VB}(Z)$ mapping $\mathcal{F}$ to $\mathcal{F}/\mathcal{F}(-Z)$.

Theorem 10.3 ([**Kahn**]).   1. *The functor res_Z is dense, i.e. every vector bundle over Z is isomorphic to $\mathrm{res}_Z\,\mathcal{F}$ for some vector bundle $\mathcal{F}$ over X.*

2. *If Z is a reduction cycle, the restriction of res_Z onto $\mathrm{VB}^f(X)$ maps non-isomorphic vector bundles to non-isomorphic ones.*

   We call a vector bundle over Z full if it is isomorphic to $\mathrm{res}_Z\,\mathcal{F}$, where $\mathcal{F} \in \mathrm{VB}^f(X)$, and denote by $\mathrm{VB}^f(Z)$ the category of full vector bundles over Z.

3. *The functor $R_Z = \mathrm{res}_Z \circ \pi^\sharp$ induces a representation equivalence $\mathrm{CM}(\mathbf{A}) \to \mathrm{VB}^f(Z)$.*

Note that this functor cannot be faithful, since Hom-spaces in the category $\mathrm{VB}(Z)$ are finite dimensional. Moreover, it can map indecomposable vector bundles to decomposable ones, cf. Theorem 11.1 below.

Kahn also gives a description of full vector bundles over a weak reduction cycle.

Theorem 10.4 ([**Kahn**]). *If Z is a weak reduction cycle, the following conditions on a vector bundle $\mathcal{E} \in \mathrm{VB}(Z)$ are equivalent:*

1) $\mathcal{E}$ *is full.*
2) $\mathcal{E}$ *is generically spanned and there is a vector bundle $\mathcal{E}_2$ over the cycle $2Z$ such that $\mathcal{E}_2|Z \simeq \mathcal{E}$ and the mapping $\mathrm{H}^0(\mathcal{E}(Z)) \to \mathrm{H}^1(\mathcal{E})$ induced by the canonical exact sequence $0 \to \mathcal{E} \to \mathcal{E}_2(Z) \to \mathcal{E}(Z) \to 0$ is injective.*

The latter sequence is obtained by tensoring with $\mathcal{E}(Z)$ the exact sequence $0 \to \mathcal{O}_Z(-Z) \to \mathcal{O}_{2Z} \to \mathcal{O}_Z \to 0$.

Actually we need a generalization of Theorems 10.2, 10.4 for *families* of vector bundles and modules based on an algebra Λ.

Definition 10.5. Let $\pi : X \to S$ be a resolution of a normal surface singularity and Λ be a $\mathbb{k}$-algebra (maybe non-commutative).

- For any coherent sheaf of $\mathcal{O}_X \otimes \Lambda$-bimodules $\mathcal{F}$ we denote by $\mathcal{F}^g$ the image of the evaluation mapping $\Gamma(X, \mathcal{F}) \otimes (\mathcal{O}_X \otimes \Lambda) \to \mathcal{F}$ and say that $\mathcal{F}$ is *generically spanned* if the support of $\mathcal{F}/\mathcal{F}^g$ is 0-dimensional, i.e. a finite set of closed points.
- We call a family of vector bundles $\mathcal{F}$ over X based on Λ *full* if it is isomorphic to $\pi^\sharp \mathcal{M} = (\pi^* \mathcal{M})^{\vee\vee}$, where $\mathcal{M}$ is a family of Cohen–Macaulay modules over S based on Λ and $\mathcal{N}^\vee = \mathcal{H}om_{\mathcal{O}_X \otimes \Lambda^{\mathrm{op}}}(\mathcal{N}, \mathcal{O}_X \otimes \Lambda^{\mathrm{op}})$ for any family $\mathcal{N}$ of $\mathcal{O}_X$-modules based on Λ.
- We call a family of vector bundles over an effective cycle C *full* if it possess a full lifting to a family of vector bundle over X.

To extend Kahn's results to families, one needs some restrictions on the base algebra Λ. For our purpose it is enough to consider algebras Λ such that gl.dim $\Lambda \leq 2$. The advantage is that in this case any kernel of a mapping between two flat Λ-modules is also flat. Especially, if $\mathcal{F}$ is a family of $\mathcal{O}_X$-modules based on Λ, $U \subseteq X$ is any open subset and $U = \bigcup_i U_i$ is its affine open covering, then $\Gamma(U, \mathcal{F})$ is the kernel of the natural mapping $\bigoplus_i \Gamma(U_i, \mathcal{F}) \to \bigoplus_{i,j} \Gamma(U_i \cap U_j, \mathcal{F})$, thus Λ-flat. As a corollary, if $\pi : X \to Y$ is a proper morphism, the direct image $\pi_* \mathcal{F}$ is Λ-flat, so is a family of $\mathcal{O}_Y$-modules based on Λ. (We need 'proper' in order $\pi_* \mathcal{F}$ to be coherent.) The same is true if we consider families based on regular schemes of dimension at most 2, since their local rings are of global dimension at most 2.

Theorem 10.6. *Suppose that Z is a weak reduction cycle for a resolution $\pi : X \to S$ of a normal surface singularity and gl.dim $\Lambda \leq 2$.*

1. *Let $\mathcal{F}$ be a family of vector bundles over X based on Λ such that*
   (a) *$\mathcal{F}$ is generically spanned;*
   (b) *the restriction $\Gamma(X, \mathcal{F}) \to \Gamma(\check{X}, \mathcal{F})$ is surjective.*
   Then $\mathcal{F}$ is full.
2. *Let $\mathcal{E}$ be a family of vector bundles over Z such that*
   (a) *$\mathcal{E}$ is generically spanned;*
   (b) *there is a lifting of $\mathcal{E}$ to a vector bundle over $2Z$ such that the induced mapping $\mathrm{H}^0(E, \mathcal{E}(Z)) \to \mathrm{H}^1(E, \mathcal{E})$ is injective.*
   Then $\mathcal{E}$ is full.

Proof. 1. Set $\mathcal{M} = \pi_* \mathcal{F}$, $\mathcal{F}' = \pi^* \mathcal{M}/(\mathcal{O}_X\text{-torsion})$. Then $\mathcal{F}'$ can be considered as a subsheaf of $\mathcal{F}$ containing $\mathcal{F}^g$ (since global sections of $\mathcal{M}$ are the same as those of $\mathcal{F}$). Note that $\mathcal{M}(L) \simeq \pi_* \mathcal{F}(L)$ is a Cohen–Macaulay module by [**EGA**, Proposition 6.3.1]. Therefore $\mathcal{M}$ is a family of Cohen–Macaulay modules. Since $\mathcal{F}$ is generically generated, its stalks $\mathcal{F}_x$ coincide with $\mathcal{F}_x^g$ provided x is not a closed point. But as X is normal, any family $\mathcal{N}^\vee$ is completely determined by its stalks at non-closed points. Thus $\mathcal{F} \simeq \mathcal{F}^{\vee\vee} \simeq (\mathcal{F}')^{\vee\vee} \simeq \pi^\sharp \mathcal{M}$.

2. Suppose that $\mathcal{E}_n$ is a family of vector bundles over nZ such that $\mathcal{E}_n \otimes_{\mathcal{O}_{nZ}} \mathcal{O}_Z \simeq \mathcal{E}$. Then the obstruction for lifting $\mathcal{E}_n$ to a family of vector

bundles over $(n+1)Z$ lies in $\mathrm{H}^2(E, \mathcal{H}om(\mathcal{E}, \mathcal{E}(-n)))$. (It can be shown just in the same way as in [**Pet**], where the complex analytic case was considered.) But this cohomology space is 0 since $\dim E = 1$. Hence such a lifting is always possible, so we can construct a sequence $\mathcal{E}_n$, where each $\mathcal{E}_n$ is a family of vector bundles over nZ, such that $\mathcal{E}_n \otimes_{\mathcal{O}_{nZ}} \mathcal{O}_{(n-1)Z} \simeq \mathcal{E}_{n-1}$. Taking inverse limit we get a family $\mathcal{F}$ of vector bundles over X based on Λ such that $\mathcal{E}_1 = \mathcal{E}$ and $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{nZ} \simeq \mathcal{E}_n$ for all n. We only have to show that $\mathcal{F}$ is full. According to the Theorem on Formal Functions [**Ha**, Theorem III.11.1], the $\mathfrak{m}$-adic completion of $\mathrm{H}^1(X, \mathcal{F}(-Z))$ coincides with $\varprojlim \mathrm{H}^1(E, \mathcal{E}_n(-Z))$. The natural exact sequence

$$0 \to \mathcal{O}_Z(-nZ) \longrightarrow \mathcal{O}_{(n+1)Z} \longrightarrow \mathcal{O}_{nZ} \to 0$$

tensored with $\mathcal{F}$ gives an exact sequence

(10.1) $$0 \to \mathcal{E}(-nZ) \longrightarrow \mathcal{E}_{n+1} \longrightarrow \mathcal{E}_n \to 0.$$

Since both $\mathcal{E}$ and $\mathcal{O}_Z(-Z)$ are generically spanned, so is $\mathcal{E}(-nZ)$ for all $n \geq 0$. It means that for every $n > 0$ there is a homomorphism $m\mathcal{O}_Z \to \mathcal{E}(-(n-1)Z)$ with the cokernel support of dimension 0. Twisting it by $-Z$ we get a homomorphism $m\mathcal{O}_Z(-Z) \to \mathcal{E}(-nZ)$ with the same property, which induces an epimorphism $m\,\mathrm{H}^1(E, \mathcal{O}_Z(-Z)) \to \mathrm{H}^1(E, \mathcal{E}(-nZ))$. Since Z is a weak reduction cycle, we get $\mathrm{H}^1(E, \mathcal{E}(-nZ)) = 0$ for all $n > 0$. The exact sequence (10.1) twisted by $-Z$ gives an exact sequence of cohomologies

$$\mathrm{H}^1(E, \mathcal{E}(-(n+1)Z)) \longrightarrow \mathrm{H}^1(E, \mathcal{E}_{n+1}(-Z)) \longrightarrow \mathrm{H}^1(E, \mathcal{E}_n(-Z)) \longrightarrow 0,$$

wherefrom we can deduce that $\mathrm{H}^1(E, \mathcal{E}_n(-Z)) = 0$ by an obvious induction. Therefore $\mathrm{H}^1(X, \mathcal{F}(-Z)) = 0$ and the exact sequence $0 \to \mathcal{F}(-Z) \to \mathcal{F} \to \mathcal{E} \to 0$ shows that every global section of $\mathcal{E}$ can be lifted to a global section of $\mathcal{F}$. But outside E every quasicoherent sheaf over X is generated by its global sections. Hence $\mathcal{F}$ is generically spanned, i.e. the condition 1(a) holds. Note also that the equality $\mathrm{H}^1(X, \mathcal{F}(-Z)) = 0$ implies that $\mathrm{H}^1(X, \mathcal{F}) \simeq \mathrm{H}^1(E, \mathcal{E})$.

To verify the condition 1(b) we use local cohomologies [**Gro2**], especially the exact sequence

$$\mathrm{H}^0(X, \mathcal{F}) \longrightarrow \mathrm{H}^0(\breve{X}, \mathcal{F}) \to \mathrm{H}^1_E(X, \mathcal{F}) \to \mathrm{H}^1(X, \mathcal{F}),$$

which shows that the condition 1(b) can be reformulated as follows:

1(b$'$) *the mapping* $\mathrm{H}^1_E(X, \mathcal{F}) \to \mathrm{H}^1(X, \mathcal{F})$ *is injective.*

Due to [**Wahl**, Lemma B.2] we can identify $\mathrm{H}^1_E(X, \mathcal{F})$ with $\varinjlim \mathrm{H}^0(E, \mathcal{E}_n(nZ))$, where the limit is taken along homomorphisms

$$\mu_n : \mathrm{H}^0(E, \mathcal{E}_{nZ}(nZ)) \to \mathrm{H}^0(E, \mathcal{E}_{n+1}((n+1)Z))$$

arising from the natural exact sequence

$$0 \to \mathcal{O}_{nZ}(-Z) \longrightarrow \mathcal{O}_{(n+1)Z} \longrightarrow \mathcal{O}_Z \to 0$$

tensored by $\mathcal{E}_{n+1}((n+1)Z)$. Especially all μ_n are injective. We shall show that under our conditions they are also surjective, or, equivalently, all homomorphisms $\mathrm{H}^0(E, \mathcal{E}((n+1)Z)) \longrightarrow \mathrm{H}^1(E, \mathcal{E}_n(nZ))$ are injective. Actually we shall prove that even their compositions with the restrictions $\mathcal{E}_n(nZ) \to \mathcal{E}(nZ)$, i.e. homomorphisms

$$\delta_n : \mathrm{H}^0(E, \mathcal{E}((n+1)Z)) \longrightarrow \mathrm{H}^1(E, \mathcal{E}(nZ))$$

are injective. The latter arise from the exact sequence

$$0 \to \mathcal{E}(nZ) \longrightarrow \mathcal{E}_2((n+1)Z) \longrightarrow \mathcal{E}((n+1)Z) \longrightarrow 0,$$

thus, by the condition 2(b), we may suppose that it is injective for $n = 0$. Since Z is a weak reduction cycle, the sheaf $\mathcal{O}_X(-Z)$ is generically spanned, hence all sheaves $\mathcal{O}_X(-nZ)$ are generically spanned too, so there is a homomorphism $m\mathcal{O}_X \to \mathcal{O}_X(-nZ)$ with the cokernel supported on a finite set of closed points. Then the dual mapping $\mathcal{O}_X(nZ) \to m\mathcal{O}_X$ is a monomorphism. Tensoring with $\mathcal{E}(Z)$, we get a monomorphism $\phi : \mathcal{E}((n+1)Z) \to m\mathcal{E}(Z)$, wherefrom we get the following commutative diagram:

$$
\begin{array}{ccc}
\mathrm{H}^0(E, \mathcal{E}((n+1)Z) & \xrightarrow{\ \delta_n\ } & \mathrm{H}^1(E, \mathcal{E}(nZ)) \\
{\scriptstyle \mathrm{H}^0(\phi)}\downarrow & & \downarrow \\
m\,\mathrm{H}^0(E, \mathcal{E}(Z)) & \xrightarrow{\ m\,\delta_0\ } & \mathrm{H}^1(E, \mathcal{E}).
\end{array}
$$

Since both δ_0 and $\mathrm{H}^0(\phi)$ are injective, so is δ_n.

Thus $\mathrm{H}^1_E(X, \mathcal{F}) \simeq \mathrm{H}^0(E, \mathcal{E}(Z))$. Since also $\mathrm{H}^1(X, \mathcal{F}) \simeq \mathrm{H}^1(E, \mathcal{E})$, we see that the condition 1(b) is actually equivalent to the condition 2(b). It accomplishes the proof. □

We also need the following important, though rather simple, observation.

Proposition 10.7. *If a family $\mathcal{F}$ of vector bundles over X based on Λ is full, so are also all its fibres $\mathcal{F}(L) = \mathcal{F} \otimes_\Lambda L$.*

Proof. First show that $\mathrm{H}^1(\breve{X}, \mathcal{F}) = 0$. Note that since E is a closed subscheme of a regular scheme X, it can be locally defined by one equation [**Ha**, Proposition II.6.11]. Therefore $\mathrm{H}^2_E(X, \mathcal{F}) = 0$ [**Gro2**] and the mapping $\mathrm{H}^1(X, \mathcal{F}) \to \mathrm{H}^1(\breve{X}, \mathcal{F})$ is surjective. The exact sequence $0 \to \mathcal{F}(-E) \to \mathcal{F} \to \mathcal{E} \to 0$, where $\mathcal{E} = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_E$, together with the obvious equalities $\mathrm{H}^i(\breve{X}, \mathcal{E}) = 0$, implies that $\mathrm{H}^i(\breve{X}, \mathcal{F}) \simeq \mathrm{H}^i(\breve{X}, \mathcal{F}(-nE))$ for all n. But X is projective over the affine scheme S and $-E$ is ample, so $\mathrm{H}^1(X, \mathcal{F}(-nE)) = 0$ for some n, hence also $\mathrm{H}^1(\breve{X}, \mathcal{F}) = 0$.

Now from the Künneth formulae [**CE**] we obtain a commutative diagram

$$
\begin{array}{ccc}
\mathrm{H}^0(X, \mathcal{F}) \otimes_\Lambda L & \longrightarrow & \mathrm{H}^0(X, \mathcal{F} \otimes_\Lambda L) \\
\downarrow & & \downarrow \\
\mathrm{H}^0(\breve{X}, \mathcal{F}) \otimes_\Lambda L & \longrightarrow & \mathrm{H}^0(\breve{X}, \mathcal{F} \otimes_\Lambda L),
\end{array}
$$

where the lower horizontal arrow is an isomorphism and the left vertical arrow is surjective. Hence the right vertical arrow is surjective too, which means that $\mathcal{F}(L)$ is full. □

Remark 10.8. Obviously, the subcategory $\mathrm{VB}^f(X) \subset \mathrm{VB}(X)$ is closed under direct summands. On the contrary, it is not the case for the subcategory $\mathrm{VB}^f(Z) \subset \mathrm{VB}(Z)$ (cf. for instance Theorem 11.1 below). The same is true for families too. That is why, even under the conditions of Theorem 10.6.2, we cannot claim that a full lifting $\mathcal{F}$ of $\mathcal{E}$ is strict if so is $\mathcal{E}$, though it is true that $\mathcal{F}(L) \simeq \mathcal{F}(L')$ implies $L \simeq L'$. On the other hand, if $\mathcal{F}$ is strict, so is the family $\mathcal{M} = \pi_* \mathcal{F}$ of Cohen–Macaulay **A**-modules, since the restriction of π_* onto $\mathrm{VB}^f(X)$ is an equivalence $\mathrm{VB}^f(X) \xrightarrow{\sim} \mathrm{CM}(\mathbf{A})$.

11. Cohen–Macaulay types: minimal elliptic case

Recall that the *fundamental cycle* of a resolution $\pi : X \to S$ of a normal surface singularity is the smallest effective cycle Z such that $(Z.E_i) \leq 0$ for each irreducible component E_i of the exceptional curve E [**Lau**]. This singularity is called *minimally elliptic* [**Lau**, **Reid**] if it is Gorenstein and $h^1(\mathcal{O}_X) = 1$ for some (hence for any) resolution $\pi : X \to S$. If π is minimal, an equivalent condition is: $(Z + K.E_i) = 0$ for all i, where Z is the fundamental cycle of this resolution and K is the canonical divisor of X. In particular, then $\omega_Z \simeq \mathcal{O}_Z$. One can easily check that Z is a reduction cycle in this case [**Kahn**, **DGK**].

For minimally elliptic singularities the criterion of Theorem 10.4 can be essentially simplified and restated in terms of $\mathcal{E}$ itself, without references to liftings.

Theorem 11.1 ([**Kahn**]). *Let $\mathcal{E}$ be a vector bundle over Z. It is full if and only if $\mathcal{E} \simeq \mathcal{G} \oplus n\mathcal{O}_Z$, where*

1) *$\mathcal{G}$ is generically spanned;*
2) *$H^1(\mathcal{G}) = 0$;*
3) *$n \geq h^0(\mathcal{G}(Z))$.*

Under these conditions the full lifting $\mathcal{F}$ of $\mathcal{E}$ is indecomposable if and only if $\mathcal{G}$ is indecomposable and either $\mathcal{E} = \mathcal{O}_Z$ or $\mathcal{G} \not\simeq \mathcal{O}_Z$ and $n = h^0(\mathcal{G}(Z))$.

We can extend this result to families as follows.

Theorem 11.2. *Let $\mathbf{A}$ be a minimally elliptic singularity, $\pi : X \to S$ be its minimal resolution, and Z be the fundamental cycle of this resolution (which is known to be a reduction cycle). Suppose that $\mathcal{G}$ is a family of vector bundles over Z based on an algebra Λ with $\mathrm{gl.dim}\,\Lambda \leq 2$ such that*

1) *$\mathcal{G}$ is generically spanned;*
2) *$H^1(E, \mathcal{G}) = 0$;*

We set $P_0 = H^0(E, \mathcal{G}(Z))$. Let also P be a projective Λ-module such that there is an embedding $P_0 \to P$. Then the family $\mathcal{E} = \mathcal{G} \oplus (\mathcal{O}_Z \otimes P)$ is full.

Proof. Obviously $\mathcal{E}$ is generically spanned. Moreover,

$$H^1(E, \mathcal{E}) = H^1(E, \mathcal{O}_Z \otimes P) \simeq H^1(E, \mathcal{O}) \otimes P \simeq P,$$

$$H^0(E, \mathcal{E}(Z)) = H^0(E, \mathcal{G}(Z)) = P_0,$$

since $H^0(E, \mathcal{O}_Z(Z)) = 0$ for any exceptional cycle Z (cf. e.g. [**Reid**, Chapter 4, Exercise 14]). We have already seen that there is a lifting $\mathcal{E}'$ of $\mathcal{E}$ to a family of vector bundles over $2Z$. It gives an exact sequence

$$0 \longrightarrow \mathcal{E}(-Z) \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow 0$$

Denote by ξ the corresponding element of $\mathrm{Ext}^1_{\mathcal{O}_{2Z} \otimes \Lambda}(\mathcal{E}, \mathcal{E}(-Z))$ and by δ the induced mapping $H^0(E, \mathcal{E}(Z)) \to H^1(E, \mathcal{E})$. One can easily see that any element from $\mathrm{Ext}^1_{\mathcal{O}_{2Z} \otimes \Lambda}(\mathcal{E}, \mathcal{E}(-Z))$ that is of the form $\xi + \eta$, where $\eta \in \mathrm{Ext}^1_{\mathcal{O}_Z \otimes \Lambda}(\mathcal{E}, \mathcal{E}(-Z)) \simeq \mathrm{Ext}^1_{\mathcal{O}_Z \otimes \Lambda}(\mathcal{E}(Z), \mathcal{E})$, also defines a lifting of $\mathcal{E}$ to a family of vector bundles. Moreover, such an element induces the mapping $\delta + \delta_\eta : H^0(E, \mathcal{E}(Z)) \to H^1(E, \mathcal{E})$, where $\delta_\eta(s) = \eta s$, the Yoneda product of

$\eta \in \mathrm{Ext}^1_{\mathcal{O}_Z \otimes \Lambda}(\mathcal{E}(Z), \mathcal{E})$ with $s \in \mathrm{H}^0(E, \mathcal{E}(Z)) \simeq \mathrm{Hom}_{\mathcal{O}_Z \otimes \Lambda}(\mathcal{O}_Z \otimes \Lambda, \mathcal{E}(Z))$. Choose η from

$$\mathrm{Ext}^1_{\mathcal{O}_Z \otimes \Lambda}(\mathcal{G}(Z), \mathcal{O}_Z \otimes P) \simeq \mathrm{Ext}^1_{\mathcal{O}_Z \otimes \Lambda}(\mathcal{O}_Z \otimes D_\Lambda P, \mathcal{G}^\vee(-Z)) \simeq$$
$$\simeq \mathrm{H}^1(E, \mathcal{G}^\vee(-Z) \otimes_\Lambda P),$$

where $\mathcal{G}^\vee = \mathcal{H}om_{\mathcal{O}_Z \otimes \Lambda}(\mathcal{G}, \mathcal{O}_Z \otimes \Lambda)$, $D_\Lambda P = \mathrm{Hom}_\Lambda(P, \Lambda)$. Due to the Künneth formulae, $\mathrm{H}^1(E, \mathcal{G}^\vee(-Z) \otimes_\Lambda P) \simeq \mathrm{H}^1(E, \mathcal{G}^\vee(-Z)) \otimes_\Lambda P$. The Serre's duality implies that $\mathrm{H}^1(E, \mathcal{G}^\vee(-Z)) \simeq D_\Lambda \mathrm{H}^0(E, \mathcal{G}(Z)) \simeq D_\Lambda P_0$. Since $D_\Lambda P_0 \otimes_\Lambda P \simeq \mathrm{Hom}_\Lambda(P_0, P)$, the mapping $(s, \eta) \to \eta s$ is just the evaluation homomorphism $P_0 \times \mathrm{Hom}_\Lambda(P_0, P) \to P$. Thus δ_η coincides with η as a homomorphism $P_0 \to P$. Therefore we can choose η so that $\delta + \delta_\eta$ becomes any prescribed homomorphism $P_0 \to P$, for instance an embedding. Then the corresponding lifting $\mathcal{E}_2$ of $\mathcal{E}$ to a family of vector bundles over $2Z$ satisfies the condition (2b) of Theorem 10.6, hence $\mathcal{E}$ is full. $\qquad\square$

Corollary 11.3. *Suppose that the conditions of Theorem 11.2 hold, as well as the following:*

3) *$\mathcal{G}$ is strict and has no fibres isomorphic to $\mathcal{O}_Z$;*
4) *$\mathrm{H}^1(E, \mathcal{G}(Z))$ is flat as Λ-module;*
5) *$P = P_0$.*

Then the full lifting $\mathcal{F}$ of the family $\mathcal{E}$ to a family of vector bundles over X is also strict, as well as the family $\mathcal{M} = \pi_* \mathcal{F}$ of Cohen–Macaulay modules over $\mathbf{A}$.

Proof. Let L be an indecomposable Λ-module. The Künneth formulae imply that

$$\mathrm{H}^0(E, \mathcal{G}(Z) \otimes_\Lambda L) \simeq \mathrm{H}^0(E, \mathcal{G}(Z)) \otimes_\Lambda L = P \otimes_\Lambda L$$

since $\mathrm{H}^1(E, \mathcal{G}(Z))$ is flat, and

$$\mathrm{H}^1(E, \mathcal{G} \otimes_\Lambda L) \simeq \mathrm{H}^1(E, \mathcal{G}) \otimes_\Lambda L = 0$$

since $\mathrm{H}^2(E, \mathcal{G}) = 0$. Hence $\mathcal{G}(L)$ satisfies the indecomposability conditions from Theorem 11.1 (with $n = \dim_\Bbbk P \otimes_\Lambda L$). Since $\mathcal{F}(L)$ is full and $\mathrm{res}_Z \mathcal{F}(L) \simeq \mathcal{E}(L)$, $\mathcal{F}(L)$ is indecomposable. Moreover, if $\mathcal{F}(L) \simeq \mathcal{F}(L')$, then $\mathcal{G}(L) \oplus n\mathcal{O}_Z \simeq \mathcal{G}(L') \oplus n'\mathcal{O}_Z$ for $n' = \dim_\Bbbk P \otimes_\Lambda L'$. As neither $\mathcal{G}(L)$ nor $\mathcal{G}(L')$ has direct summands isomorphic to $\mathcal{O}_Z$, the Krull–Schmidt theorem for vector bundles implies that $\mathcal{G}(L) \simeq \mathcal{G}(L')$, thus $L \simeq L'$, so $\mathcal{F}$ and hence $\mathcal{M}$ are strict. $\qquad\square$

Now we can use the results on vector bundle types to define Cohen–Macaulay types of minimally elliptic singularities. Recall some definitions.

Definition 11.4. A normal surface singularity with a minimal resolutions $\pi : X \to S$ and exceptional curve E is called

- *simple elliptic* if E is a smooth elliptic curve [**Sai**];
- *cusp* if E is a projective configuration of type $\widetilde{\mathrm{A}}$.

The latter is not the original definition (in the case $\mathrm{char}\,\Bbbk = 0$), but is equivalent to it [**Hir**, **Kar**]. We accept it as a *definition* of cusp singularities in the case $\mathrm{char}\,\Bbbk > 0$ too. It is easy to see that both simple elliptic and cusp singularities are minimally elliptic; moreover the fundamental cycle in these cases coincides with the exceptional curve E.

Note that, according to the Ésnault–Auslander Theorem 9.3, neither minimally elliptic singularity can be Cohen–Macaulay finite (it also follows directly from the next theorem, even if char $\Bbbk > 0$).

Theorem 11.5. *A minimally elliptic singularity* **A** *is*

- *Cohen–Macaulay* tame bounded *if it simple elliptic;*
- *Cohen–Macaulay* tame unbounded *if it is a cusp singularity;*
- *Cohen–Macaulay* wild *otherwise.*

Proof. Suppose first that **A** is neither simple elliptic nor cusp. Then the exceptional curve E, hence also the fundamental cycle Z is vector bundle wild, i.e. possess a strict family $\mathcal{G}$ of vector bundles over the polynomial ring $\mathbf{R} = \Bbbk[x, y]$. Replacing $\mathcal{G}$ by $\mathcal{G}(m)$ for m big enough we may suppose that $\mathcal{G}$ is generically (even globally) spanned and $\mathrm{H}^1(E, \mathcal{G}) = 0$. Set $P = \mathrm{H}^0(E, \mathcal{G}(Z))$ and $Q = \mathrm{H}^1(E, \mathcal{G}(Z))$. There is an element $g \in \mathbf{R}$ such that $Q[g^{-1}]$ is flat over $\mathbf{R}[g^{-1}]$. Moreover, there is at most one point $z \in \mathbb{A}^2$ such that $\mathcal{G}(\Bbbk(z)) \simeq \mathcal{O}_Z$. Choose $x \in \mathbb{A}^2$ such that $x \neq z$ and $g(x) \neq 0$. Let $\mathfrak{x}$ be the corresponding maximal ideal of $\mathbf{R}$. Set $\Lambda = \hat{\mathbf{R}}_{\mathfrak{x}}$ (the $\mathfrak{x}$-adic completion) and $\hat{\mathcal{G}} = \mathcal{G} \otimes_{\mathbf{R}} \Lambda$. Then $\hat{\mathcal{G}}$ is a family of vector bundles over Z based on Λ that satisfied conditions of Corollary 11.3. Thus the family $\hat{\mathcal{G}} \oplus \mathcal{O}_Z \otimes \hat{P}_{\mathfrak{x}}$ has a full lifting $\mathcal{F}$ to X, which is strict, and the family of Cohen–Macaulay **A**-modules $\mathcal{M} = \pi_* \mathcal{F}$ based on Λ is strict too. Since $\Lambda \simeq \Bbbk[[x, y]]$, **A** is Cohen–Macaulay wild (cf. Remark 5.2.3).

If **A** is simple elliptic, we can use the Atiyah–Oda description of vector bundles over E together with the calculations from [**At**]. The latter give the following values of cohomologies for vector bundles $\mathcal{P}_{r,d}(nx) = p_{1*} i_{nx}^* \mathcal{P}_{r,d}$ from Theorem 3.1:

$$\mathrm{h}^0(E, \mathcal{P}_{r,d}(nx)) = \begin{cases} nd, & \text{if } d > 0, \\ 1, & \text{if } d = 0 \text{ and } x = o, \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathrm{h}^1(E, \mathcal{P}_{r,d}(nx)) = \begin{cases} nd, & \text{if } d < 0, \\ 1, & \text{if } d = 0 \text{ and } x = o, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $\mathcal{P}_{r,d}(nx)$ is generically spanned if and only if either $d \geq r$ or $d = 0, r = n = 1, x = o$ (the latter gives the trivial bundle $\mathcal{O}_E$), and $Q = \mathrm{H}^1(E, \mathcal{P}_{r,d}(Z))$ is flat for $d \neq br$, where $b = -(E.E)$ (recall that in this case $Z = E$, hence $\mathcal{P}(Z) = \mathcal{P}(-bo)$). If $d = br$ the sheaf Q is no more flat, since its fibre at the point o jumps, but its restriction onto $E^o = E \setminus o$ is flat. Denote by $\mathcal{P}_{r,d}^o$ the restriction of $\mathcal{P}_{r,d}$ onto E^o and define the families $\mathcal{E}_{r,d}$ of vector bundles over E, where $r \leq d$, $(r, d) = 1$, as follows:

$$\mathcal{E}_{r,d} = \begin{cases} \mathcal{P}_{r,d}, & \text{if } r \leq d < br, \\ \mathcal{P}_{r,d} \oplus \mathcal{O}_E \otimes (d - br)\mathcal{O}_E, & \text{if } d > br, \\ \mathcal{P}_{1,b}^o, & \text{if } r = 1, d = b. \end{cases}$$

In the first two cases this family is based on E, in the last one it is based on E^o. These families satisfy the conditions of Corollary 11.3, hence they can be lifted to families $\mathcal{F}_{r,d}$ of vector bundles over X. Denote $\mathcal{M}_{r,d} = \pi_* \mathcal{F}_{r,d}$.

Then Theorems 3.1 and 11.2 directly imply a description of Cohen–Macaulay $\mathbf{A}$-modules (also obtained in [$\mathbf{Kahn}$]). Below we denote by a^+ and a^- respectively the *positive* and *negative part* of a number a defined as

$$a^+ = \begin{cases} a & \text{if } a > 0, \\ 0 & \text{otherwise;} \end{cases} \qquad a^- = \begin{cases} -a & \text{if } a < 0, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 11.6. *If $\mathbf{A}$ is simply elliptic, all indecomposable Cohen–Macaulay $\mathbf{A}$-modules are:*

- $\mathbf{A}$, *of rank 1;*
- $\mathcal{M}_{r,d}(nx)$, *of rank $n(r + (d - br)^+)$, where $r \le d$, $(r,d) = 1$ and $x \in E$, $x \ne o$ if $r = 1$, $d = b$;*
- M_n, *of rank $n + 1$, where $R_E M_n \simeq \mathcal{P}_{1,b}(no) \oplus \mathcal{O}_E$.*

In particular $\mathbf{A}$ is Cohen–Macaulay tame bounded, since the number of families $\mathcal{M}_{r,d}$ with $r + (d - br)^+ = m$ is at most $b\varphi(m)$, where $\varphi(m)$ is the Euler function.

Let now $\mathbf{A}$ be a cusp singularity, $E = \bigcup_{i=1}^s E_i$ be the exceptional curve of its minimal resolution, E_i being the irreducible components so arranged that $E_i \cap E_{i+1} \ne \emptyset$ (as before, we set $E_{s+i} = E_i$), $b_i = -(E_i.E_i)$, $\mathbf{b} = (b_1, b_2, \ldots, b_s)$ and $\mathbf{b}^r$ be the r-fold repetition of $\mathbf{b}$, i.e.

$$\mathbf{b}^r = (b_1, b_2, \ldots, b_s, b_1, b_2, \ldots, b_s, \ldots, b_1, b_2, \ldots, b_s) \quad (r \text{ times}).$$

In this case $Z = E$ and $\mathcal{G}(Z) \simeq \mathcal{G}(-\mathbf{b})$. If $\mathcal{G}$ is a vector bundle over E, $\widetilde{\mathcal{G}}$ is its lifting to the normalization $\widetilde{E}$, one can calculate cohomologies of $\mathcal{G}$ using the long exact sequence of cohomologies corresponding to the exact sequence $0 \to \mathcal{G} \to \widetilde{\mathcal{G}} \to \widetilde{\mathcal{G}}/\mathcal{G} \to 0$ and the known values of $\mathrm{H}^i(\widetilde{E}, \widetilde{\mathcal{G}})$ (recall that $\widetilde{E}$ is just a union of projective lines). Applying this procedure to the vector bundles $\mathcal{V}(\mathbf{d}, m, \lambda)$ from page 17, one gets that these values can be calculated as follows (cf. [$\mathbf{DGK}$] for details). Define a *positive part* of $\mathbf{d} = (d_1, d_2, \ldots, d_{rs})$ as a subsequence $\mathbf{p} = (d_{k+1}, d_{k+2}, \ldots, d_{k+l})$, where $0 \le k < rs$, $1 \le l \le rs$, such that $d_i \ge 0$ for all $i = k+1, \ldots, k+l$, but either $l = rs$ or both $d_k < 0$ and $d_{k+l+1} < 0$ (again we set $d_{rs+j} = d_j$). Set $\theta(\mathbf{p}) = l$ if either $l = rs$ or $\mathbf{p} = (0, 0, \ldots, 0)$ and $\theta(\mathbf{p}) = l + 1$ otherwise; $\theta(\mathbf{d}) = \sum_{\mathbf{p}} \theta(\mathbf{p})$. At last, set $\delta(\mathbf{d}, \lambda) = 1$ if $\mathbf{d} = \mathbf{0} = (0, 0, \ldots, 0)$, $\lambda = 1$, and $\delta(\mathbf{d}, \lambda) = 0$ otherwise. Then

$$\mathrm{h}^0(E, \mathcal{V}(\mathbf{d}, m, \lambda)) = m \left(\sum_{i=1}^{rs} (d_i + 1)^+ - \theta(\mathbf{d}) \right) + \delta(\mathbf{d}, \lambda),$$

$$\mathrm{h}^1(E, \mathcal{V}(\mathbf{d}, m, \lambda)) = m \left(\sum_{i=1}^{rs} (d_i + 1)^- + rs - \theta(\mathbf{d}) \right) + \delta(\mathbf{d}, \lambda).$$

Moreover, $\mathcal{V}(\mathbf{d}, m, \lambda)$ is generically spanned if and only if either $\mathbf{d} = \mathbf{0}$, $m = 1$ and $\lambda = 1$ (i.e. $\mathcal{V}(\mathbf{d}, m, \lambda) \simeq \mathcal{O}_E$) or the following conditions hold[2]:

1. $\mathbf{d} > \mathbf{0}$ (it means that all $d_i \ge 0$ and at least one inequality is strict).

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[2]I must note a mistake in the preprint [$\mathbf{DGK}$], where we claimed that $\mathbf{d} > \mathbf{0}$ is enough for $\mathcal{V}(\mathbf{d}, m, \lambda)$ to satisfy Kahn's conditions. It has been improved in the final version. We are thankful to Igor Burban who has noticed this mistake.

2. If $\mathbf{d}'$ is a shift of $\mathbf{d}$, i.e. $\mathbf{d}' = (d_{k+1}, \ldots, d_{rs}, d_1, \ldots, d_k)$ for some k, $\mathbf{d}'$ contains no subsequence $(0, 1, 1, \ldots, 1, 0)$, in particular $(0, 0)$.

3. No shift of $\mathbf{d}$ is of the form $(0, 1, 1, \ldots, 1)$.

We call a sequence $\mathbf{d}$ satisfying the conditions 1–3 a *suitable sequence*. The inequality $\mathbf{d} > \mathbf{0}$ implies that $\mathrm{H}^1(E, \mathcal{V}(\mathbf{d}, m, \lambda)) = 0$, so the vector bundles $\mathcal{V}(\mathbf{d}, m, \lambda)$ with suitable $\mathbf{d}$ satisfy Kahn's conditions. Moreover, in this case $Q = \mathrm{H}^1(E, \mathcal{V}_\mathbf{d}(Z)) \simeq \mathrm{H}^1(E, \mathcal{V}_{\mathbf{d}-\mathbf{b}^r})$ is flat over $\mathbf{R} = \Bbbk[x, x^{-1}]$ if $\mathbf{d} \neq \mathbf{b}$. If $\mathbf{d} = \mathbf{b}$, Q is no more flat, but $Q[(x-1)^{-1}]$ is flat over $\mathbf{R}' = \Bbbk[x, x^{-1}, (x-1)^{-1}]$. Set $\mathcal{V}'_\mathbf{b} = \mathcal{V}_\mathbf{b} \otimes_\mathbf{R} \mathbf{R}'$ and $n_\mathbf{d} = \sum_{i=1}^{rs}(d_i - b_i + 1)^+ - \theta(\mathbf{d} - \mathbf{b}^r)$. Note that $n_\mathbf{d} = 0$ if and only if every positive part of $\mathbf{d} - \mathbf{b}^r$ contains 1 at most once, all other entries of it being 0 (for instance $\mathbf{d} = \mathbf{b}$). Define, for each suitable sequence $\mathbf{d}$, a family $\mathcal{E}_\mathbf{d}$ of vector bundles over E based on $\mathbf{R}$ if $\mathbf{d} \neq \mathbf{b}$ and on $\mathbf{R}'$ if $\mathbf{d} = \mathbf{b}$. Namely,

$$\mathcal{E}_\mathbf{d} = \mathcal{V}_\mathbf{d} \oplus n_\mathbf{d}\mathcal{O}_E \otimes \mathbf{R}, \text{ if } \mathbf{d} \neq \mathbf{b};$$

$$\mathcal{E}_\mathbf{b} = \mathcal{V}'_\mathbf{b}.$$

These families satisfy the conditions of Corollary 11.3, hence can be lifted to full families $\mathcal{F}_\mathbf{d}$ of vector bundles over X, which give rise to families $\mathcal{M}_\mathbf{d} = \pi_*\mathcal{F}_\mathbf{d}$ of Cohen–Macaulay $\mathbf{A}$-modules. Now Theorems 8.1 and 11.2 directly imply the following description of Cohen–Macaulay $\mathbf{A}$-modules.

Theorem 11.7. *If $\mathbf{A}$ is a cusp singularity, all indecomposable $\mathbf{A}$-modules are:*

- $\mathbf{A}$, *of rank 1;*
- $\mathcal{M}_\mathbf{d}(m, \lambda) = \mathcal{M}_\mathbf{d} \otimes_\mathbf{R} \mathbf{R}/(x - \lambda)^m$, *of rank $m(r + n_\mathbf{d})$, where $\mathbf{d}$ is a suitable sequence and $\lambda \neq 1$ if $\mathbf{d} = \mathbf{b}$;*
- $\mathcal{M}_\mathbf{b}(m, 1)$, *of rank $m + 1$, where $R_E\mathcal{M}_\mathbf{b}(m, 1) \simeq \mathcal{V}(\mathbf{b}, m, 1) \oplus \mathcal{O}_E$.*

In particular $\mathbf{A}$ is Cohen–Macaulay tame unbounded.

$\square$

12. Cohen–Macaulay types: Q-elliptic case

Using Corollary 9.2 we can extend the results of Section 11 to a wider class of surface singularities.

Definition 12.1.   1. A surface singularity $\mathbf{A}$ will be called *Q-Gorenstein*, if the order g of its canonical divisor K in the Picard group is finite and prime to char $\Bbbk$.

Just as in the proof of Theorem 9.3, one can construct the *Gorenstein cover* $\mathbf{B} = \bigoplus_{i=1}^{g} J^{-i}t^i$, where J is an ideal of class K and $t^n = \alpha^{-1}$ such that $J^{-n} = \alpha\mathbf{A}$.

2. A Q-Gorenstein singularity $\mathbf{A}$ is called
   - *Q-elliptic* if its Gorenstein cover is minimally elliptic;
   - *simple Q-elliptic* if its Gorenstein cover is simple elliptic;
   - *Q-cusp* if its Gorenstein cover is a cusp singularity.

Note that if $\mathbf{A}$ is Q-Gorenstein and $\mathbf{B}$ is its Gorenstein cover, the cyclic group of order g acts on $\mathbf{B}$ so that $\mathbf{A} = \mathbf{B}^G$ and the extension $\mathbf{A} \subseteq \mathbf{B}$ is unramified in codimension 1 (cf. the proof of Theorem 9.3). Therefore Theorem 11.5 and Corollary 9.2 immediately imply

Corollary 12.2. *A Q-elliptic singularity is*

- *Cohen–Macaulay* tame bounded *if it is simple Q-elliptic;*
- *Cohen–Macaulay* tame unbounded *if it a Q-cusp singularity;*
- *Cohen–Macaulay* wild *otherwise.*

Remark 12.3. If $\Bbbk = 0$, simple Q-elliptic and Q-cusp singularities arise as the so called *log-canonical* singularities [**Kaw**]. Recall that a normal surface singularity with the minimal resolution $\pi : X \to S$ is called *log-canonical* if $K_X = \pi^* K_S + \sum_{i=1}^{s} a_i E_i$ with $a_i \geq -1$, where $\pi^* K_S$ denotes the *numerical pullback* of K_S. The latter means that $\pi^* K_S = K'_S + \sum_{i=1}^{s} k_i E_i$, where K'_S is the strict transform of K_S (cf. [**Ha**, Section II.7]) and $k_i \in \mathbb{Q}$ are so chosen that $(\pi^* K_S, E_i) = 0$ for all i. In other words, K_X coincides with K_S outside E, while $(K_X.E_j) = \sum_{i=1}^{s} a_i (E_i.E_j)$ for all j. It is proved in [**Kaw**, Theorem 9.6] that, if char $\Bbbk = 0$, a log-canonical singularity is always Q-Gorenstein; moreover, it is either quotient, or simple Q-elliptic, or Q-cusp.

We shall describe Cohen–Macaulay modules over Q-cusp singularities. Namely, let $\mathbf{B}$ be a cusp singularity, $T = \operatorname{Spec} \mathbf{B}$, $\phi : Y \to T$ be the minimal resolution of T, and $F = \bigcup_{i=1}^{t} F_i$ be the exceptional curve of ϕ, where F_i are the irreducible components of F so arranged that $F_i \cap F_{i+1} = \emptyset$ (as usually we set $F_{t+i} = F_i$ and use analogous identification everywhere). Suppose that char $\Bbbk \neq 2$ and the group $G = \{1, \sigma\}$ of order 2 acts on $\mathbf{B}$ so that the lifting of this action onto Y is free outside F and reverse the orientation of F (Q-cusp singularities fit this situation: it can be shown just as in [**Kaw**, Theorem 9.6]). Then σ induces a reflection of $\Delta(F)$. If we choose F_1 to be its fixed component, σ maps F_i onto F_{t+2-i} with the coordinate transformation $x \mapsto 1/x$. It induces the action of σ on vector bundles over F such that $\mathcal{V}(\mathbf{d}, m, \lambda)^\sigma \simeq \mathcal{V}(\mathbf{d}^\sigma, m, 1/\lambda)$, where $(d_1, d_2, \ldots, d_{rt})^\sigma = (d_1, d_{rt}, d_{rt-1}, \ldots, d_2)$. Therefore its action on Cohen–Macaulay $\mathbf{B}$ modules is (using notation of Theorem 11.7):

$$\mathcal{M}_{\mathbf{d}}(m, \lambda)^\sigma \simeq \mathcal{M}_{\mathbf{d}^\sigma}(m, 1/\lambda), \ \mathbf{B}^\sigma \simeq \mathbf{B} \ \text{ and } \ M_n^\sigma \simeq M_n$$

(since necessarily $\mathbf{b}^\sigma = \mathbf{b}$). But it is known from generalities about group actions that the restriction onto $\mathbf{A}$ of an indecomposable $\mathbf{B}$-module M is indecomposable if $M^\sigma \not\simeq M$ and decomposes $M = M' \oplus M''$, where M', M'' are indecomposable and non-isomorphic, if $M^\sigma \simeq M$. Moreover, if N is another module, $N \not\simeq M$ and $N \not\simeq M^\sigma$, then M^σ and N^σ have no common direct summands. Denote by $\mathcal{N}_{\mathbf{d}}$ the following families of Cohen–Macaulay $\mathbf{A}$-modules:

- the restriction of $\mathcal{M}_{\mathbf{d}}$ onto $\mathbf{A}$ if $\mathbf{d}^\sigma$ does not coincide with any t-shift of $\mathbf{d}$;
- the restriction of $\mathcal{M}_{\mathbf{d}}[(x^2 - 1)^{-1}]$ onto $\mathbf{A}$ if $\mathbf{d}^\sigma$ is a t-shift of $\mathbf{d}$,

where $\mathbf{R} = \Bbbk[x, x^{-1}]$. For any sequence $\mathbf{d}$ such that $\mathbf{d}^\sigma$ coincides with a t-shift of $\mathbf{d}$ denote by $N'_{\mathbf{d}}(m, \pm 1)$ and $N''_{\mathbf{d}}(m, \pm 1)$ the indecomposable direct summands of $\mathcal{M}_{\mathbf{d}}(m, \pm 1)$. Note that always $\mathbf{b}^\sigma = \mathbf{b}$. The previous observations together with Theorem 11.7 imply

Theorem 12.4. *In the above situation all indecomposable Cohen–Macaulay* $\mathbf{A}$*-modules are:*

- $\mathcal{N}_{\mathbf{d}}(m, \lambda) = \mathcal{N}_{\mathbf{d}} \otimes_{\mathbf{R}} \mathbf{R}/(x - \lambda)^m$, *where* $\lambda \neq 0$, *and* $\lambda \neq \pm 1$ *if* $\mathbf{d}^\sigma$ *is a t-shift of* $\mathbf{d}$;
- $\mathcal{N}'_{\mathbf{d}}(m, \pm 1)$ *and* $\mathcal{N}''_{\mathbf{d}}(m, \pm 1)$ *for such* $\mathbf{d}$ *that* $\mathbf{d}^\sigma$ *is a t-shift of* $\mathbf{d}$;
- $\mathbf{A}$ *and* $\mathbf{B}^- = \{\, b \in \mathbf{B} \,|\, \sigma(b) = -b \,\}$.

Here $\mathbf{d}$ always denotes a suitable sequence.

13. APPLICATION TO HYPERSURFACES AND CURVES.

We can apply the results of Section 11 to *hypersurface singularities*, i.e. rings of the shape $\mathbf{A} = \Bbbk[[x_1, x_2, \ldots, x_n]]/(f)$, using the results of Knörrer [**Kn**] (see also [**Yo**]) on the relations between Cohen–Macaulay modules over $\mathbf{A}$ and over its *suspension* $\mathbf{A}^\sharp = \Bbbk[[x_1, x_2, \ldots, x, z]]/(f + z^2)$. (In this section we suppose that char $\Bbbk \neq 2$.) Namely, for every $\mathbf{A}$-module M denote by syz M its first syzygy as of $\mathbf{A}$-module and by ΩM its first syzygy as of $\mathbf{A}^\sharp$-module. For every $\mathbf{A}^\sharp$-module N denote by res N the $\mathbf{A}$-module N/zN. These operations map Cohen–Macaulay modules to Cohen–Macaulay ones, so we consider them as functors between categories of Cohen–Macaulay $\mathbf{A}$- and $\mathbf{A}^\sharp$-modules. As $\mathbf{A}$ is Gorenstein, syz can be considered as an automorphism of the *stable category* $\underline{\mathrm{CM}}(\mathbf{A})$, which is the factor of $\mathrm{CM}(\mathbf{A})$ modulo free modules. We also denote by σ the automorphism of $\mathbf{A}^\sharp$ mapping z to $-z$ and leaving all x_i fixed, and by N^σ the $\mathbf{A}^\sharp$-module obtained from N by twisting with σ. Then the results of [**Kn**] can be formulated as follows.

Theorem 13.1.   1. *Let* $M \not\simeq \mathbf{A}$ *be an indecomposable Cohen–Macaulay* $\mathbf{A}$*-module.*
   (a) *If* $M \not\simeq \mathrm{syz}\, M$, *the module* ΩM *is indecomposable.*
   (b) *If* $M \simeq \mathrm{syz}\, M$, $\Omega M \simeq \Omega_1 M \oplus \Omega_2 M$, *where* $\Omega_1 M$ *and* $\Omega_2 M$ *are indecomposable and non-isomorphic.*
   Every indecomposable Cohen–Macaulay $\mathbf{A}^\sharp$*-module is isomorphic to one of those described in items* (a),(b).
2. *Let* $N \not\simeq \mathbf{A}^\sharp$ *be an indecomposable Cohen–Macaulay* $\mathbf{A}^\sharp$*-module.*
   (a) *If* $N \not\simeq N^\sigma$, *the module* res N *is indecomposable.*
   (b) *If* $N \simeq N^\sigma$, res $N = \mathrm{res}_1 N \oplus \mathrm{res}_2 N$, *where* $\mathrm{res}_1 N$ *and* $\mathrm{res}_2 N$ *are indecomposable and non-isomorphic.*
   Every indecomposable Cohen–Macaulay $\mathbf{A}$*-module is isomorphic to one of those described in items* (a),(b).

We apply these results to singularities of type T_{pqr}. Namely, denote by $\mathbf{T}_{pqr}$ the factor

$$\Bbbk[[x, y, z]]/(x^p + y^q + z^r + \lambda xyz), \quad \text{where} \ \ 1/p + 1/q + 1/r \leq 1 \ \ \text{and} \ \ \lambda \neq 0$$

(we may suppose that $p \geq q \geq r$). Moreover, we demand this singularity to be *isolated*, which imposes restrictions on λ in the *quasi-homogeneous* cases, when $(p, q, r) \in \{\, (2, 3, 6), (2, 4, 4), (3, 3, 3) \,\}$. Note that in all other cases the isomorphism class of $\mathbf{T}_{pqr}$ does not depend on λ. A hypersurface singularity that is an (iterated) suspension of $\mathbf{T}_{pqr}$, i.e.

$$\Bbbk[[x, y, z, t_1, t_2, \ldots, t_m]]/\left(x^p + y^q + z^r + \lambda xyz + \sum_{i=1}^{m} t_i^2 \right)$$

is called *a singularity of type* T_{pqr}. It is known [**Lau**] that the surface singularities $\mathbf{T}_{pqr}$ are simple elliptic in quasi-homogeneous case and cusp singularities in all other cases. Therefore they are tame, so Theorem 13.1 implies

Corollary 13.2. *All hypersurface singularities of type* T_{pqr} *are Cohen–Macaulay tame.*

Unfortunately, we do not have precise formulae for syzygies of $\mathbf{T}_{pqr}$-modules, so we cannot give an explicit description of Cohen–Macaulay modules over their suspensions.

We can also use Knörrer's correspondence to obtain a description of Cohen–Macaulay modules over curve singularities of types T_{pq}, i.e. the factors

$$\mathbf{T}_{pq} = \Bbbk[[x,y]]/(x^p + y^q + \lambda x^2 y^2), \quad \text{where} \quad 1/p + 1/q \le 1/2 \quad \text{and} \quad \lambda \ne 0$$

(again in quasi-homogeneous cases, when $(p,q) \in \{(3,6),(4,4)\}$, some conditions must be imposed on λ in order this singularity to be isolated). To do it one only has to note that the suspension of $\mathbf{T}_{pq}$ is isomorphic to a surface singularities $\mathbf{T}_{pq2}$ (change z to $z + \lambda xy/2$). Moreover, in this case one can explicitly calculate the action of σ on the minimal resolution of $\mathbf{T}_{pq2}$, thus get an explicit list of indecomposable $\mathbf{T}_{pq}$-modules (cf. [**DGK**] for details). It accomplishes the study of Cohen–Macaulay modules over tame curve singularities, filling the flaw in [**DG1**], where no explicit description of $\mathbf{T}_{pq}$-modules was obtained (their tameness was established using deformation theory).

14. Some conjectures and remarks

We end up with some conjectures and remarks (cf. [**DGK**]).

Conjecture 14.1. *In the following cases the ring* $\mathbf{A}$ *is Cohen–Macaulay wild:*
  1) $\mathbf{A}$ *is a surface singularity that is neither quotient, nor simple Q-elliptic, nor Q-cusp;*
  2) $\mathbf{A}$ *is a hypersurface singularity that is neither simple (i.e. of types* A-D-E [**AGV**]*) nor of type* T_{pqr}*;*
  3) $\mathbf{A}$ *is non-isolated and the dimension of its singular locus is greater than 1 (i.e.* $\mathbf{A}_{\mathfrak{p}}$ *is not regular for some prime ideal* $\mathfrak{p}$ *of depth 2).*

If this conjecture is true, we shall have a complete description of Cohen–Macaulay types of isolated surface and hypersurface singularities. The result is given in Table 1 (the conjectured cases marked with '?'). Unfortunately, we have now no further conjectures, not even examples, for non-isolated singularities with 1-dimensional singular locus. Probably, very few of them can be Cohen–Macaulay tame.

Remark 14.2. All known examples of Cohen–Macaulay tame unbounded singularities, in particular those from Table 1, are actually of exponential growth. It seems very plausible that it is always so. Nevertheless, just as in the case of finite dimensional algebras, it can only be shown *a posteriori*, when one has a description of modules. We do not see any "natural" way

Table 1.

Cohen–Macaulay types of singularities

CM type	curves	surfaces	hypersurfaces
finite	dominate A-D-E	quotient	simple (A-D-E)
tame bounded	dominate $\mathbf{T}_{pq}$ with $1/p + 1/q = 1/2$	simple Q-elliptic (only ?)	of type T_{pqr} with $1/p + 1/q + 1/r = 1$ (only ?)
tame unbounded	dominate $\mathbf{T}_{pq}$ with $1/p + 1/q < 1/2$	Q-cusp (only ?)	of type T_{pqr} with $1/p + 1/q + 1/r < 1$ (only ?)
wild	all other	all other ?	all other ?

to prove this conjecture without such calculations. That is why we prefer to say 'bounded' and 'unbounded' instead of more usual 'of polynomial growth' and 'of exponential growth.'

Remark 14.3. In the complex analytic case, Artin's Approximation Theorem [**Art**] implies that the list of Cohen–Macaulay modules remains the same if **A** denotes the ring of germs of analytic functions on a simple elliptic or cusp singularity. The lifting of families is more cumbersome. It is always possible if the base is a finite dimensional algebra. If it is an algebraic variety T, we can only claim that for each point $t \in T$ a lifting is possible over a neighbourhood U of t in T. It gives a lifting of an appropriate family to an etale covering $\widetilde{T}$ of T. If T is a smooth curve or surface, so is $\widetilde{T}$, therefore the results on tameness and wildness from Sections 11,12 remain valid. On the other hand, in the case of cusps it seems credible that the families $\mathcal{E}_{\mathbf{d}}$ from the proof of Theorem 11.5 can actually be lifted over T, just as in [**Kahn**] for simple elliptic case.

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