

MAXIMAL COHEN-MACAULAY MODULES OVER SURFACE SINGULARITIES

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ABSTRACT. This is a survey article about properties of Cohen-Macaulay modules over surface singularities. We discuss properties of the Macaulayfication functor, reflexive modules over simple, quotient and minimally elliptic singularities, geometric and algebraic McKay Correspondence. Finally, we describe matrix factorizations corresponding to indecomposable Cohen-Macaulay modules over the non-isolated singularities A_∞ and D_∞ .

1. INTRODUCTION AND HISTORICAL REMARKS

The study of Cohen-Macaulay modules over Noetherian local rings originates from the theory of integral representations of finite groups (which, on its side, grew up from a very classical problem of classification of crystallographic groups, considered by Bieberbach, Fedorov, Schoenflies and others at the end of the 19th century). Namely, let G be a finite group, p a prime number and $A = \mathbb{Z}_{(p)}$ be the ring of p -adic integers. Then the category of representations of the group G over the ring A is equivalent to the category of finitely generated $A[G]$ -modules, which are free as A -modules. This category is not abelian. However, it is extension-closed in $A[G] - \text{mod}$ and has Krull-Remak-Schmidt property. Although the ring $A[G]$ is in general not commutative, it is finite as a module over its center, i.e. it is a so-called *order*. A finitely-generated $A[G]$ -module which is free as A -module, is called a *lattice*.

If G is a finite commutative group, then the ring $A[G]$ is commutative of Krull dimension one. In this case, the category of lattices coincides with the category of *maximal Cohen-Macaulay modules* this article deals with.

In algebraic geometry, the property of an algebraic variety to be Cohen-Macaulay (respectively, Gorenstein) describes a restricted class of singular varieties, for which one has a duality theory in the way it exists for smooth varieties. For certain classes of local Noetherian rings of small dimensions the category of Cohen-Macaulay modules can be defined in a very simple way.

For example, if (A, \mathfrak{m}) is a reduced Noetherian local ring of Krull dimension one then the category of Cohen-Macaulay modules coincides with the category of torsion free modules. Note that the study of torsion free modules over curve singularities is important from the point of view of moduli problems (for example, to investigate the singularities of the compactified Jacobian of an irreducible projective curve, see [1] and [39]). If (A, \mathfrak{m}) is a *normal surface singularity*, then the Cohen-Macaulay modules over A are precisely the Noetherian reflexive modules, i.e. finitely generated A -modules such that $M \cong M^{**}$.

First results about the representation type of the category of Cohen-Macaulay modules were obtained by Drozd-Roiter [34] and Jacobinski [56] in the 60-ies in the framework of integral representations of finite groups. A suggestion to study homological properties of the category of Cohen-Macaulay modules over a Gorenstein local ring of any Krull dimension as well as the first fundamental results going in this direction (existence of almost split sequences, Serre duality in the stable category) are due to Auslander [7], see also [8]. These ideas have found its further development in a work of Buchweitz [20], who suggested to study the stable category of Cohen-Macaulay modules over a Gorenstein local ring as a triangulated category, see also [76].

In 1978 Herzog has shown that the two-dimensional quotient singularities have finitely many indecomposable Cohen-Macaulay modules [50]. This results was a starting point and a motivation for a whole bunch of interesting results dedicated to the so-called McKay correspondence. This

theory can be divided (although a little bit artificially) into two parts: geometric (due to Artin, Esnault, Gonzalez-Springer, Knörrer, Verdier and others) and algebraic (Auslander, Reiten and others). A combination of both approaches yields a construction of explicit bijections between the set of isomorphism classes of irreducible representations of a finite subgroup $G \subset \mathrm{SL}(2, \mathbb{C})$, the irreducible components of the exceptional divisor of a minimal resolution of $\mathrm{Spec}(\mathbb{C}[[x, y]]^G)$ and the set of isomorphism classes of indecomposable Cohen-Macaulay modules over $\mathbb{C}[[x, y]]^G$. A discussion of these and related results is the main goal of our article.

The geometric approach of the study of Cohen-Macaulay modules over the quotient surface singularities was generalized by Kahn [60] to the case of *minimally elliptic* surface singularities. In the case of *simply elliptic* singularities he used a classification of Atiyah [6] of indecomposable vector bundles on elliptic curves to prove they have tame Cohen-Macaulay representation type. His programme was completed by Drozd, Greuel and Kashuba [33] who elaborated and extended Kahn's results on the case of *cuspidal singularities*.

Recently, Kapustin and Li [59] revived the interest of mathematicians to a study of Cohen-Macaulay modules over isolated hypersurface singularities because of their relations with the so-called *Landau-Ginzburg models* arising in string theory. In [77] Orlov established an exact connection between the derived category of coherent sheaves on a projective variety and the stable category of graded Cohen-Macaulay modules over its homogeneous coordinate algebra. This gave a rigorous meaning to the physical equivalence between Landau-Ginzburg models and non-linear sigma models, predicted by string theorists a long time ago. Another interesting application of Cohen-Macaulay modules to link invariants was discovered by Khovanov and Rozansky [63].

The plan of this article is the following. First, we recall some basic results from commutative algebra related to our study of Cohen-Macaulay modules. Then we discuss some “folklore results” on the Macaulayfication functor in the case of two-dimensional Cohen-Macaulay surface singularities. Next, we describe both the algebraic and the geometric approaches to McKay correspondence for quotient surface singularities as well as its generalization for simply elliptic and cusp singularities. Finally, we give a new proof of a result of Buchweitz, Greuel and Schreyer stating that the surface singularities A_∞ and D_∞ have countable (also called discrete) Cohen-Macaulay representation type.

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2. GENERALITIES ABOUT COHEN-MACAULAY MODULES

Let (A, \mathfrak{m}) be a Noetherian local ring, $k = A/\mathfrak{m}$ its residue field and $d = \mathrm{kr. dim}(A)$ its Krull dimension. Throughout the paper $A\text{-mod}$ denotes the category of Noetherian (i.e. finitely generated) A -modules, whereas $A\text{-Mod}$ stands for the category of all A -modules. In this section we collect some basic facts about Cohen-Macaulay modules.

Definition 2.1. For a Noetherian A -module M its *depth* is defined as

$$\mathrm{depth}_A(M) = \inf_{i \geq 0} \{i \mid \mathrm{Ext}_A^i(k, M) \neq 0\}.$$

Lemma 2.2. *Let M be a Noetherian A -module. Then we have:*

$$\mathrm{depth}_A(M) \leq \dim(M) := \mathrm{kr. dim}(A/\mathrm{ann}(M)),$$

in particular, the depth of a Noetherian module is always finite. Moreover, if $I = \mathrm{ann}_A(M)$ then $\mathrm{depth}_A(M) = \mathrm{depth}_{A/I}(M)$. Hence, in what follows, we shall write $\mathrm{depth}(M)$ for the depth of M , omitting the subscript.

For a proof of this lemma we refer to [19, Proposition 1.2.12].

Definition 2.3. A Noetherian A -module M is called *maximal Cohen-Macaulay* if $\text{depth}(M) = d$. In what follows, we simply call such modules Cohen-Macaulay.

The following properties follow immediately from the definition.

Proposition 2.4. Let $\text{CM}(A)$ denote the full subcategory of Cohen-Macaulay modules. Then

- The category $\text{CM}(A)$ is closed under extensions and direct summands.
- $\text{CM}(A) = \{M \in A\text{-mod} \mid \text{Ext}_A^i(T, M) = 0 \text{ for all } T \in \text{art}(A), 0 \leq i < d\}$, where $\text{art}(A)$ is the category of Noetherian modules of finite length.
- We have an exact functor $\text{CM}(A) \longrightarrow \text{CM}(\hat{A})$.
- If A is Artinian then $\text{CM}(A) = A\text{-mod}$.

The given definition of Cohen-Macaulay modules might look like a little bit artificial. The following lemma explains why this notion is natural in the context of the commutative algebra.

Lemma 2.5. Let (A, \mathfrak{m}) be a reduced Noetherian ring of Krull dimension one. Then an A -module M is Cohen-Macaulay if and only if it is torsion free.

Proof. Indeed, since the ring A is one-dimensional, the torsion part of a finitely generated A -module is zero-dimensional, hence of finite length. Moreover, a Noetherian A -module M is torsion free if and only if $\text{Hom}_A(k, M) = 0$, what is equivalent to the condition $\text{depth}(M) = 1$. \square

Definition 2.6. Let (A, \mathfrak{m}) be a Noetherian local ring of Krull dimension d .

- The ring A is Cohen-Macaulay if it is Cohen-Macaulay as an A -module.
- The ring A is Gorenstein if it is Cohen-Macaulay and $\text{Ext}_A^d(k, A) \cong k$.

Remark 2.7. A Cohen-Macaulay ring need not be reduced. For example, the local ring $k[[x, y]]/(y^2)$ is Cohen-Macaulay.

Example 2.8. Consider the following examples of Cohen-Macaulay and Gorenstein rings.

- A regular local ring is always Gorenstein.
- Let (A, \mathfrak{m}) be regular and $f \in \mathfrak{m}$, then A/f is Gorenstein, see [19, Proposition 3.1.19].
- Any reduced curve singularity is Cohen-Macaulay.
- The curve singularity $k[[x, y, z]]/(xy, xz, yz)$ is Cohen-Macaulay but not Gorenstein.
- The ring $k[[x, y, z]]/(xy, xz)$ is not Cohen-Macaulay.
- Let k be an algebraically closed field of characteristic zero, $G \subset \text{GL}_n(k)$ a finite subgroup without pseudo-reflections, then the invariant ring $k[[x_1, \dots, x_n]]^G$ is always Cohen-Macaulay, see Theorem 4.1 below. It is Gorenstein if and only if $G \subseteq \text{SL}_n(k)$, see [89] and [90].

Example 2.9. Let k be an algebraically closed field of characteristic zero, $A = k[[x, y]]/(y^2 - x^{n+1})$ a simple curve singularity of type A_n . Then the ideals $I_1 = \langle x, y \rangle, I_2 = \langle x^2, y \rangle, \dots, I_n = \langle x^n, y \rangle$ are Cohen-Macaulay as A -modules. The modules I_2, \dots, I_n are always indecomposable and $I_1 \cong R$, where R is the normalization of A . The module I_1 is indecomposable for even n and $I_1 \cong A/(y - x^{\frac{n+1}{2}}) \oplus A/(y + x^{\frac{n+1}{2}})$ for odd n . Moreover, this is a complete list of indecomposable A -modules, see for example [93].

Theorem 2.10 (Auslander-Buchsbaum formula). Let (A, \mathfrak{m}) be a Noetherian ring, M a Noetherian A -module such that $\text{pr. dim}(M) < \infty$. Then

$$\text{pr. dim}(M) + \text{depth}(M) = \text{depth}(A).$$

This theorem implies that the structure of Cohen-Macaulay modules over a regular local ring is particularly easy.

Corollary 2.11. *Let (A, \mathfrak{m}) be a regular local ring, then a Cohen-Macaulay module M is free.*

Proof. Since A is regular, $\text{kr. dim}(A) = \text{gl. dim}(A) = \text{depth}(A)$, see for example [84, Section IV.D]. In particular, the projective dimension of a finitely generated module M is always finite. From the condition $\text{depth}(M) = \text{kr. dim}(A)$ it follows $\text{pr. dim}(M) = 0$, i.e. M is projective, hence free. \square

Theorem 2.12 (Depth Lemma). *Let $0 \rightarrow M \rightarrow N \rightarrow K \rightarrow 0$ be a short exact sequence of A -modules. Then we have:*

- $\text{depth}(N) \geq \min(\text{depth}(M), \text{depth}(K))$
- *If $\text{depth}(N) > \min(\text{depth}(M), \text{depth}(K))$ then $\text{depth}(M) = \text{depth}(K) + 1$.*

A proof of this statement can be found for example in [19, Proposition 1.2.9]. As its immediate corollary we obtain.

Proposition 2.13. *Let (A, \mathfrak{m}) be a Cohen-Macaulay local ring of Krull dimension d and T any A -module. Then the d -th syzygy module $\text{syz}^d(T)$ is always Cohen-Macaulay.*

In the literature one can find other definitions of Cohen-Macaulay modules. One possibility to introduce them uses regular sequences.

Proposition 2.14. *Let (A, \mathfrak{m}) be a Noetherian local ring of Krull dimension d .*

- *A module M is Cohen-Macaulay if and only if there exists an M -regular sequence, i.e. a sequence $f_1, \dots, f_d \in \mathfrak{m}$ such that the homomorphism*

$$M/(f_1, \dots, f_i)M \xrightarrow{f_{i+1}} M/(f_1, \dots, f_i)M$$

is injective for all $0 \leq i < d$.

- *If A is Cohen-Macaulay, then any A -regular sequence is also M -regular.*

For a proof of this proposition see [19, Chapter 2].

The following result says that the category of Cohen-Macaulay modules behaves well under basic functorial operations of the commutative algebra.

Proposition 2.15. *Let (A, \mathfrak{m}) be a Noetherian local ring.*

- *For any non-zero divisor $f \in \mathfrak{m}$ we have a functor $\text{CM}(A) \rightarrow \text{CM}(A/f)$, $M \mapsto M/fM$.*
- *For any prime ideal $\mathfrak{p} \in \text{Spec}(A)$ we have a functor $\text{CM}(A) \rightarrow \text{CM}(A_{\mathfrak{p}})$, $M \mapsto M_{\mathfrak{p}}$.*

For a proof of this proposition, see [19, Theorem 2.1.3].

Definition 2.16. Let (A, \mathfrak{m}) be a Cohen-Macaulay ring of Krull dimension d . A Cohen-Macaulay module K is called canonical if $\text{Ext}_A^d(k, K) \cong k$ and $\text{inj. dim}(K) < \infty$.

Theorem 2.17. *Let (A, \mathfrak{m}) be a Cohen-Macaulay ring of Krull dimension d .*

- *If Cohen-Macaulay modules K and K' are canonical then $K \cong K'$. Hence, we may use the notation $K = K_A$.*
- *A Noetherian A -module K is canonical if and only if $\dim_k \text{Ext}_A^i(k, K) = \delta_{i,d}$.*
- *If A is Gorenstein then the regular module A is canonical.*
- *A canonical module exists in $\text{CM}(A)$ if and only if there exists a local Gorenstein ring (B, \mathfrak{n}) and a surjective ring homomorphism $B \rightarrow A$. In this case*

$$K_A \cong \text{Ext}_B^t(A, B),$$

where $t = \text{kr. dim}(B) - \text{kr. dim}(A)$.

- *If $B \rightarrow A$ is a homomorphism of local Cohen-Macaulay rings such that A is finite as B -module and B has a canonical module, then $K_A \cong \text{Ext}_B^t(A, K_B)$, where $t = \text{kr. dim}(B) - \text{kr. dim}(A)$. In particular, if k is a field, $A = k[[x_1, x_2, \dots, x_n]]/I$ a complete k -algebra and $B \rightarrow A$ its Noether normalization, then $K_A = \text{Hom}_B(A, B)$.*

For a proof of these results we refer to [19, Section 3.3].

The importance of the canonical module becomes clear after the following theorem.

Theorem 2.18. *Let (A, \mathfrak{m}) be a Cohen-Macaulay ring having a canonical module K .*

- *For any Cohen-Macaulay module M and any integer $t > 0$ we have: $\text{Ext}_A^t(M, K) = 0$. In particular, K is an injective object in the exact category of Cohen-Macaulay modules.*
- *For any Cohen-Macaulay module M the dual module $M^\vee = \text{Hom}_A(M, K)$ is again Cohen-Macaulay. Moreover, the canonical morphism $M \rightarrow M^{\vee\vee}$ is an isomorphism.*
- *The canonical module K behaves well under basic functorial operations of the commutative algebra. In particular,*
 - $\widehat{K}_A \cong K_{\widehat{A}}$,
 - *for any $\mathfrak{p} \in \text{Spec}(A)$ we have: $(K_A)_{\mathfrak{p}} \cong K_{A_{\mathfrak{p}}}$*
 - *for any non-zero divisor $f \in \mathfrak{m}$ we have: $K_A/fK_A \cong K_{A/f}$.*

For a proof of this theorem, see [19, Section 3.3].

By Theorem 2.17 the canonical module of a Cohen-Macaulay ring exists under very general assumptions. Nevertheless, there are examples of Noetherian local Cohen-Macaulay rings without the canonical module.

Example 2.19. In [40, Proposition 3.1] the authors construct an integral local ring (A, \mathfrak{m}) of Krull dimension one such that its completion \widehat{A} contains a minimal prime ideal \mathfrak{p} such that $\mathfrak{p}^2 = 0$ and $\mathfrak{p} \cong (A/\mathfrak{p})^n$ for $n \geq 2$. Since A is a one-dimensional integral domain, it is Cohen-Macaulay. Our goal is to show that A does not have the canonical module.

Indeed, assume K is the canonical module of A . Since the quotient ring $Q = Q(A)$ is a field, by Theorem 2.18 we have: $K \otimes_A Q \cong K_Q = Q$. Hence, K is isomorphic to an ideal in A . Moreover, the completion \widehat{A} is also Cohen-Macaulay and its canonical module exists and is an ideal. By [52, Korollar 6.7] the total ring of quotients $\widehat{Q} = Q(\widehat{A})$ is Gorenstein.

Let $\widetilde{\mathfrak{p}} = \widehat{Q} \otimes_{\widehat{A}} \mathfrak{p}$. Since localization is an exact functor, we have: $\widetilde{\mathfrak{p}}$ is a minimal prime ideal in \widehat{Q} , $\widetilde{\mathfrak{p}}^2 = 0$ and $\widetilde{\mathfrak{p}} \cong (\widehat{Q}/\widetilde{\mathfrak{p}})^n$. By [19, Lemma 1.2.19] the top of a Gorenstein Artinian ring is isomorphic to its socle. However, the top of \widehat{Q} contains the simple module $\widehat{Q}/\widetilde{\mathfrak{p}}$ with multiplicity one. However, the socle of \widehat{Q} contains a semi-simple submodule $\widetilde{\mathfrak{p}} = (\widehat{Q}/\widetilde{\mathfrak{p}})^n$. Hence, the ring \widehat{Q} can not be Gorenstein. Contradiction. \square

In what follows, we shall need the notion of the local cohomologies of a Noetherian module M .

Definition 2.20. Let (A, \mathfrak{m}) be a Noetherian local ring, then the functor $\Gamma_{\mathfrak{m}} : A\text{-mod} \rightarrow A\text{-Mod}$ is left exact:

$$\Gamma_{\mathfrak{m}}(M) = \varinjlim \text{Hom}_A(A/\mathfrak{m}^t, M) = \{x \in M \mid \mathfrak{m}^t x = 0 \text{ for some } t > 0\}.$$

By the definition, $H_{\mathfrak{m}}^i(M) := R^i \Gamma_{\mathfrak{m}}(M)$.

Remark 2.21. Since taking a direct limit preserves exactness, we have a functorial isomorphism:

$$H_{\mathfrak{m}}^i(M) \cong \varinjlim \text{Ext}_A^i(A/\mathfrak{m}^t, M).$$

Theorem 2.22. *Let (A, \mathfrak{m}) be a Noetherian local ring, M a Noetherian A -module, $t = \text{depth}(M)$ and $d = \dim(M)$. Then we have the following properties:*

- *All local cohomologies $H_{\mathfrak{m}}^i(M)$ are Artinian A -modules, $i \geq 0$.*
- *$H_{\mathfrak{m}}^i(M) = 0$ for $i < t$ and $i > d$. In particular, $H_{\mathfrak{m}}^i(M) = 0$ for $i > \text{kr.dim}(A)$ (Grothendieck's vanishing).*
- *$H_{\mathfrak{m}}^t(M) \neq 0$ and $H_{\mathfrak{m}}^d(M) \neq 0$ (Grothendieck's non-vanishing).*

For a proof of this theorem we refer to [19, Section 3.5] and [13, Chapter 6].

In particular, we obtain an alternative characterization of Cohen-Macaulay modules.

Corollary 2.23. *Let (A, \mathfrak{m}) be a Noetherian local ring of Krull dimension d , M a Noetherian A -module. Then M is Cohen-Macaulay if and only if $H_{\mathfrak{m}}^i(M) = 0$ for $i \neq d$.*

Using this characterization, the following lemma is easy to show.

Lemma 2.24. *Let $(A, \mathfrak{m}) \subset (B, \mathfrak{n})$ be a finite extension of local Noetherian rings, M a Noetherian B -module. Then M is Cohen-Macaulay over B if and only if it is Cohen-Macaulay over A .*

Proof. Note that the forgetful functor $B\text{-mod} \rightarrow A\text{-mod}$ is exact and $\Gamma_{\mathfrak{n}}(M) \cong \Gamma_{\mathfrak{m}}(M)$ as A -modules. Hence, $H_{\mathfrak{n}}^i(M) \cong H_{\mathfrak{m}}^i(M)$ as A -modules, what implies the claim. \square

Corollary 2.25. *Let $A = k[[x_1, x_2, \dots, x_n]]/I$ be Cohen-Macaulay of Krull dimension d and $B = k[[y_1, y_2, \dots, y_d]] \rightarrow A$ be its Noether normalization. Then a Noetherian A -module M is Cohen-Macaulay if and only if it is free as B -module.*

Proof. Indeed, by Corollary 2.11 a Cohen-Macaulay B -module is free. It remains to apply Lemma 2.24. \square

Let $E = E(k)$ be the injective envelope of the residue field $k = A/\mathfrak{m}$, $\mathbb{D} = \text{Hom}_A(-, E(k)) : A\text{-Mod} \rightarrow A\text{-Mod}$.

Theorem 2.26 (Matlis Duality). *Let (A, \mathfrak{m}) be a local Noetherian ring, $\mathbf{art}(A)$ the category of Noetherian A -modules of finite length and $\mathbf{Art}(A)$ the category of Artinian A -modules.*

- *The functor $\mathbb{D} : \mathbf{art}(A) \rightarrow \mathbf{art}(A)$ is exact and fully faithful, moreover, $\mathbb{D}^2 \cong \text{Id}$.*
- *If the ring A is complete, then $\mathbb{D} : A\text{-mod} \rightarrow \mathbf{Art}(A)$ and $\mathbb{D} : \mathbf{Art}(A) \rightarrow A\text{-mod}$ are exact fully faithful functors and $\mathbb{D}^2 \cong \text{Id}$.*

For a proof of this theorem we refer to [19, Proposition 3.2.12 and Theorem 3.2.13] and [13, Section 10.2].

The duality functor \mathbb{D} enters in the formulation of the following fundamental result of the commutative algebra.

Theorem 2.27 (Grothendieck's Local Duality). *Let (A, \mathfrak{m}) be a Noetherian local ring with a canonical module K_A . Then there exists an isomorphism of δ -functors*

$$\phi_i : H_{\mathfrak{m}}^i \cong \mathbb{D} \text{Ext}_A^{d-i}(-, K_A) : A\text{-mod} \rightarrow \mathbf{Art}(A), \quad i \geq 0.$$

A proof of this theorem can be found in [13, Section 11.2.8].

Corollary 2.28. *Let (A, \mathfrak{m}) be a Cohen-Macaulay local ring having a canonical module K . Then the Cohen-Macaulay A -modules are precisely those Noetherian modules M for which the complex $\text{RHom}_A(M, K) \in \text{Ob}(D^+(A\text{-mod}))$ has exactly one non-vanishing cohomology.*

It turns out that in the case of *hypersurface* singularities the Cohen-Macaulay modules have the following convenient description.

Proposition 2.29 (Eisenbud [35]). *Let (S, \mathfrak{n}) be a regular local ring of Krull dimension $d \geq 2$, $f \in \mathfrak{n}^2$, $A = S/f$ and M a Cohen-Macaulay A -module without free direct summands. Then M considered as an A -module, has a 2-periodic minimal free resolution.*

Proof. Since the ring S is regular and $\text{depth}(M) = \text{kr. dim}(A) = d-1$, by the Auslander-Buchsbaum formula, we get: $\text{pr. dim}_S(M) = 1$. Hence, M viewed as an S -module has a free resolution

$$0 \rightarrow S^n \xrightarrow{\alpha} S^n \rightarrow M \rightarrow 0,$$

where $\alpha \in \text{Mat}_n(\mathfrak{n})$. Since M is annihilated by f , we have the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S^n & \xrightarrow{\alpha} & S^n & \longrightarrow & M & \longrightarrow & 0 \\ & & \cdot f \downarrow & \swarrow \beta & \downarrow \cdot f & & \downarrow \cdot f & & \\ 0 & \longrightarrow & S^n & \xrightarrow{\alpha} & S^n & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

where β is a chain homotopy $(\cdot f, \cdot f) \sim 0$. Hence, we have found a matrix $\beta \in \text{Mat}_n(S)$ such that $\alpha\beta = \beta\alpha = fI_n$. In particular, it implies that $\beta \in \text{Mat}_n(\mathfrak{n})$.

Let $\bar{\alpha}$ and $\bar{\beta}$ be the images of α and β in $\text{Mat}_n(\mathfrak{m})$, where \mathfrak{m} is the maximal ideal of A , then the sequence $A^n \xrightarrow{\bar{\alpha}} A^n \rightarrow M \rightarrow 0$ is exact. Moreover, we claim that

$$\dots \xrightarrow{\bar{\alpha}} A^n \xrightarrow{\bar{\beta}} A^n \xrightarrow{\bar{\alpha}} A^n \rightarrow M \rightarrow 0$$

is a minimal free resolution of M . Indeed, let $\bar{x} \in A^n$ be such that $\bar{\alpha}\bar{x} = 0$. This means that there exists an element $y \in S^n$ such that $\alpha x = y$, where x is some preimage of \bar{x} in S^n . This implies: $f x = \beta \alpha x = \beta y$, hence $x = \beta y$ and $\bar{x} = \bar{\beta}\bar{y}$. We have shown: $\ker(\alpha) = \text{im}(\beta)$. The remaining part is analogous. \square

Remark 2.30. The pair (α, β) is a *matrix factorization*, which corresponds to the Cohen-Macaulay module M . In these terms one can write: $M = M(\alpha, \beta)$.

Example 2.31. Let k be a field of characteristic zero and $A = k[[x, y]]/(y^2 - x^{2n})$ a simple curve singularity of type A_{2n+1} . Then the module $M_{\pm} = A/(y \pm x^n)$ has the minimal free resolution

$$\dots \longrightarrow A \xrightarrow{y \pm x^n} A \xrightarrow{y \mp x^n} A \longrightarrow M_{\pm} \longrightarrow 0.$$

The rank one modules $I_i = \langle y, x^i \rangle$, $2 \leq i \leq 2n-1$, have minimal free resolutions of the following form:

$$\dots \longrightarrow A^2 \xrightarrow{\begin{pmatrix} y & x^{2n-i} \\ x^i & y \end{pmatrix}} A^2 \xrightarrow{\begin{pmatrix} y & x^{2n-i} \\ x^i & y \end{pmatrix}} A^2 \longrightarrow I_i \longrightarrow 0.$$

\square

Recall the following standard result from the commutative algebra.

Lemma 2.32. *Assume a Noetherian ring A is reduced. Then the total ring of fractions $Q(A)$ is isomorphic to a direct product of fields $Q_1 \times Q_2 \times \dots \times Q_t$, where $Q_i = Q(A/\mathfrak{p}_i)$ is the field of fractions of A/\mathfrak{p}_i and $\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_t$ are the minimal prime ideals of A .*

A proof of this lemma can be found in [17, Proposition 10, Section 2.5, Chapter IV] or [26, Proposition 1.4.27 and Theorem 1.5.20].

Definition 2.33. Let A be a reduced Noetherian ring, $Q = Q_1 \times Q_2 \times \dots \times Q_t$ its total ring of fractions and M a Noetherian A -module. Then the *multi-rank* of M is the tuple of non-negative integers (r_1, r_2, \dots, r_t) such that $Q \otimes_A M \cong Q_1^{r_1} \times Q_2^{r_2} \times \dots \times Q_t^{r_t}$.

Let (S, \mathfrak{n}) be a regular local ring of Krull dimension $d \geq 2$, $A = S/f$ a reduced hypersurface singularity. Since a regular ring is factorial (see [84, Corollary IV.D.4]), we can write $f = f_1 f_2 \dots f_t$, where all elements $f_i \in \mathfrak{n}$ are irreducible. In these terms

$$Q(A) \cong Q(S/f_1) \times Q(S/f_2) \times \dots \times Q(S/f_t).$$

Lemma 2.34. *Let (S, \mathfrak{n}) be a regular local ring, $A = S/f$ a reduced hypersurface singularity, M a Cohen-Macaulay module over A and $(\alpha, \beta) \in \text{Mat}_n(\mathfrak{n})$ the corresponding matrix factorization. Then $\det(\alpha) = u f_1^{r_1} f_2^{r_2} \dots f_t^{r_t}$, where u is an invertible element in S and (r_1, r_2, \dots, r_t) is the multi-rank of M .*

Proof. Since $\alpha\beta = \beta\alpha = f^n I_n$, it follows that $\det(\alpha) = u f_1^{r_1} f_2^{r_2} \dots f_t^{r_t}$ for some unit $u \in S$ and non-negative integers r_1, r_2, \dots, r_t .

By the definition of matrix factorizations, an A -module M viewed as an S -module has a free resolution $0 \rightarrow S^n \xrightarrow{\alpha} S^n \rightarrow M \rightarrow 0$. For any $i \in \{1, 2, \dots, t\}$ consider the prime ideal $\mathfrak{p}_i = (f_i) \subseteq S$. Then the localization $S_{\mathfrak{p}_i}$ is a discrete valuation ring and we have a free resolution

$$0 \rightarrow S_{\mathfrak{p}_i}^n \xrightarrow{\alpha_{\mathfrak{p}_i}} S_{\mathfrak{p}_i}^n \rightarrow M_{\mathfrak{p}_i} \rightarrow 0.$$

Since $(\det(\alpha))_{\mathfrak{p}_i} = \det(\alpha_{\mathfrak{p}_i})$, it is easy to see that $\text{length}_{A_{\mathfrak{p}_i}}(M_{\mathfrak{p}_i}) = r_i$. Since the residue field $k(\mathfrak{p}_i)$ of the ring $S_{\mathfrak{p}_i}$ is isomorphic to $Q(S/f_i)$, it remains to note that

$$k(\mathfrak{p}_i)^{r_i} \cong M_{\mathfrak{p}_i} \otimes_{S_{\mathfrak{p}_i}} Q(S/f_i) \cong M \otimes_S Q(S/f_i) \cong M \otimes_A Q(S/f_i).$$

□

Definition 2.35. Let (A, \mathfrak{m}) be a Noetherian local ring. It is called an *isolated singularity* if for any $\mathfrak{p} \in \text{Spec}(A)$, $\mathfrak{p} \neq \mathfrak{m}$ the ring $A_{\mathfrak{p}}$ is regular. In particular, a regular local ring is an isolated singularity.

Lemma 2.36. Let k be a field and $A = k[[x_1, \dots, x_n]]/f$, where $f \in \mathfrak{m}^2$. Then A is an isolated singularity if and only if the Tyurina number

$$\tau(f) = \dim_k(k[[x_1, \dots, x_n]]/(f, j(f)))$$

is finite, where $j(f) = \langle \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \rangle_A$ is the Jacobi ideal of f .

Proof. First of all note that A is an isolated singularity if and only if $A \otimes_k \bar{k}$ is, where \bar{k} is the algebraic closure of k . Hence, we may without loss of generality assume k is algebraically closed.

The singular locus of $\text{Spec}(A)$ viewed as a subscheme of $\text{Spec}(k[[x_1, x_2, \dots, x_n]])$ is $V(f, j(f))$. The singularity A is isolated if and only if $V(f, j(f)) = (0, 0, \dots, 0)$. By Hilbert-Rückert's Nullstellensatz this is equivalent for the ideal $(f, j(f))$ to be (x_1, x_2, \dots, x_n) -primary in $k[[x_1, \dots, x_n]]$. The last condition is equivalent to the finiteness of $\tau(f)$. □

Remark 2.37. Let $A = k[[x_1, \dots, x_n]]/f$, where $f \in \mathfrak{m}^2$ and k be a field of characteristic zero. Then A is an isolated singularity if and only if the *Milnor number*

$$\mu(f) = \dim_k(k[[x_1, \dots, x_n]]/j(f))$$

is finite, see for example [45, Lemma 23].

Note that this is no longer true if the characteristic of k is positive. For example, let $\text{char}(k) = 3$ and $A = k[[x, y]]/(x^2 + y^3)$. Since $\tau(f) = 3$, A is an isolated singularity. However, $\mu(f) = \infty$.

Example 2.38. Consider the so-called $T_{2,3,\infty}$ hypersurface singularity $A = k[[x, y, z]]/(x^2 + y^3 + xyz)$. Then the singular locus of $\text{Spec}(A)$ is given by the ideal $I = (x, y)$. Since $V(I) = \text{Spec}(k[[t]])$, this singularity is not isolated.

Definition 2.39. Let (A, \mathfrak{m}) be a Cohen-Macaulay ring. A Noetherian module M is called *locally free on the punctured spectrum* if the $A_{\mathfrak{p}}$ -module $M_{\mathfrak{p}}$ is free for any prime ideal $\mathfrak{p} \neq \mathfrak{m}$.

The following lemma is straightforward.

Lemma 2.40. Let (A, \mathfrak{m}) be a Cohen-Macaulay isolated singularity. Then any Cohen-Macaulay A -module is locally free on the punctured spectrum.

Proof. Let M be a Cohen-Macaulay A -module, then for any $\mathfrak{p} \in \text{Spec}(A) \setminus \{\mathfrak{m}\}$ the ring $A_{\mathfrak{p}}$ is regular and the module $M_{\mathfrak{p}}$ is Cohen-Macaulay, hence free by Corollary 2.11. □

Definition 2.41. Let k be a field. Its *real valuation* is a function $\|\cdot\| : k \rightarrow \mathbb{R}_{\geq 0}$ such that

- $\|ab\| = \|a\| \cdot \|b\|$
- $\|a + b\| \leq \|a\| + \|b\|$.

For a formal power series $f = \sum_{\alpha \in \mathbb{N}^n} c_\alpha x^\alpha \in k[[x_1, x_2, \dots, x_n]]$ and $\varepsilon \in \mathbb{R}_{>0}$ we denote

$$\|f\|_\varepsilon := \sum_{\alpha \in \mathbb{N}^n} \|c_\alpha\| \varepsilon^{|\alpha|} \in \mathbb{R} \cup \{\infty\}.$$

A power series f is called *convergent* with respect to a valuation $\|\cdot\|$ if there exists $\varepsilon \in \mathbb{R}_{>0}$ such that $\|f\|_\varepsilon < \infty$. Let $k\{x_1, x_2, \dots, x_n\}$ denote the ring of convergent power series. A k -algebra A is called *analytic* if it is isomorphic to an algebra of the form $k\{x_1, x_2, \dots, x_n\}/I$.

Remark 2.42. If $\|a\| = 0$ for all $a \in k$ then the ring of convergent power series coincides with the ring of formal power series.

Definition 2.43. Let (A, \mathfrak{m}) be a Noetherian local ring and $k = A/\mathfrak{m}$ be its residue field. The ring A is called *Henselian* if for any polynomial $p = p(t) \in A[t]$ such that $p(t) \equiv \bar{p}_1(t)\bar{p}_2(t) \pmod{\mathfrak{m}}$, where $\bar{p}_1(t), \bar{p}_2(t)$ are coprime in $k[t]$, there exist polynomials $p_1(t), p_2(t) \in A[t]$ such that $p(t) = p_1(t)p_2(t)$ and $p_i \equiv \bar{p}_i \pmod{\mathfrak{m}}$ for $i = 1, 2$.

Theorem 2.44. *Let k be a field and $A = k\{x_1, x_2, \dots, x_n\}/I$ be a local analytic k -algebra. Then it is Henselian.*

A proof of this result can be found in [45, Theorem 1.17].

Let (A, \mathfrak{m}) be a Henselian Cohen-Macaulay local ring having a canonical module. First of all note that the category $A\text{-mod}$ of all Noetherian A -modules is a *Krull-Remak-Schmidt category*, see [25, Proposition 30.6] and [88, Theorem A.3]. Hence, the category $\text{CM}(A)$ of Cohen-Macaulay modules has Krull-Remak-Schmidt property, too.

The main goal of this article is to describe the surface singularities having finite, discrete and tame Cohen-Macaulay representation type. In particular, one can pose a question about the existence of almost split sequences.

Theorem 2.45 (Auslander). *Let (A, \mathfrak{m}) be a Henselian local ring having a canonical module, M a non-free indecomposable Cohen-Macaulay A -module, which is locally free on the punctured spectrum. Then there exists an almost split sequence*

$$0 \rightarrow \tau(M) \rightarrow N \rightarrow M \rightarrow 0$$

ending at M . Moreover, the category $\text{CM}(A)$ admits almost split sequences if and only if A is an isolated singularity.

For a proof of this Theorem, see [7] and [93, Chapter 3].

This theorem naturally raises the question about an explicit description of the Auslander-Reiten translation τ .

Definition 2.46. Let (A, \mathfrak{m}) be a local Cohen-Macaulay ring, M a Noetherian A -module and $G \xrightarrow{\varphi} F \rightarrow M \rightarrow 0$ a free presentation of M . The Auslander-transpose of M is defined via the following exact sequence:

$$0 \rightarrow M^* \rightarrow F^* \xrightarrow{\varphi^*} G^* \rightarrow \text{Tr}(M) \rightarrow 0.$$

In these terms, we have the following theorem.

Theorem 2.47 (Auslander). *Let (A, \mathfrak{m}) be a Henselian local ring of Krull dimension d , K the canonical A -module and M a non-free indecomposable Cohen-Macaulay A -module, which is locally free on the punctured spectrum. Then*

$$\tau(M) \cong \mathrm{syz}^d(\mathrm{Tr}(M))^\vee,$$

where $N^\vee = \mathrm{Hom}_A(N, K)$. If A is moreover Gorenstein, then $\tau(M) \cong \mathrm{syz}^{2-d}(M)$.

For a proof we refer to [7, Proposition 8.7, page 105] and [7, Proposition 1.3, page 205], see also [93, Proposition 3.11].

3. COHEN-MACAULAY MODULES OVER SURFACE SINGULARITIES

Throughout this section, let (A, \mathfrak{m}) be a *reduced* Cohen-Macaulay singularity of Krull dimension two, having a canonical module K_A . Let \mathcal{P} denote the set of prime ideals in A of height 1.

Lemma 3.1. *Let N be a Cohen-Macaulay A -module and M a Noetherian A -module. Then the A -module $\mathrm{Hom}_A(M, N)$ is Cohen-Macaulay.*

Proof. From a free presentation $A^n \xrightarrow{\varphi} A^m \rightarrow M \rightarrow 0$ of M we obtain an exact sequence:

$$0 \rightarrow \mathrm{Hom}_A(M, N) \rightarrow N^m \xrightarrow{\varphi^*} N^n \rightarrow \mathrm{coker}(\varphi^*) \rightarrow 0.$$

Since $\mathrm{depth}_A(N) = 2$, applying the Depth Lemma twice we obtain:

$$\mathrm{depth}_A(\mathrm{Hom}_A(M, N)) \geq 2.$$

Hence, $\mathrm{Hom}_A(M, N)$ is Cohen-Macaulay. \square

Proposition 3.2. *In the notations of this section, the canonical embedding functor $\mathrm{CM}(A) \rightarrow A\text{-mod}$ has a left adjoint*

$$M \mapsto M^\dagger := M^{\vee\vee} = \mathrm{Hom}_A(\mathrm{Hom}_A(M, K_A), K_A).$$

Proof. Note that for any Noetherian module M the A -module M^\dagger is Cohen-Macaulay by Lemma 3.1. Next, for any Noetherian A -module M there exists an exact sequence

$$0 \rightarrow \mathrm{tor}(M) \rightarrow M \xrightarrow{i_M} M^\dagger \rightarrow T \rightarrow 0,$$

where $\mathrm{tor}(M)$ is the torsion part of M and i_M is the canonical morphism.

- Let us first assume M to be torsion free, so we have a short exact sequence

$$0 \rightarrow M \xrightarrow{i_M} M^\dagger \rightarrow T \rightarrow 0.$$

Since for any $\mathfrak{p} \in \mathcal{P}$ the ring $A_{\mathfrak{p}}$ is reduced and Cohen-Macaulay of Krull dimension one and the module $M_{\mathfrak{p}}$ is torsion free, it is Cohen-Macaulay. By Theorem 2.18 we have: $(K_A)_{\mathfrak{p}} \cong K_{A_{\mathfrak{p}}}$, hence the morphism $(i_M)_{\mathfrak{p}} : M_{\mathfrak{p}} \rightarrow (M^\dagger)_{\mathfrak{p}}$ is an isomorphism. This means that for any $\mathfrak{p} \in \mathcal{P}$ we have: $T_{\mathfrak{p}} = 0$, hence T is a finite length module. If N is a Cohen-Macaulay module, it follows from the exact sequence

$$\mathrm{Hom}_A(T, N) \rightarrow \mathrm{Hom}_A(M^\dagger, N) \rightarrow \mathrm{Hom}_A(M, N) \rightarrow \mathrm{Ext}_A^1(T, N)$$

and equalities $\mathrm{Hom}_A(T, N) = 0 = \mathrm{Ext}_A^1(T, N)$, that the canonical morphism $(i_M)_* : \mathrm{Hom}_A(M^\dagger, N) \rightarrow \mathrm{Hom}_A(M, N)$ is an isomorphism.

- Now let M be an arbitrary A -module. Since a Cohen-Macaulay module N is always torsion-free, we have a canonical isomorphism

$$\mathrm{Hom}_A(M/\mathrm{tor}(M), N) \cong \mathrm{Hom}_A(M, N).$$

In the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_A((M/\mathrm{tor}(M))^\dagger, N) & \longrightarrow & \mathrm{Hom}_A(M/\mathrm{tor}(M), N) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_A(M^\dagger, N) & \longrightarrow & \mathrm{Hom}_A(M, N) \end{array}$$

both vertical arrows and the first horizontal arrow are isomorphisms. Hence, the canonical morphism $(i_M)_* : \mathrm{Hom}_A(M^\dagger, N) \rightarrow \mathrm{Hom}_A(M, N)$ is an isomorphism for any A -module M and a Cohen-Macaulay module N . \square

Definition 3.3. Let M and N be two Noetherian A -modules. A morphism $f : M \rightarrow N$ is called *birational isomorphism* if the induced map $1 \otimes f : Q(A) \otimes M \rightarrow Q(A) \otimes N$ is an isomorphism of $Q(A)$ -modules.

Lemma 3.4. For a Noetherian module M the canonical morphism $M \rightarrow M^\dagger$ is a birational isomorphism.

Proof. Indeed, since in the exact sequence

$$0 \rightarrow \mathrm{tor}(M) \rightarrow M \xrightarrow{i_M} M^\dagger \rightarrow T \rightarrow 0$$

the modules $\mathrm{tor}(M)$ and T are torsion Noetherian modules and $Q(A)$ is a flat A -module, the claim follows. \square

Lemma 3.5. A Noetherian module M is Cohen-Macaulay if and only if the canonical morphism $M \rightarrow M^\dagger$ is an isomorphism.

Proof. Since M^\dagger is always Cohen-Macaulay, one direction is clear. Assume now that M is Cohen-Macaulay. From the canonical isomorphism $\mathrm{End}_A(M) \cong \mathrm{Hom}_A(M^\dagger, M)$ we obtain a morphism $\pi_M : M^\dagger \rightarrow M$ such that $\pi_M \circ i_M = 1_M$. It is clear that π_M is an epimorphism. Since it is a birational isomorphism, the multi-rank of $\ker(\pi_M)$ is zero, hence it is a torsion module. But M is torsion free, hence π_M is a monomorphism. \square

Lemma 3.6. Let M and N be two Cohen-Macaulay A -modules. Then a morphism $f : M \rightarrow N$ is an isomorphism if and only if for all $\mathfrak{p} \in \mathcal{P}$ the morphism $f_{\mathfrak{p}} : M_{\mathfrak{p}} \rightarrow N_{\mathfrak{p}}$ is an isomorphism.

Proof. If $f_{\mathfrak{p}}$ is an isomorphism for all $\mathfrak{p} \in \mathcal{P}$, then f is a birational isomorphism, hence $\ker(f)$ has zero multi-rank. Since M is torsion free, $\ker(f) = 0$ and f is injective.

Next, consider the short exact sequence $0 \rightarrow M \xrightarrow{f} N \rightarrow T \rightarrow 0$. Since $T_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \in \mathcal{P}$, the module T is of finite length. Moreover, since $\mathrm{Hom}_A(T, K_A) = 0 = \mathrm{Ext}_A^1(T, K_A)$, the induced morphism $\mathrm{Hom}_A(N, K_A) \rightarrow \mathrm{Hom}_A(M, K_A)$ is an isomorphism, hence $f^\dagger : M^\dagger \rightarrow N^\dagger$ is an isomorphism, too. The claim follows now from the commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \downarrow & & \downarrow \\ M^\dagger & \xrightarrow{f^\dagger} & N^\dagger \end{array}$$

and the fact that the vertical arrows and f^\dagger are isomorphisms. \square

Proposition 3.7. Let (A, \mathfrak{m}) be a Noetherian local ring of Krull dimension two, which is Gorenstein in codimension one. Then for any Noetherian A -module M we have a functorial isomorphism $M^* \rightarrow M^\vee$, where $X^* = \mathrm{Hom}_A(X, A)$. In particular, $M^\dagger \simeq M^{**}$, so the Cohen-Macaulay modules over A are precisely the reflexive modules.

Proof. The ring A is Gorenstein in codimension one if and only if for all $\mathfrak{p} \in \mathcal{P}$ we have: $K_{A_{\mathfrak{p}}} \cong A_{\mathfrak{p}}$. Let $Q = Q(A)$ be the total ring of fractions of A . Since $Q = Q(A_{\mathfrak{p}})$ for $\mathfrak{p} \in \mathcal{P}$, by Theorem 2.18 we get: Q is a Gorenstein ring of Krull dimension zero and

$$K_A \otimes Q \cong K_Q \cong Q \cong A \otimes_A Q.$$

In particular, the modules A and K_A are birationally isomorphic. Moreover, for any birational isomorphism $j : A \rightarrow K_A$ and any Noetherian module M the induced morphism of Cohen-Macaulay modules $j_* : \mathrm{Hom}_A(M, A) \rightarrow \mathrm{Hom}_A(M, K_A)$ is an isomorphism in codimension one. By Lemma 3.6 it is an isomorphism. \square

Recall the following well-known result of Serre.

Theorem 3.8 (Serre). *A two-dimensional singularity (A, \mathfrak{m}) is normal if and only if it is Cohen-Macaulay and isolated.*

For a proof, see [84, Theorem IV.D.11].

From Proposition 3.7 and Serre's theorem we immediately obtain the following corollary.

Corollary 3.9. *Let (A, \mathfrak{m}) be a normal two-dimensional singularity. Then a Noetherian A -module is Cohen-Macaulay if and only if it is reflexive.*

Let (A, \mathfrak{m}) be a reduced Cohen-Macaulay ring of Krull dimension two, $X = \mathrm{Spec}(A)$, $U = X \setminus \{\mathfrak{m}\}$ and $i : U \rightarrow X$ the inclusion map. Let $M \mapsto \widetilde{M}$ and $\mathcal{F} \mapsto \Gamma(\mathcal{F})$ denote the quasi-inverse equivalences $A\text{-mod} \rightarrow \mathrm{Coh}(X)$ and $\mathrm{Coh}(X) \rightarrow A\text{-mod}$.

Proposition 3.10. *Let M be a torsion free Noetherian A -module. Then there is a functorial isomorphism $M^\dagger \rightarrow \Gamma(i_* i^* \widetilde{M})$.*

Proof. First of all, if M is a Cohen-Macaulay A -module then the canonical morphism $M \rightarrow \Gamma(i_* i^* \widetilde{M})$ is an isomorphism. Indeed, by [47, Corollaire 2.9] we have an exact sequence

$$0 \rightarrow H_{\mathfrak{m}}^0(M) \rightarrow M \rightarrow \Gamma(i_* i^* \widetilde{M}) \rightarrow H_{\mathfrak{m}}^1(M) \rightarrow 0.$$

It remains to note that $H_{\mathfrak{m}}^0(M) = 0 = H_{\mathfrak{m}}^1(M)$ for a Cohen-Macaulay module M .

Next, if M is torsion free then we have an exact sequence $0 \rightarrow M \rightarrow M^\dagger \rightarrow T \rightarrow 0$ where T is a module of finite length. Since i is an open embedding, the functor i^* is exact and $i^* T = 0$, so we obtain an isomorphism $i^* M \rightarrow i^* M^\dagger$. The claim follows from the commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\quad} & M^\dagger \\ \downarrow & & \downarrow \cong \\ \Gamma(i_* i^* \widetilde{M}) & \xrightarrow{\cong} & \Gamma(i_* i^* \widetilde{M}^\dagger). \end{array}$$

\square

Corollary 3.11. *Let (A, \mathfrak{m}) be a reduced Cohen-Macaulay ring of Krull dimension two, M and N two Cohen-Macaulay modules. Then a morphism $f : M \rightarrow N$ is an epimorphism if and only if for all $\mathfrak{p} \in \mathcal{P}$ the morphism $f_{\mathfrak{p}} : M_{\mathfrak{p}} \rightarrow N_{\mathfrak{p}}$ is an epimorphism.*

Proof. The second condition is equivalent to the fact that $T := \mathrm{coker}(f)$ is a module of finite length. Since the functor i^* is exact and $i^*(T) = 0$, the result follows from the functorial isomorphisms $M \rightarrow \Gamma(i_* i^* \widetilde{M})$ and $N \rightarrow \Gamma(i_* i^* \widetilde{N})$. \square

The following result is also a consequence of Proposition 3.10.

Corollary 3.12. *Let (A, \mathfrak{m}) be a normal surface singularity, $\mathrm{VB}(U)$ the category of locally free \mathcal{O}_U -modules. Then the functor $i^* : \mathrm{CM}(A) \rightarrow \mathrm{VB}(U)$, mapping a Cohen-Macaulay module M to the locally free sheaf $i^* \widetilde{M}$, is an equivalence of categories.*

Proof. Let M be a Cohen-Macaulay A -module. Since for any $\mathfrak{p} \in \mathcal{P}$ the ring $A_{\mathfrak{p}}$ is regular, the module $M_{\mathfrak{p}}$ is free. Hence, the coherent sheaf $i^*\widetilde{M}$ is indeed locally free.

Let \mathcal{F} be a locally free sheaf on U , then the direct image sheaf $\mathcal{G} := i_*\mathcal{F}$ is quasi-coherent. However, any quasi-coherent sheaf on a Noetherian scheme can be written as the direct limit of an increasing sequence of coherent subsheaves $\mathcal{G}_1 \subseteq \mathcal{G}_2 \subseteq \cdots \subseteq \mathcal{G}$. Since the functor i^* is exact, we obtain an increasing filtration $i^*\mathcal{G}_1 \subseteq i^*\mathcal{G}_2 \subseteq \cdots \subseteq i^*\mathcal{G}$. But $i^*\mathcal{G} = i^*i_*\mathcal{F} \cong \mathcal{F}$. Since the scheme U is Noetherian and \mathcal{F} is coherent, it implies that $\mathcal{F} \cong i^*\mathcal{G}_t$ for some $t \geq 1$. Moreover, since the module $G_t = \Gamma(\mathcal{G}_t)$ is torsion-free on the punctured spectrum, the morphism $G_t \rightarrow G_t^\dagger =: M$ induces an isomorphism $i^*G_t \cong i^*G_t^\dagger \cong \mathcal{F}$. Hence, the functor i^* is dense. Moreover, by Proposition 3.10 we have: $i_*\mathcal{F} \cong i_*i^*\widetilde{M} \cong \widetilde{M}$, hence $i_*\mathcal{F}$ is always coherent. Since $i^*i_*\mathcal{F} \cong \mathcal{F}$ for any \mathcal{O}_U -module \mathcal{F} , it is easy to see that the functor $\text{VB}(U) \rightarrow \text{CM}(A)$ given by $\mathcal{F} \mapsto \Gamma(i_*\mathcal{F})$ is quasi-inverse to i^* . \square

Remark 3.13. It can be shown that for an isolated surface singularity (A, \mathfrak{m}) the abelian category $\text{Coh}(U)$ is hereditary. Hence, the category of Cohen-Macaulay modules on a normal surface singularity can be interpreted as the category of vector bundles on a certain “non-compact” smooth curve.

Definition 3.14. Let (A, \mathfrak{m}) be a reduced Cohen-Macaulay ring of Krull dimension two and M be a Cohen-Macaulay module over A . The interior tensor product functor $M \boxtimes_A - : \text{CM}(A) \rightarrow \text{CM}(A)$ is defined as $M \boxtimes N := (M \otimes_A N)^\dagger$.

Proposition 3.15. *Let N be a Cohen-Macaulay module. Then the interior tensor product functor $N \boxtimes_A -$ is left adjoint to the interior Hom-functor $\text{Hom}_A(N, -)$. Moreover, one also has the following canonical isomorphisms in the category $\text{CM}(A)$:*

$$M_1 \boxtimes M_2 \cong M_2 \boxtimes M_1, \quad (M_1 \boxtimes M_2) \boxtimes M_3 \cong M_1 \boxtimes (M_2 \boxtimes M_3).$$

Proof. For any A -modules M, N and K there is a canonical isomorphism

$$\text{Hom}_A(M \otimes N, K) \cong \text{Hom}_A(M, \text{Hom}_A(N, K)).$$

If K is Cohen-Macaulay then $\text{Hom}_A(N, K)$ is Cohen-Macaulay, too. Moreover, we have a natural isomorphism $\text{Hom}_A((M \otimes N)^\dagger, K) \cong \text{Hom}_A(M \otimes N, K)$ implying that

$$\text{Hom}_A(M \boxtimes_A N, K) \cong \text{Hom}_A(M, \text{Hom}_A(N, K))$$

for any Cohen-Macaulay modules M, N and K .

In order to prove the second part of the proposition, consider the composition map

$$\iota : (M_1 \otimes M_2) \otimes M_3 \rightarrow (M_1 \otimes M_2)^\dagger \otimes M_3 \rightarrow ((M_1 \otimes M_2)^\dagger \otimes M_3)^\dagger$$

which is a birational isomorphism. Moreover, the cokernel of ι is a module of finite length. By the universal property of the Macaulayfication functor we obtain a commutative diagram

$$\begin{array}{ccc} M_1 \otimes M_2 \otimes M_3 & \xrightarrow{\iota} & ((M_1 \otimes M_2)^\dagger \otimes M_3)^\dagger \\ \downarrow j & \nearrow \varphi & \\ (M_1 \otimes M_2 \otimes M_3)^\dagger & & \end{array}$$

where all morphisms ι, j and φ are birational isomorphisms. Moreover, the cokernel of φ is a quotient of the cokernel of ι , hence it has finite length. By Lemma 3.6 this implies that φ is an isomorphism. Hence,

$$(M_1 \boxtimes M_2) \boxtimes M_3 \cong (M_1 \otimes M_2 \otimes M_3)^\dagger \cong M_1 \boxtimes (M_2 \boxtimes M_3),$$

implying the claim. The proof of the remaining statement is similar. \square

Remark 3.16. Let (A, \mathfrak{m}) be a normal surface singularity. Then the equivalence of categories $i^* : \text{CM}(A) \rightarrow \text{VB}(U)$ additionally satisfies the following property:

$$i^*(M_1 \boxtimes M_2) \cong i^*M_1 \otimes i^*M_2.$$

Let (A, \mathfrak{m}) be a Henselian Cohen-Macaulay local ring and $A \subseteq B$ a finite ring extension. Then the ring B is semi-local. Moreover, $B \cong (B_1, \mathfrak{n}_1) \times (B_2, \mathfrak{n}_2) \times \cdots \times (B_t, \mathfrak{n}_t)$, where all (B_i, \mathfrak{n}_i) are local. Assume all the rings B_i are Cohen-Macaulay.

Proposition 3.17. *The functor $B \boxtimes_A - : \text{CM}(A) \rightarrow \text{CM}(B)$ mapping a Cohen-Macaulay module M to $(B \otimes_A M)^\dagger$ is left adjoint to the forgetful functor $\text{CM}(B) \rightarrow \text{CM}(A)$.*

Proof. Let N be a Cohen-Macaulay B -module and M a Cohen-Macaulay A -module. Then we have functorial isomorphisms:

$$\text{Hom}_A(M, N) \cong \text{Hom}_B(B \otimes_A M, N) \cong \text{Hom}_B((B \otimes_A M)^\dagger, N),$$

implying the claim. \square

Proposition 3.18. *Let (A, \mathfrak{m}) be a reduced Henselian Cohen-Macaulay local ring of Krull dimension two, $A \subseteq B$ a finite ring extension such that B is reduced and Cohen-Macaulay. Let M be a Noetherian B -module, then $M^{\dagger A} \cong M^{\dagger B}$ as A -modules.*

Proof. By Corollary 2.24, the B -module $M^{\dagger B}$ is Cohen-Macaulay as an A -module. Let $\{\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_n\}$ be the set of minimal prime ideals of A , then $Q(A) \cong A_{\mathfrak{p}_1} \times A_{\mathfrak{p}_2} \times \cdots \times A_{\mathfrak{p}_n}$. Since the ring extension $A \subseteq B$ is finite, for a given minimal prime ideal $\mathfrak{p} \in \text{Spec}(A)$ the set $I(\mathfrak{p}) := \{\mathfrak{q} \in \text{Spec}(B) \mid \mathfrak{p} \subseteq \mathfrak{q}, \text{ht}(\mathfrak{q}) = 0\}$ is non-empty and finite. Moreover, we have: $B_{\mathfrak{p}} \cong \prod_{\mathfrak{q} \in I(\mathfrak{p})} B_{\mathfrak{q}}$. This implies that $Q(B) \cong Q(A) \otimes_A B$, hence $Q(A) \otimes_A M \cong Q(B) \otimes_B M$ and the torsion part $\text{tor}_B(M)$ of the B -module M coincide with the torsion part $\text{tor}_A(M)$ of M viewed as an A -module. Hence, we may without loss of generality assume M is torsion free, both as A - and B -module. Moreover, by the universal property of the Macaulayfication functor we obtain a morphism $\varphi : M^{\dagger A} \rightarrow M^{\dagger B}$ making the following diagram

$$\begin{array}{ccc} & M & \\ \iota \swarrow & & \searrow j \\ M^{\dagger A} & \xrightarrow{\varphi} & M^{\dagger B} \end{array}$$

commutative. However, the cokernels of ι and j have finite length over A , hence the cokernel of φ has finite length, too. Moreover, φ is a birational isomorphism, hence it is a monomorphism. By Lemma 3.6 it is an isomorphism. \square

Proposition 3.19. *Let k be an algebraically closed field of characteristic zero, (A, \mathfrak{m}) be a normal k -algebra of Krull dimension two such that $k = A/\mathfrak{m}$ and Ω_A^1 be the module of Kähler differentials and $\Omega_A^2 = \Omega_A^1 \wedge \Omega_A^1$. Then the canonical module $K = K_A$ is isomorphic to $(\Omega_A^2)^\dagger \cong (\Omega_A^2)^{**}$. In the geometrical terms, if $X = \text{Spec}(A)$ and $U = X \setminus \{\mathfrak{m}\}$ then $K \cong \Gamma(i_* \Omega_U^2)$, where Ω_U^2 is the sheaf of regular differential two-forms on the smooth two-dimensional scheme U .*

Proof. Since the scheme U is smooth and two-dimensional, we have: $\omega_U \cong \Omega_U^2$, where ω_U is the canonical module of U . Moreover, $\tilde{K}|_U \cong \omega_U$, see [49], hence $K = \Gamma(i_* i^* \tilde{K}) \cong \Gamma(i_* \Omega_U^2)$. \square

Having this description of K_A in mind, one may ask about a possible interpretation of the module $(\Omega_A^1)^{**}$. It turns out that this question is closely related with the theory of almost split sequences of Cohen-Macaulay modules over normal surface singularities.

Proposition 3.20. *Let (A, \mathfrak{m}) be a normal two-dimensional Noetherian ring having a canonical module K . Consider the exact sequence*

$$(1) \quad 0 \rightarrow K \rightarrow D \rightarrow A \rightarrow k \rightarrow 0$$

corresponding to a generator of $\text{Ext}_A^2(k, K) \cong k$. Then the module D is Cohen-Macaulay.

Proof. From the exact sequence $0 \rightarrow K \rightarrow D \rightarrow \mathfrak{m} \rightarrow 0$ it follows that $H_{\mathfrak{m}}^0(D) = 0$. Moreover, since $\text{Ext}_A^1(A, K) = 0 = \text{Ext}_A^2(A, K)$, we get: $\text{Ext}_A^1(\mathfrak{m}, K) \cong \text{Ext}_A^2(k, K) \cong k$.

Let $\mathbb{D} = \text{Hom}_A(-, E(k))$ be the Matlis functor. If A is not complete, it need not be a duality, but anyway it is exact and by the Local Duality theorem we have a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_{\mathfrak{m}}^1(D) & \longrightarrow & H_{\mathfrak{m}}^1(\mathfrak{m}) & \longrightarrow & H_{\mathfrak{m}}^2(K) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbb{D}(\text{Ext}_A^1(D, K)) & \longrightarrow & \mathbb{D}(\text{Ext}_A^1(\mathfrak{m}, K)) & \longrightarrow & \mathbb{D}(\text{Hom}_A(K, K)), \end{array}$$

where all horizontal maps are induced by (1) and all vertical arrows are isomorphisms. But the morphism $\text{Hom}_A(K, K) \rightarrow \text{Ext}_A^1(\mathfrak{m}, K) \cong k$ is obviously non-zero, hence it is surjective and the corresponding dual morphism is injective. Thus, $k \cong H_{\mathfrak{m}}^1(\mathfrak{m}) \rightarrow H_{\mathfrak{m}}^2(K)$ is a monomorphism and $H_{\mathfrak{m}}^1(D) = 0$. \square

Remark 3.21. The constructed exact sequence (1) is called *fundamental*, the corresponding Cohen-Macaulay module D is called *fundamental module* or *Auslander module*. The reason for this terminology will be explained below in Remark 4.17.

Lemma 3.22. *Let (A, \mathfrak{m}) be a local normal domain of Krull dimension two having a canonical module K . Then we have: $(D \wedge D)^{**} \cong K$.*

A proof of this lemma can be found in [94, Lemma 1.2].

Proposition 3.23 (Martsinkovsky). *Let k be an algebraically closed field of characteristic zero, $A = k\{x, y, z\}/f$ an isolated analytic hypersurface singularity. Then $D \cong (\Omega_A^1)^{**}$ if and only if A is quasi-homogeneous.*

For a proof of this result, see [72, Theorem 1].

Remark 3.24. If (A, \mathfrak{m}) is a quasi-homogeneous analytic normal surface singularity over an algebraically closed field k of characteristic zero then $D \cong (\Omega_A^1)^{**}$, see [11, Proposition 2.1] and [61, Proposition 2.35]. It was shown by Herzog [51] that if the ring A is Gorenstein, $D \cong (\Omega_A^1)^{**}$ and the canonical morphism $\Omega_A^1 \otimes_A k \rightarrow (\Omega_A^1)^{**} \otimes_A k$ is injective, then A is quasi-homogeneous.

Moreover, the following conjecture was posed by Martsinkovsky.

Conjecture 3.25. *Let A be a normal analytic algebra of Krull dimension two over an algebraically closed field of characteristic zero. Then the isomorphism $D \cong (\Omega_A^1)^{**}$ is equivalent to the quasi-homogeneity of A .*

Remark 3.26. Let (S, \mathfrak{n}) be a regular Noetherian ring of Krull dimension three, $(A, \mathfrak{m}) = S/f$ a normal hypersurface singularity and $k = A/\mathfrak{m}$ the residue field. Then we have: $D \cong \text{syz}^3(k)$, see [94, Lemma 1.5].

The interest to the fundamental module is explained by the following result of Auslander.

Theorem 3.27 (Auslander). *Let k be an algebraically closed field of characteristic zero, (A, \mathfrak{m}) be an analytic local normal k -algebra and M an indecomposable non-free Cohen-Macaulay A -module. Then the almost split sequence ending in M has the following form:*

$$0 \rightarrow K \boxtimes_A M \rightarrow D \boxtimes_A M \rightarrow M \rightarrow 0,$$

in particular, $\tau(M) \cong K \boxtimes_A M$.

For a proof of this Theorem, see [9, Theorem 6.6] and [93, Chapter 11].

4. COHEN-MACAULAY MODULES OVER TWO-DIMENSIONAL QUOTIENT SINGULARITIES

In this section we deal with Cohen-Macaulay modules over quotient two-dimensional singularities. First let us recall some basic properties of rings of invariants with respect to an action of a finite group.

Theorem 4.1. *Let (A, \mathfrak{m}) be a Noetherian local normal domain, $G \subset \text{Aut}(A)$ a finite group of invariants of A such that the order of the group $t = |G|$ is invertible in A . Then the ring of invariants A^G is again*

- (1) *a Noetherian local normal domain;*
- (2) *the ring extension $A^G \subset A$ is finite;*
- (3) *moreover, if the ring A is complete then A^G is complete, too;*
- (4) *if A is Cohen-Macaulay then A^G is Cohen-Macaulay as well.*

Proof. We prove this theorem step by step. Without loss of generality we assume the action of G is effective, i.e. for any $g \in G$ there exists $a \in A$ such that $g(a) \neq a$.

1. First of all we show the ring A^G is local and $\mathfrak{n} = \mathfrak{m} \cap A^G$ is its unique maximal ideal. Indeed, let $x \notin \mathfrak{n}$ then x is invertible in A . Since its inverse is again G -invariant, x is invertible in A^G . Hence, \mathfrak{n} is the unique maximal ideal of A^G .

Now we prove the ring A^G is Noetherian. For this it is sufficient to show that for any ideal I in A^G we have: $(IA) \cap A^G = I$. Indeed, for any increasing chain of ideals

$$I_1 \subseteq I_2 \subseteq \dots \subseteq I_n \subseteq \dots \subseteq A^G$$

we consider the induced chain $I_1A \subseteq I_2A \subseteq \dots \subseteq I_nA \subseteq \dots \subseteq A$. Since A is Noetherian, $I_mA = I_nA$ for some big n and all $m \geq n$. Hence, $I_m = (I_mA) \cap A^G = (I_nA) \cap A^G = I_n$.

Let $f_1, f_2, \dots, f_m \in I$ and $r_1, r_2, \dots, r_m \in A$ be such that $f = \sum_{i=1}^m f_i r_i \in A^G$. Then we have:

$$tf = \sum_{\sigma \in G} \sigma(f) = \sum_{i=1}^m \sum_{\sigma \in G} \sigma(f_i r_i) = t \sum_{i=1}^m f_i \frac{1}{t} \sum_{\sigma \in G} \sigma(r_i) = t \sum_{i=1}^m f_i \tilde{r}_i,$$

where $\tilde{r}_i = \frac{1}{t} \sum_{\sigma \in G} \sigma(r_i) \in A^G$. Hence,

$$f = \sum_{i=1}^m f_i \tilde{r}_i$$

and $(IA) \cap A^G = I$.

2. In order to prove that the ring of invariants A^G is normal observe first that its ring of fractions $K := Q(A^G)$ coincides with L^G , where $L = Q(A)$. Indeed, one inclusion $K = Q(A^G) \subset Q(A)^G =: L$ is clear. To prove another one, take any fraction $\frac{a}{b} \in Q(A)^G$. Let $G = \{g_1 = e, g_2, \dots, g_t\}$ then

$$\frac{a}{b} = \frac{ag_2(b) \dots g_t(b)}{bg_2(b) \dots g_t(b)} = \frac{\tilde{a}}{\tilde{b}}$$

where $\tilde{b} \in A^G$. Since $\frac{a}{b}$ is invariant under the action of G , $\tilde{a} \in A^G$ and $\frac{\tilde{a}}{\tilde{b}} \in Q(A^G)$. Now, assume $x \in K$ is such that there exist elements $c_1, c_2, \dots, c_l \in A^G$ such that $x^l + c_1 x^{l-1} + \dots + c_l = 0$. Then $x \in A \cap Q(A^G) = A^G$, hence A^G is normal.

To prove the ring extension $A^G \subset A$ is finite, first note that by Artin's Lemma the field extension $K \subset L$ is Galois, hence separable, see [67, Theorem VI.1.8]. Let $a \in L$ and $\varphi_a(y) = y^t + c_1 y^{t-1} + \dots + c_t \in K[y]$ be its characteristic polynomial. Recall that $\text{tr}(a) = \text{tr}_{L/K}(a) = -c_1 \in K$ is K -linear. Moreover, since the field extension $K \subset L$ is separable, the K -bilinear form

$$L \times L \rightarrow K, \quad (a, b) \mapsto \text{tr}(ab)$$

is non-degenerate, see [67, Theorem VI.5.2]. Since the field extension $K \subseteq L$ is Galois, the characteristic polynomial of $a \in A$ is

$$\varphi_a(y) = (y - h_1(a))(y - h_2(a)) \dots (y - h_t(a)),$$

where $G = \{h_1, h_2, \dots, h_t\}$. In particular, $\text{tr}(a) \in A^G$.

Let $\alpha_1, \alpha_2, \dots, \alpha_n \in L$ be a basis of L over K . Note that without loss of generality we may assume all elements α_i actually belong to A . Denote

$$A^! := \{x \in L \mid \text{tr}(xa) \in A^G \text{ for all } a \in A\}.$$

Since tr is A^G -linear, $A^!$ is an A^G -module; moreover, $A \subseteq A^!$. Let

$$M := \langle \alpha_1, \alpha_2, \dots, \alpha_n \rangle_{A^G} \subset A.$$

It is easy to see that $A^! \subseteq M^! := \langle \alpha_1^*, \alpha_2^*, \dots, \alpha_n^* \rangle_{A^G}$, where $\alpha_i^*(\alpha_j) = \delta_{i,j}$. Summing everything up, we see that A is a submodule of a finitely generated A^G -module $M^!$. Since A^G is Noetherian, A is a Noetherian A^G -module, hence finite over A^G .

3. Assume the ring A is complete. We already know the ring A^G is local and $\mathfrak{n} = \mathfrak{m} \cap A^G$ is its unique maximal ideal. In order to show the local ring (A^G, \mathfrak{n}) is complete, take any sequence $(a_n)_{n \geq 1}$ of elements of A^G such that $a_n - a_m \in \mathfrak{n}^l$ for all $n, m \geq 1$, where $l = \min(m, n)$. By completeness of A there exists a unique element $a \in A$ such that $a \equiv a_l \pmod{\mathfrak{m}^l}$ for all $l \geq 1$. Since for all $g \in G$ we have $g(a_l) = a_l$, by Krull's intersection theorem

$$g(a) - a \in \bigcap_{l \geq 1} \mathfrak{m}^l = 0.$$

4. Since $t = |G|$ is invertible in A , we can consider the Reynolds's operator $p : A \rightarrow A^G$, given by the rule $a \mapsto p(a) := \frac{1}{|G|} \sum_{g \in G} g(a)$. It is clear that p is A^G -linear and $p(a) = a$ for $a \in A^G$, hence $A \cong A^G \oplus A'$ viewed as an A^G -module. Moreover, since the ring extension $A^G \subset A$ is known to be finite, $H_{\mathfrak{n}}^t(A) = H_{\mathfrak{n}}^t(A)$ for all $t \geq 0$. If A is Cohen-Macaulay, then by Corollary 2.23 we have $H_{\mathfrak{n}}^t(A) = 0$ for $t \neq d$. Thus, $H_{\mathfrak{n}}^t(A^G) = 0$ and A^G is Cohen-Macaulay, too. \square

Remark 4.2. If the order of the group G is not invertible in A , then the ring of invariants A^G can be *not* Noetherian! Such an example was constructed for the first time by Nagata in [74]. Moreover, Fogarty gave an example of a finite group G acting on a Cohen-Macaulay ring A such that the ring of invariants A^G is Noetherian but not Cohen-Macaulay, see [41].

Remark 4.3. If (A, \mathfrak{m}) is a Noetherian local ring and G a finite group of automorphisms of A such that $|G|$ is invertible in A and A^G is Cohen-Macaulay, then A is *not* necessary Cohen-Macaulay. Indeed, let k be a field of characteristic different from two, $A = k[[x, y]]/(xy, y^2)$ and $G = \langle \sigma \mid \sigma^2 = e \rangle$ acts on A k -linearly by the rule $\sigma(x) = x$ and $\sigma(y) = -y$. Then $A^G = k[[x]]$ is Cohen-Macaulay, but A is not.

Let k be an algebraically closed field, $R = k[[x_1, x_2, \dots, x_n]]$ the ring of formal power series, \mathfrak{n} its maximal ideal and $G \subset \text{Aut}(R)$ a finite group of ring automorphisms of R such that $|G|$ is invertible in k . Then G acts on the cotangent space $V := \mathfrak{n}/\mathfrak{n}^2$ and we obtain a group homomorphism $\rho : G \rightarrow \text{GL}(V)$.

Proposition 4.4. *In the notations as above denote $G' = \text{im}(\rho)$. Then we have: $R^G \cong R^{G'}$.*

Proof. The following argument is due to Cartan [24]. For an element $g \in G$ let \bar{g} be the corresponding element of G' . Let

$$\bar{g}x_i = \sum_{j=1}^n \bar{g}_{ij}x_j$$

be the linearized action of G on R . Consider the ring endomorphism $\tau : R \rightarrow R$ given by the rule

$$\tau(x_i) = \frac{1}{|G|} \sum_{g \in G} \bar{g}^{-1} g(x_i), \quad 1 \leq i \leq n.$$

Since the induced action of τ on the cotangent space $\mathfrak{n}/\mathfrak{n}^2$ is identity, it is an automorphism. Moreover, it is easy to see that for any element $h \in G$ we have: $\bar{h} \circ \tau = \tau \circ h$, so these two actions are conjugate. Therefore, they have isomorphic rings of invariants, namely, $a \mapsto \tau(a)$ gives an isomorphism $R^G \rightarrow R^{G'}$. \square

Remark 4.5. Proposition 4.4 remains true if we replace R by the ring of convergent power series $\mathbb{C}\{x_1, x_2, \dots, x_n\}$. Moreover, it was shown by Cartan that if $A = \mathbb{C}\{x_1, x_2, \dots, x_n\}/I$ is an analytic algebra over \mathbb{C} and G a finite group of invariants of A then A^G is again analytic, see [24, Théorème 4].

In other words, Proposition 4.4 and Remark 4.5 mean that dealing with quotient singularities we may without loss of generality assume that group G acts linearly on R . Moreover, it is sufficient to consider the so-called small subgroups.

Definition 4.6. Let $G \subset \mathrm{GL}_n(k)$ be a finite subgroup. An element $g \in G$ is a *pseudo-reflection* if $\mathrm{rank}(1 - g) = 1$. A subgroup G is called *small* if it contains no pseudo-reflections.

Example 4.7. Let $R = k[[x_1, x_2]]$ and $G = \mathbb{Z}_2 = \langle \sigma \rangle$, where $\sigma(x_1) = x_2$ and $\sigma(x_2) = x_1$. Then σ is a pseudo-reflection and $R^G = k[[x_1 + x_2, x_1 x_2]] \cong k[[u, v]]$. Hence, if a finite group is not small, one can obtain a smooth ring as a quotient.

The following theorem lists some basic results on quotient singularities.

Theorem 4.8. *Let k be an algebraically closed field of characteristic zero, $R = k[[x_1, x_2, \dots, x_n]]$ and $G \subset \mathrm{GL}_n(k)$ be a finite subgroup. Then the following properties hold:*

- (1) *The invariant ring R^G is always normal and Cohen-Macaulay.*
- (2) *There exists a finite small group $G' \subset \mathrm{GL}_n(k)$ such that $R^G \cong R^{G'}$.*
- (3) *Let G_1 and G_2 be two finite small subgroups of $\mathrm{GL}_n(k)$. Then $R^{G_1} \cong R^{G_2}$ if and only if there exists an element $g \in \mathrm{GL}_n(k)$ such that $g^{-1}G_1g = G_2$.*
- (4) *Let $G \subset \mathrm{GL}_n(k)$ be a finite small subgroup. Then R^G is Gorenstein if and only if $G \subset \mathrm{SL}_n(k)$.*

The property (1) follows is proven in Theorem 4.1, (2) and (3) are due to Prill, see [78, Proposition 6] and [78, Theorem 2], respectively. Finally, the part (4) is a result of Watanabe [89] and [90], see also [53].

Let $R = \mathbb{C}[[x_1, x_2]]$. All finite subgroups of $\mathrm{SL}_2(\mathbb{C})$ modulo conjugation are known and the corresponding quotient singularities are precisely the *simple hypersurface singularities*.

- (1) If $G = \langle g | g^n = e \rangle \cong \mathbb{Z}_n$ is a cyclic subgroup of order $n \geq 2$, where

$$g = \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}, \quad \xi = \exp\left(\frac{\pi i}{n}\right)$$

then the corresponding ring of invariants is

$$\mathbb{C}[[x_1^n, x_2^n, x_1 x_2]] \cong \mathbb{C}[[u, v, w]]/(uv - w^n).$$

This is the so-called simple singularity of type A_{n-1} .

(2) Binary dihedral group \mathbb{D}_n is generated by two elements a and b , where

$$a = \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}, \quad \xi = \exp\left(\frac{\pi i}{n}\right), \quad b = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

It corresponds to the D_n -singularity ($n \geq 4$), given by the equation

$$u^{n-1} + uv^2 + w^2.$$

(3) Binary tetrahedral group \mathbb{T} is generated by three elements σ, τ and μ where

$$\sigma = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \tau = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mu = \frac{1}{\sqrt{2}} \begin{pmatrix} \xi^7 & \xi^7 \\ \xi^5 & \xi \end{pmatrix}, \quad \xi = \exp\left(\frac{2\pi i}{8}\right).$$

It corresponds to E_6 -singularity

$$u^3 + v^4 + w^2 = 0.$$

(4) Binary octahedral group \mathbb{O} is generated by the matrices σ, τ, μ occurring in the description of \mathbb{T} and by

$$\kappa = \begin{pmatrix} \xi & 0 \\ 0 & \xi^7 \end{pmatrix}.$$

The corresponding singularity is

$$E_7 : \quad u^3v + v^3 + w^2.$$

(5) Finally, we have the binary icosahedral subgroup $\mathbb{I} = \langle \sigma, \tau \rangle$, where

$$\sigma = - \begin{pmatrix} \xi^3 & 0 \\ 0 & \xi^2 \end{pmatrix}, \quad \tau = \frac{1}{\sqrt{5}} \begin{pmatrix} -\xi + \xi^4 & \xi^2 - \xi^3 \\ \xi^2 - \xi^3 & \xi - \xi^4 \end{pmatrix}, \quad \xi = \exp\left(\frac{2\pi i}{5}\right).$$

The corresponding singularity is

$$E_8 : \quad u^3 + v^5 + w^2.$$

Remark 4.9. In fact, all finite subgroups of $\mathrm{SL}_2(\mathbb{C})$ can be described in the following elegant way: they are parameterized by triples of positive integers

$$\left\{ (p, q, r) \mid p \leq q \leq r, \quad \frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1 \right\}$$

and are given by the presentation

$$G_{p,q,r} = \langle x, y, z \mid x^p = y^q = z^r = xyz \rangle.$$

Moreover,

- If $p = 1$ then $G_{1,q,r} \cong \mathbb{Z}_{q+r}$;
- $G_{2,2,n}$ is the binary dihedral group \mathbb{D}_n ;
- $G_{2,3,3}$ is the binary tetrahedral group \mathbb{T} ;
- $G_{2,3,4}$ is the binary octahedral group \mathbb{O} ;
- $G_{2,3,5}$ is the binary icosahedral group \mathbb{I} .

Remark 4.10. A classification of all small subgroups of $\mathrm{GL}_2(\mathbb{C})$ modulo conjugation is also known, see for example [18, Satz 2.3].

Theorem 4.11 (Herzog [50]). *Let k be an algebraically closed field of characteristic zero, $R = k[[x_1, x_2]]$ or $k\{x_1, x_2\}$, $G \subseteq \mathrm{GL}_2(k)$ be a finite small subgroup and $A = R^G$. Then $\mathrm{CM}(A) = \mathrm{add}_A(R)$, i.e. any indecomposable Cohen-Macaulay A -module is a direct summand of R viewed as A -module.*

Proof. The inclusion map $\iota : A \rightarrow R$ has an inverse: $p : R \rightarrow A$ given by the formula

$$p(r) = \frac{1}{|G|} \sum_{g \in G} g(r).$$

Moreover, p is a morphism of A -modules, hence $R \cong A \oplus R'$, where R' is a certain A -module. Hence, for any A -module M we have:

$$R \otimes_A M \cong M \oplus N,$$

where $N = R \otimes_A R'$. This implies:

$$(R \otimes_A M)^\dagger \cong M \oplus N^\dagger$$

in the category of A -modules. Since $R \otimes_A M$ is also an R -module, by Proposition 3.18 $(R \otimes_A M)^\dagger_R \cong (R \otimes_A M)^\dagger_A$ as A -modules. Moreover, R is regular, hence by Corollary 2.11 we have: $(R \otimes_A M)^\dagger_R \cong R^n$ for some positive integer n . Since the category $A\text{-mod}$ is Krull-Schmidt, M is a direct summand of R . \square

Theorem 4.11 shows that a quotient surface singularity always has finite Cohen-Macaulay representation type. However, one would wish a more explicit description of indecomposable Cohen-Macaulay modules.

Recall the following easy fact from category theory.

Lemma 4.12. *Let \mathcal{A} be an additive category, X an object of \mathcal{A} and $\Gamma = \text{End}_{\mathcal{A}}(X)$ its endomorphism ring. Then the functor $\text{Hom}_{\mathcal{A}}(X, -) : \mathcal{A} \rightarrow \text{mod} - \Gamma$ induces an equivalence of categories*

$$\text{Hom}_{\mathcal{A}}(X, -) : \text{add}(X) \longrightarrow \text{pro}(\Gamma),$$

where $\text{pro}(\Gamma)$ is the category of finitely generated projective right Γ -modules.

Since by Herzog's result we know that $\text{CM}(A) = \text{add}_A(R)$ for a quotient singularity $A = R^G$, it raises a question about a description of the ring $\text{End}_A(R)$.

Definition 4.13. Let G be a finite group acting on a ring S . Then the skew group ring $S * G$ is a free left S -module

$$S * G = \left\{ \sum_{g \in G} a_g [g] \mid a_g \in S \right\}$$

and the multiplication is given by the rule : $a_g [g] \cdot a_h [h] = a_g g(a_h) [gh]$.

Theorem 4.14 (Auslander). *Let k be an algebraically closed field of characteristic zero, $G \subset \text{GL}_2(k)$ a small subgroup, $R = k[[x_1, x_2]]$ and $A = R^G$. Then the k -linear map*

$$\theta : R * G \longrightarrow \text{End}_A(R)$$

mapping an element $s[g]$ to the morphism $r \mapsto sg(r)$, is an isomorphism of algebras.

We refer to [9] (see also [93, Proposition 10.9]) for a proof of this theorem.

Since $\text{char}(k) = 0$, the Jacobson's radical of $k[[x_1, x_2]] * G$ is $\mathfrak{m} * G$ and the semi-simple algebra $R * G / \text{rad}(R * G)$ is isomorphic to the group algebra of G . Moreover, the functor

$$R * G\text{-mod} \longrightarrow (R * G / \text{rad}(R * G))\text{-mod}$$

mapping an $R * G$ -module M to $M / \text{rad}(M)$ induces a bijection between indecomposable projective $R * G$ -modules and irreducible representations of the group G .

Theorem 4.15 (Auslander). *Let $G \subset \text{GL}_2(k)$ be a small subgroup, $R = k[[x_1, x_2]]$ and $A = R^G$. Then the exact functor $\mathbb{F} : R * G\text{-mod} \rightarrow A\text{-mod}$ mapping an $R * G$ -module M to $M^G = \{m \in M \mid gm = m \text{ for all } g \in G\}$ and a morphism of $R * G$ -modules $f : M \rightarrow N$ to $f|_{M^G}$, induces an equivalence of categories $\text{pro}(R * G) \rightarrow \text{CM}(A)$.*

Proof. Let us check the functor \mathbb{F} is exact. Let

$$0 \rightarrow M \xrightarrow{\varphi} N \xrightarrow{\psi} K \rightarrow 0$$

be an exact sequence of $R * G$ -modules. It is clear that φ^G is injective and $\ker(\psi^G) = \text{im}(\varphi^G)$. Let us check ψ^G is an epimorphism. For $z \in K^G$ take $y \in N : \psi(y) = z$ and put $\tilde{y} = \frac{1}{|G|} \left(\sum_{g \in G} g \cdot y \right) \in N^G$. Then $\psi(\tilde{y}) = z$, hence \mathbb{F} is exact.

Next, it is easy to see that $\mathbb{F}(R * G) = (R * G)^G = S_1 := \left\{ \tilde{r} = \sum_{g \in G} g(r)[g] \mid r \in R \right\}$ is isomorphic to R as an A -module. Moreover, in the category of left $R * G$ -modules we have:

$$\text{End}_{R * G}(R * G) \cong (R * G)^{\text{op}}.$$

Consider the following chain of morphisms of algebras:

$$R * G \xrightarrow{\alpha} (R * G)^{\text{op}} \xrightarrow{\beta} \text{End}_{R * G}(R * G) \xrightarrow{\gamma} \text{End}_A(S_1)$$

where $\alpha(p[g]) = g^{-1}(p)[g^{-1}]$, $\beta(\eta)(\zeta) = \zeta\eta$ and γ is the morphism induced by \mathbb{F} . Let $r \in R$ and $\tilde{r} = \sum_{h \in G} h(r)[h]$. Then

$$\begin{aligned} ((\gamma \circ \beta \circ \alpha)(p[g]))(\tilde{r}) &= \left(\sum_{h \in G} h(r)[h](g^{-1}(p)[g^{-1}]) \right) = \sum_{h \in G} h(rg^{-1}(p))hg^{-1} = \\ &= \sum_{h \in G} hg^{-1}(g(r)p)hg^{-1} = \widetilde{pg(r)}. \end{aligned}$$

Hence, the morphism of k -algebras $\gamma \circ \beta \circ \alpha$ coincides with the morphism θ from Theorem 4.14, which is known to be an isomorphism. Since $\mathbb{F}(R * G) \cong R$ and $\mathbb{F} : \text{End}_{R * G}(R * G) \rightarrow \text{End}_A(R)$ is an isomorphism, by Theorem 4.11 we get a chain of equivalences of categories

$$\text{pro}(R * G) = \text{add}_{R * G}(R * G) \cong \text{add}_A(R) \cong \text{CM}(A).$$

□

Let V be an irreducible representation of the group G , then $P = P_V := k[[x_1, x_2]] \otimes_k V$ is an indecomposable projective left $R * G$ -module such that $P/\text{rad}(P) \cong V$, where $p[g] \in R * G$ acts on a simple tensor $q \otimes v$ by the rule $p[g](q \otimes v) = pg(q) \otimes g(v)$.

Corollary 4.16 (McKay Correspondence à la Auslander). *Let $G \subset \text{GL}_2(k)$ be a finite small subgroup, $R = k[[x_1, x_2]]$ and $A = R^G$. Then there exists a bijection between irreducible representations of G and indecomposable Cohen-Macaulay A -modules given by the functor*

$$\text{Rep}(G) \ni V \mapsto (k[[x_1, x_2]] \otimes_k V)^G \in A\text{-mod}.$$

Remark 4.17. In the notations of Corollary 4.16, let $V = \langle e_1, e_2 \rangle_k \cong k^2$ be a two-dimensional vector space. The given embedding $\rho : G \subset \text{GL}(V) = \text{GL}_2(k)$ defines a two-dimensional representation $g \in G \mapsto \rho(g)$ called the *fundamental representation*. By a result of Watanabe [89] and [90], see also [53], the Cohen-Macaulay A -module $K = (k[[x_1, x_2]] \otimes_k V_{\det})^G$ is canonical, where $V_{\det} = k$ and $\det(g) := \det(\rho(g))$. Note that we have $V_{\det} = \wedge^2(V)$. For $f = \alpha_1 e_1 + \alpha_2 e_2 \in V$ denote $\tilde{f} = \alpha_1 x_1 + \alpha_2 x_2$. Then the Koszul resolution

$$0 \rightarrow k[[x_1, x_2]] \otimes \wedge^2(V) \xrightarrow{\alpha} k[[x_1, x_2]] \otimes V \xrightarrow{\beta} k[[x_1, x_2]] \xrightarrow{\phi} k \rightarrow 0$$

where $\alpha(p \otimes (f_1 \otimes f_2 - f_2 \otimes f_1)) = p\tilde{f}_1 \otimes f_2 - p\tilde{f}_2 \otimes f_1$, $\beta(q \otimes f) = q\tilde{f}$ and $\phi(t) = t(0, 0)$ is also a minimal free resolution of k in the category of $k[[x_1, x_2]] * G$ -modules. Hence, taking G -invariants we obtain an exact sequence

$$\omega : 0 \rightarrow K \rightarrow D \rightarrow A \rightarrow k \rightarrow 0,$$

where ω denotes the corresponding element in $\text{Ext}_A^2(k, K) \cong k$. Moreover, since the morphism $\alpha : k[[x_1, x_2]] \otimes \wedge^2(V) \xrightarrow{\alpha} k[[x_1, x_2]] \otimes V$ is non-split, the sequence ω is non-split, too, and $\omega \neq 0$. Hence, the module $D = (k[[x_1, x_2]] \otimes V)^G$ is the fundamental module of the quotient singularity $A = k[[x_1, x_2]]^G$.

Remark 4.18. Let W be a non-trivial irreducible $k[G]$ -module. Then its minimal free projective resolution as $k[[x_1, x_2]] * G$ -module is

$$0 \rightarrow k[[x_1, x_2]] \otimes (\wedge^2(V) \otimes W) \xrightarrow{\alpha \otimes 1} k[[x_1, x_2]] \otimes (V \otimes W) \xrightarrow{\beta \otimes 1} k[[x_1, x_2]] \otimes W \rightarrow W \rightarrow 0.$$

Since morphisms $\alpha \otimes 1$ and $\beta \otimes 1$ are almost split in the category of projective $k[[x_1, x_2]] * G$ -modules, $W^G = 0$ and the functor $\text{pro}(k[[x_1, x_2]] * G) \rightarrow \text{CM}(A)$ is an equivalence of categories, we obtain an exact sequence of Cohen-Macaulay A -modules

$$0 \rightarrow (k[[x_1, x_2]] \otimes (\wedge^2(V) \otimes W))^G \xrightarrow{\alpha \otimes 1} (k[[x_1, x_2]] \otimes (V \otimes W))^G \xrightarrow{\beta \otimes 1} (k[[x_1, x_2]] \otimes W)^G \rightarrow 0,$$

which is precisely the almost split sequence ending at the indecomposable Cohen-Macaulay module $(k[[x_1, x_2]] \otimes W)^G$.

Corollary 4.19. Let W be an irreducible representation of G over k and $M_W = (k[[x_1, x_2]] \otimes W)^G$ the corresponding indecomposable Cohen-Macaulay A -module. Then $\text{rank}(M_W) = \dim_k(W)$.

Example 4.20. Let k be an algebraically closed field of characteristic zero, $G = \langle g \rangle = \mathbb{Z}_n$ be a cyclic group of order $n \geq 2$, $\varepsilon \in k$ a primitive n -th root of unity and $0 < m < n$ an integer such that $\text{gcd}(n, m) = 1$. Define an embedding $\rho : G \rightarrow \text{GL}_2(k)$ by the rule

$$\rho(g) = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^m \end{pmatrix}.$$

Let $A = k[[x_1, x_2]]^G$ be the corresponding quotient singularity. Since the group G is cyclic, all irreducible representations of G are one-dimensional. Moreover, there are precisely n non-isomorphic irreducible representations V_1, V_2, \dots, V_n of G defined by the action $g \cdot 1 = \varepsilon^{-l}$, $1 \leq l \leq n$. By Theorem 4.15, the corresponding indecomposable Cohen-Macaulay A -modules are

$$k[[x_1, x_2]] \supseteq M_l := \left\{ \sum_{i,j=0}^{\infty} a_{ij} x_1^i x_2^j \mid a_{ij} \in k, i + mj \equiv l \pmod{n} \right\}, \quad 1 \leq l \leq n.$$

Remark 4.21. In the notations of Example 4.20, put $m = n - 1$. Then the corresponding invariant ring $A = k[[x_1, x_2]]^G$ is the simple singularity $k[[u, v, w]] / (uv - w^n)$ of type A_{n-1} , $n \geq 2$. In this case all non-free indecomposable Cohen-Macaulay A -modules are given by the matrix factorizations $M = M(\varphi_l, \psi_l)$, where

$$\varphi_l = \begin{pmatrix} u & -w^{n-l} \\ -w^l & v \end{pmatrix}, \quad \psi_l = \begin{pmatrix} v & w^{n-l} \\ w^l & u \end{pmatrix}, \quad 1 \leq l < n.$$

Matrix factorizations describing indecomposable Cohen-Macaulay modules over the other two-dimensional simple hypersurface singularities can be found for example in [57].

By Theorem 4.11 all quotient singularities have finite Cohen-Macaulay representation type. Moreover, Auslander [9] and Esnault [36] have shown that the converse is true in the case $k = \mathbb{C}$.

Theorem 4.22. Let (X, x) be a normal complex two-dimensional singularity of finite Cohen-Macaulay representation type. Then (X, x) is a quotient singularity.

A proof of this theorem can be found in [93, Chapter 11].

It is interesting to note that the normal complex two-dimensional singularities of finite Cohen-Macaulay representation type can be described in purely topological terms.

Theorem 4.23 (Prill). *Let (X, x) be a complex normal surface singularity. Then the following statements are equivalent:*

- (X, x) is a quotient singularity;
- The local fundamental group of (X, x) is finite.

For a proof of this theorem we refer to [78, Theorem 3], see also [18, Satz 2.8].

Combining theorems of Prill and Auslander, we obtain the following interesting corollary.

Corollary 4.24. *Let (X, x) be a complex normal surface singularity. Then the following statements are equivalent:*

- (X, x) is a quotient singularity;
- The local fundamental group of (X, x) is finite.
- (X, x) has finite Cohen-Macaulay representation type.

5. COHEN-MACAULAY MODULES OVER CERTAIN NON-ISOLATED SINGULARITIES

Let k be an algebraically closed field of arbitrary characteristic. The aim of this section is to classify indecomposable Cohen-Macaulay modules over non-isolated singularities $k[[x, y, z]]/xy$ (type A_∞) and $k[[x, y, z]]/(x^2y - z^2)$ (type D_∞). By a result of Buchweitz, Greuel and Schreyer these singularities have *discrete* (also called *countable*) Cohen-Macaulay representation type. Note that in their paper [21] the authors obtain a complete classification of indecomposable Cohen-Macaulay modules over the corresponding *curve* singularities $k[[x, y]]/x^2$ and $k[[x, y]]/x^2y$. Although, by Knörrer's correspondence [66] the surface singularities A_∞ and D_∞ have the same Cohen-Macaulay representation type as the corresponding curve singularities, the problem to describe all indecomposable matrix factorizations for $k[[x, y, z]]/xy$ and $k[[x, y, z]]/(x^2y - z^2)$ remained to be done.

Let k be an algebraically closed field and (A, \mathfrak{m}) be a reduced analytic Cohen-Macaulay k -algebra of Krull dimension two, which is not an isolated singularity, and let R be its normalization. It is well-known that R is again analytic and the ring extension $A \subset R$ is finite, see [45] or [26]. Moreover, the ring R is isomorphic to the product of a finite number of normal local rings:

$$R \cong (R_1, \mathfrak{n}_1) \times (R_1, \mathfrak{n}_1) \times \cdots \times (R_t, \mathfrak{n}_t).$$

Note that all rings R_i are automatically Cohen-Macaulay, see [84, Theorem IV.D.11].

Let $I = \text{ann}(R/A) \cong \text{Hom}_A(R, A)$ be the conductor ideal. Note that I is also an ideal in R , denote $\bar{A} = A/I$ and $\bar{R} = R/I$. By Lemma 3.1 the ideal I is Cohen-Macaulay, both as A - and R -module. Moreover, $V(I) \subset \text{Spec}(A)$ is exactly the locus where the ring A is not normal. It is not difficult to show that both rings \bar{A} and \bar{R} have Krull dimension one and are Cohen-Macaulay (but not necessary reduced). Let $Q(\bar{A})$ and $Q(\bar{R})$ be the corresponding total rings of fractions, then the inclusion $\bar{A} \rightarrow \bar{R}$ induces an inclusion $Q(\bar{A}) \rightarrow Q(\bar{R})$.

Let M be a Cohen-Macaulay A -module. Recall that $R \boxtimes_A M = (R \otimes_A M)^\dagger$ and for any Noetherian R -module N we have an exact sequence

$$0 \rightarrow \text{tor}(N) \rightarrow N \xrightarrow{i_N} N^\dagger \rightarrow T \rightarrow 0,$$

where T is an R -module of finite length. Hence, the canonical morphism

$$\theta_M : Q(\bar{R}) \otimes_A M = Q(\bar{R}) \otimes_{Q(\bar{A})} (Q(\bar{A}) \otimes_A M) \longrightarrow Q(\bar{R}) \otimes_R (R \otimes_A M) \longrightarrow Q(\bar{R}) \otimes_R (R \boxtimes_A M)$$

is an epimorphism. Moreover, one can show that the canonical morphism

$$\eta_M : Q(\bar{A}) \otimes_A M \rightarrow Q(\bar{R}) \otimes_A M \xrightarrow{\theta_M} Q(\bar{R}) \otimes_R (R \boxtimes_A M)$$

is a monomorphism provided M is Cohen-Macaulay.

Definition 5.1. In the notations of this section, consider the following *category of triples* $\text{Tri}(A)$. Its objects are triples $(\widetilde{M}, V, \theta)$, where \widetilde{M} is a Cohen-Macaulay R -module, V is a Noetherian

$Q(\bar{A})$ -module and $\theta : Q(\bar{R}) \otimes_{Q(\bar{A})} V \rightarrow Q(\bar{R}) \otimes_R \widetilde{M}$ is an epimorphism of $Q(\bar{R})$ -modules such that the induced morphism of $Q(\bar{A})$ -modules

$$V \rightarrow Q(\bar{R}) \otimes_{Q(\bar{A})} V \xrightarrow{\theta} Q(\bar{R}) \otimes_R \widetilde{M}$$

is a monomorphism. A morphism between two triples $(\widetilde{M}, V, \theta)$ and $(\widetilde{M}', V', \theta')$ is given by a pair (F, f) , where $F : \widetilde{M} \rightarrow \widetilde{M}'$ is a morphism of R -modules and $f : V \rightarrow V'$ is a morphism of $Q(\bar{A})$ -modules such that the following diagram

$$\begin{array}{ccc} Q(\bar{R}) \otimes_{Q(\bar{A})} V & \longrightarrow & Q(\bar{R}) \otimes_R \widetilde{M} \\ 1 \otimes f \downarrow & & \downarrow 1 \otimes F \\ Q(\bar{R}) \otimes_{Q(\bar{A})} V' & \longrightarrow & Q(\bar{R}) \otimes_R \widetilde{M}' \end{array}$$

is commutative.

The definition is motivated by the following theorem.

Theorem 5.2 (Burban-Drozd). *Let k be an algebraically closed field and (A, \mathfrak{m}) a reduced analytic two-dimensional Cohen-Macaulay ring which is a non-isolated singularity. Then in the notations of Definition 5.1 we have: the functor $\mathbb{F} : \text{CM}(A) \rightarrow \text{Tri}(A)$ mapping a Cohen-Macaulay module M to the triple $(R \boxtimes_A M, Q(\bar{A}) \otimes_A M, \theta_M)$ is an equivalence of categories.*

Moreover, the full subcategory $\text{CM}^{\text{lf}}(A)$ consisting of Cohen-Macaulay modules which are locally free on the punctured spectrum of A , is equivalent to the full subcategory $\text{Tri}^{\text{lf}}(A)$ consisting of those triples $(\widetilde{M}, V, \theta)$ for which the morphism θ is an isomorphism.

The details of the proof of this theorem will appear in a forthcoming joint paper of both authors [23]. Next, we need an explicit description of a functor $\mathbb{G} : \text{Tri}(A) \rightarrow \text{CM}(A)$ (or, at least its description on objects), which is quasi-inverse to \mathbb{F} . The construction is as follows. Let $T = (\widetilde{M}, V, \theta)$ be an object of $\text{Tri}(A)$. Then one can find an \bar{A} -module X , a morphism of \bar{A} -modules $\phi : X \rightarrow \bar{R} \otimes_R \widetilde{M}$ and an isomorphism $\psi : Q(\bar{R}) \otimes_{\bar{A}} X \rightarrow Q(\bar{R}) \otimes_{Q(\bar{A})} V$ such that the following diagram

$$\begin{array}{ccc} Q(\bar{R}) \otimes_{\bar{A}} X & \xrightarrow{1 \otimes \phi} & Q(\bar{R}) \otimes_R \widetilde{M} \\ & \searrow \psi & \nearrow \theta \\ & & Q(\bar{R}) \otimes_{Q(\bar{A})} V \end{array}$$

is commutative. Consider the following commutative diagram with exact rows in the category of A -modules:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I\widetilde{M} & \longrightarrow & M & \longrightarrow & X & \longrightarrow & 0 \\ & & \downarrow = & & \downarrow & & \downarrow \phi & & \\ 0 & \longrightarrow & I\widetilde{M} & \longrightarrow & \widetilde{M} & \longrightarrow & \bar{R} \otimes_R \widetilde{M} & \longrightarrow & 0. \end{array}$$

Then $\mathbb{G}(T) \cong M^\dagger$.

The aim of this section is to apply Theorem 5.2 for a classification of indecomposable Cohen-Macaulay modules over non-isolated singularities $k[[x, y, z]]/xy$ and $k[[x, y, z]]/(x^2y - z^2)$.

Theorem 5.3. *Let $A = k[[x, y, z]]/xy$. Then the indecomposable non-free Cohen-Macaulay A -modules are described by the following matrix factorizations:*

- $M(x, y)$ and $M(y, x)$. Let $R = R_1 \times R_2 = k[[x, z_1]] \times k[[y, z_2]]$ be the normalization of A then $M(x, y) \cong R_2$ and $M(y, x) \cong R_1$.

- $M(\varphi_n, \psi_n)$ and $M(\psi_n, \varphi_n)$, where

$$\varphi_n = \begin{pmatrix} y & z^n \\ 0 & x \end{pmatrix} \quad \text{and} \quad \psi_n = \begin{pmatrix} x & -z^n \\ 0 & y \end{pmatrix}, \quad n \geq 1.$$

Proof. Let $\pi : k[[x, y, z]]/xy \rightarrow k[[x, z_1]] \times k[[y, z_2]]$ be the normalization map, where $\pi(z) = z_1 + z_2$. Then, keeping the notations of Theorem 5.2, we have:

- $I = \text{ann}_A(R/A) = (x, y)$;
- $\bar{A} = A/I = k[[z]]$, $\bar{R} = k[[z_1]] \times k[[z_2]]$.

Since the categories $\text{CM}(A)$ and $\text{Tri}(A)$ are equivalent, it suffices to describe indecomposable objects of the category of triples.

Let $T = (\bar{M}, V, \theta)$ be an object of $\text{Tri}(A)$. Since the semi-local ring R is regular, by Corollary 2.11 any Cohen-Macaulay R -module \bar{M} has the form $R_1^p \oplus R_2^q$. Moreover, since $Q(\bar{A}) = k((z))$ is a field, we have: $V = k((z))^n$. Note that $\bar{R} \otimes_R \bar{M} = k((z_1))^p \oplus k((z_2))^q$. Therefore, the gluing morphism $\theta : Q(\bar{R}) \otimes_{Q(\bar{A})} V \rightarrow Q(\bar{R}) \otimes_R \bar{M}$ is given by a pair of matrices $(\theta_1(z_1), \theta_2(z_2)) \in \text{Mat}_{p \times n}(k((z_1))) \times \text{Mat}_{q \times n}(k((z_2)))$. Additional assumptions on θ imply that

- both matrices $\theta_1(z_1)$ and $\theta_2(z_2)$ have full row rank;
- the matrix $\begin{pmatrix} \theta_1(z) \\ \theta_2(z) \end{pmatrix} \in \text{Mat}_{p+q, n}(k((z)))$ has full column rank.

Note that $\text{Aut}_R(R_1^p) = \text{GL}(p, R_1)$ and $\text{Aut}_R(R_2^q) = \text{GL}(q, R_2)$. Hence,

$$\left(R_1^p \oplus R_2^q, k((z))^n, (\theta_1(z_1), \theta_2(z_2)) \right) \cong \left(R_1^p \oplus R_2^q, k((z))^n, (\theta'_1(z_1), \theta'_2(z_2)) \right)$$

in the category $\text{Tri}(A)$ if and only if there exists an element

$$(F_1(z_1), F_2(z_2), f(z)) \in \left((\text{GL}(p, k[[z_1]]) \times \text{GL}(q, k[[z_2]]) \times \text{GL}(n, k((z)))) \right)$$

such that

$$\theta'_1(z_1) = F_1^{-1}(z_1)\theta_1(z_1)f(z_1), \quad \theta'_2(z_2) = F_2^{-1}(z_2)\theta_2(z_2)f(z_2).$$

Observe that the obtained matrix problem is almost equivalent to the problem of classification of indecomposable representations of the quiver $\bullet \longleftarrow \bullet \longrightarrow \bullet$ over the field $k((z))$ and the following lemma is true.

Lemma 5.4. *The indecomposable objects of the category $\text{Tri}(A)$ are the following:*

- (1) $\left(R_1, k((z)), ((1), (\emptyset)) \right)$ and $\left(R_2, k((z)), ((\emptyset), (1)) \right)$;
- (2) $\left(R, k((z)), ((1), (1)) \right)$;
- (3) $\left(R, k((z)), ((1), (z_2^n)) \right)$ and $\left(R, k((z)), ((z_1^n), (1)) \right)$, $n \geq 1$.

Note that by Theorem 5.2 the triples of type (2) and (3) correspond to Cohen-Macaulay modules which are locally free on the punctured spectrum. It is not difficult to see that the indecomposable triples of type (1) correspond to both components R_1 and R_2 of the normalization R and the indecomposable triple of type (2) is exactly the regular module A . An interesting problem is to describe matrix factorizations corresponding to the triples of type (3).

Let $T = \left(R, k((z)), ((z_1^n), (z_2^m)) \right)$, where $n, m \geq 1$. Note that

$$e_1 = \frac{x}{x+y} \quad \text{and} \quad e_2 = \frac{y}{x+y}$$

are idempotents in $Q(A)$ and

$$z_1 = \frac{xz}{x+y} \quad \text{and} \quad z_2 = \frac{yz}{x+y}$$

where $e_1, e_2 \in R$ are viewed as elements of $Q(A) = Q(R)$. Let $X = k[[z]]$ and $\phi = (z_1^n, z_2^m)$. Consider the pull-back diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & IR & \longrightarrow & M & \longrightarrow & X \longrightarrow 0 \\ & & \downarrow = & & \downarrow & & \downarrow \phi \\ 0 & \longrightarrow & IR & \longrightarrow & R & \longrightarrow & \bar{R} \longrightarrow 0 \end{array}$$

then $\mathbb{G}(T) \cong M^\dagger$. Note that $M^\dagger = (I, z_1^n + z_2^m)_A^\dagger$, where we view $(I, z_1^n + z_2^m)_A$ as a submodule of $Q(A)$. Since the element $x + y$ is a non-zero divisor in A , we have:

$$(I, z_1^n + z_2^m)_A \cong (x + y)^{n+m} (I, z_1^n + z_2^m)_A \cong (x^{n+m+1}, y^{n+m+1}, x^{n+m}z^n + y^{n+m}z^m)_A.$$

Note that without loss of generality we may assume that either $n = 0$ or $m = 0$. For $m = 0, n \geq 1$ we have: $M_n := (x^{n+1}, y^{n+1}, x^n z^n + y^n)_A = (-x^{n+1}, x^n z^n + y^n)_A$. The minimal free resolution of M_n is

$$\dots \longrightarrow A^2 \xrightarrow{\begin{pmatrix} y & z^n \\ 0 & x \end{pmatrix}} A^2 \xrightarrow{\begin{pmatrix} x & -z^n \\ 0 & y \end{pmatrix}} A^2 \xrightarrow{\begin{pmatrix} y & z^n \\ 0 & x \end{pmatrix}} A^2 \longrightarrow M_n \longrightarrow 0.$$

In particular, $M_n \cong M_n^\dagger$ is already Cohen-Macaulay and $M_n = M(\varphi_n, \psi_n)$. \square

Remark 5.5. Using Lemma 2.34 and keeping the notations of Theorem 5.3 it is easy to see that for all $n \geq 1$ the modules $M(\varphi_n, \psi_n)$ are locally free of rank one on the punctured spectrum.

Remark 5.6. The statement of Theorem 5.3 remains true in the case of an analytical algebra $k\{x, y, z\}/xy$ with respect to an arbitrary valuation of k .

Theorem 5.7. Let $A = k[[x, y, z]]/(x^2y - z^2)$ be the coordinate ring of a surface singularity of type D_∞ . Then the indecomposable non-free Cohen-Macaulay A -modules are given by the following matrix factorizations:

(1) $M(\alpha^+, \alpha^-)$ where

$$\alpha^+ = \begin{pmatrix} z & xy \\ x & z \end{pmatrix} \quad \text{and} \quad \alpha^- = \begin{pmatrix} -z & xy \\ x & -z \end{pmatrix}.$$

Observe that $M(\alpha^+, \alpha^-) \cong M(\alpha^-, \alpha^+) \cong R$, where R is the normalization of A .

(2) $M(\beta^+, \beta^-)$, where

$$\beta^+ = \begin{pmatrix} x^2 & z \\ z & y \end{pmatrix} \quad \text{and} \quad \beta^- = \begin{pmatrix} y & -z \\ -z & x^2 \end{pmatrix}.$$

(3) $M(\gamma_m^+, \gamma_m^-)$ ($m \geq 1$), where

$$\gamma_m^+ = \begin{pmatrix} z & xy & 0 & -y^{m+1} \\ x & z & y^m & 0 \\ 0 & 0 & z & xy \\ 0 & 0 & x & z \end{pmatrix} \quad \text{and} \quad \gamma_m^- = \begin{pmatrix} -z & -xy & 0 & y^{m+1} \\ x & z & y^m & 0 \\ 0 & 0 & -z & -xy \\ 0 & 0 & x & z \end{pmatrix}.$$

(4) $M(\delta_m^+, \delta_m^-)$ ($m \geq 1$), where

$$\delta_m^+ = \begin{pmatrix} z & xy & -y^m & 0 \\ x & z & 0 & y^m \\ 0 & 0 & z & xy \\ 0 & 0 & x & z \end{pmatrix} \quad \text{and} \quad \delta_m^- = \begin{pmatrix} -z & -xy & -y^m & 0 \\ x & z & 0 & -y^m \\ 0 & 0 & -z & -xy \\ 0 & 0 & x & z \end{pmatrix}.$$

Note that in all four cases we have: $M(\phi, \psi) \cong M(\psi, \phi)$. Moreover, the indecomposable modules of types (2), (3) and (4) are locally free on the punctured spectrum.

Proof. In the notations of Theorem 5.2 we have

$$\pi : A = k[[x, y, z]]/(x^2y - z^2) \rightarrow k[[x, t]] =: R,$$

where $\pi(x) = x, \pi(y) = t^2$ and $\pi(z) = tx$, is the normalization of A . It is easy to see that

$$I = \text{ann}_A(R/A) = (x, z)A = xR$$

is the conductor ideal. Hence, $\bar{A} = A/I = k[[y]] = k[[t^2]]$ and $\bar{R} = R/I = k[[t]]$.

Let $T = (\bar{M}, V, \theta)$ be an object of the category of triples $\text{Tri}(A)$. Since the ring R is regular, by Corollary 2.11 any Cohen-Macaulay R -module \bar{M} has the form $R^n, n \geq 1$. Moreover, because $Q(\bar{A}) = k((t^2))$ is a field, we have: $V = k((t^2))^m, m \geq 1$. Note that $\bar{R} \otimes_R \bar{M} = k[[t]]^n$. Therefore, the gluing morphism $\theta : Q(\bar{R}) \otimes_{Q(\bar{A})} V \rightarrow Q(\bar{R}) \otimes_R \bar{M}$ is simply given by a matrix $\theta(t) \in \text{Mat}_{m \times n}(k((t)))$. Additional constrains on morphism θ imply that

- the matrix $\theta(t)$ has full row rank
- if $\theta(t) = \theta_0 + t\theta_1$, where $\theta_0, \theta_1 \in \text{Mat}_{m \times n}(k((t^2)))$, then the matrix $\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$ has full column rank.

Note that $\text{Aut}_R(R^m) = \text{GL}(m, R)$. Moreover, $F \in \text{Mat}_{m \times m}(R)$ is invertible if and only if $F(0)$ is invertible. Therefore, the existence of an isomorphism

$$(R^m, k((t^2))^n, \theta(t)) \cong (R^m, k((t^2))^n, \theta'(t))$$

in the category $\text{Tri}(A)$ is equivalent to the existence of an element

$$(F(t), f(t^2)) \in \left(\text{GL}(m, k[[t]]) \times \text{GL}(n, k((t^2))) \right)$$

such that

$$(2) \quad \theta' = F^{-1}\theta f.$$

A classification of indecomposable Cohen-Macaulay modules over A follows from the following lemma.

Lemma 5.8. *Let a matrix $\theta \in \text{Mat}_{m \times n}(k((t)))$ be of full row rank. Then applying the transformation rule (2) we can decompose θ to a direct sum of the following matrices:*

$$(1), (t), (1 \ t)$$

and

$$\begin{pmatrix} 1 & t \\ t^d & 0 \end{pmatrix}, \quad d \geq 1.$$

Assume Lemma 5.8 is proven. Since the isomorphism classes of indecomposable Cohen-Macaulay A -modules stand in bijection with the equivalence classes of matrices $\theta(t)$ modulo the transformation rule (2), the problem reduces to a description of modules corresponding to the canonical forms listed above.

It is clear that $\theta = 1$ corresponds to the free module A . In a similar way, it is not difficult to show that R corresponds to $\theta = (1 \ t)$. Now let us describe modules corresponding to the matrices

$$\theta = \begin{pmatrix} 1 & t \\ t^d & 0 \end{pmatrix}, \quad d \geq 1.$$

Denote $X = k[[t^2]] \oplus k[[t^2]]$, put $\phi = \theta$ and consider the pull-back diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & IR^2 & \longrightarrow & M & \longrightarrow & X \longrightarrow 0 \\ & & \downarrow = & & \downarrow & & \downarrow \phi \\ 0 & \longrightarrow & IR^2 & \longrightarrow & R^2 & \longrightarrow & \bar{R}^2 \longrightarrow 0. \end{array}$$

Then $\mathbb{G}(T) \cong M^\dagger$. Since $t = \frac{z}{x}$, we have

$$\begin{aligned} M &= \left(I(R^2), \left(\begin{array}{c} 1 \\ \frac{z^d}{x^d} \end{array} \right), \left(\begin{array}{c} \frac{z}{x} \\ 0 \end{array} \right) \right)_A \\ &= \left(\left(\begin{array}{c} x \\ 0 \end{array} \right), \left(\begin{array}{c} z \\ 0 \end{array} \right), \left(\begin{array}{c} 0 \\ x \end{array} \right), \left(\begin{array}{c} 0 \\ z \end{array} \right), \left(\begin{array}{c} 1 \\ \frac{z^d}{x^d} \end{array} \right), \left(\begin{array}{c} \frac{z}{x} \\ 0 \end{array} \right) \right)_A, \end{aligned}$$

where M is considered as a submodule of $Q(A)^2$. Note that

$$\left(\begin{array}{c} z \\ 0 \end{array} \right) = x \left(\begin{array}{c} \frac{z}{x} \\ 0 \end{array} \right),$$

hence the second generator can be omitted. Since A is an integral domain, we have:

$$\left(\begin{array}{cc} x & 0 \\ 0 & x^d \end{array} \right) M \cong M.$$

This implies:

$$M \cong \left(\left(\begin{array}{c} x^2 \\ 0 \end{array} \right), \left(\begin{array}{c} 0 \\ x^{d+1} \end{array} \right), \left(\begin{array}{c} 0 \\ x^d z \end{array} \right), \left(\begin{array}{c} x \\ z^d \end{array} \right), \left(\begin{array}{c} z \\ 0 \end{array} \right) \right)_A.$$

Next, we show that the first generator can be expressed via the remaining four. First note that

$$\left(\begin{array}{c} x^2 \\ 0 \end{array} \right) = x \left(\begin{array}{c} x \\ z^d \end{array} \right) - \left(\begin{array}{c} 0 \\ x z^d \end{array} \right).$$

Now distinguish two cases.

- Let $d = 2m, m \geq 1$, be even. Since $xz^{2m} = x^{2m+1}y^{2m} = x^{d+1}y^m$, we have:

$$\left(\begin{array}{c} 0 \\ x z^{2m} \end{array} \right) = y^m \left(\begin{array}{c} 0 \\ x^{2m+1} \end{array} \right) = y^m \left(\begin{array}{c} 0 \\ x^{d+1} \end{array} \right).$$

- Similarly, if $d = 2m + 1, m \geq 0$, is odd, we have $xz^d = xz^{2m+1} = xx^{2m}y^m z = y^m x^d z$ and

$$\left(\begin{array}{c} 0 \\ x z^d \end{array} \right) = y^m \left(\begin{array}{c} 0 \\ x^d z \end{array} \right).$$

Thus, we obtain:

$$M \cong \left(\left(\begin{array}{c} 0 \\ x^{d+1} \end{array} \right), \left(\begin{array}{c} 0 \\ x^d z \end{array} \right), \left(\begin{array}{c} x \\ z^d \end{array} \right), \left(\begin{array}{c} z \\ 0 \end{array} \right) \right)_A \subset A^2.$$

It turns out that this module is already Cohen-Macaulay. Indeed, a straightforward calculation shows that $\text{syz}(M) = \rho$, where

$$\rho = \begin{pmatrix} z & xy & 0 & -y^{m+1} \\ -x & -z & -y^m & 0 \\ 0 & 0 & z & xy \\ 0 & 0 & -x & -z \end{pmatrix}$$

in the case $d = 2m, m \geq 1$, is even, and

$$\rho = \begin{pmatrix} z & xy & -y^{m+1} & 0 \\ -x & -z & 0 & -y^{m+1} \\ 0 & 0 & z & xy \\ 0 & 0 & -x & -z \end{pmatrix} \sim \begin{pmatrix} z & xy & -y^{m+1} & 0 \\ x & z & 0 & y^{m+1} \\ 0 & 0 & z & xy \\ 0 & 0 & x & z \end{pmatrix}$$

if $d = 2m + 1, m \geq 0$, is odd. Moreover, we have the following equalities

$$\begin{pmatrix} z & xy & 0 & -y^{m+1} \\ x & z & y^m & 0 \\ 0 & 0 & z & xy \\ 0 & 0 & x & z \end{pmatrix} \begin{pmatrix} z & xy & 0 & -y^{m+1} \\ -x & -z & -y^m & 0 \\ 0 & 0 & z & xy \\ 0 & 0 & -x & -z \end{pmatrix} = (x^2 y - z^2) I_4$$

and

$$\begin{pmatrix} z & xy & y^{m+1} & 0 \\ -x & -z & 0 & y^{m+1} \\ 0 & 0 & z & xy \\ 0 & 0 & -x & -z \end{pmatrix} \begin{pmatrix} z & xy & -y^{m+1} & 0 \\ x & z & 0 & -y^{m+1} \\ 0 & 0 & z & xy \\ 0 & 0 & x & z \end{pmatrix} = (x^2y - z^2)I_4$$

which imply that the A -module M has a 2-periodic resolution, hence is Cohen-Macaulay.

It remains to identify the Cohen-Macaulay A -module corresponding to the canonical form $\theta = (t)$. Since $t = \frac{z}{x}$, the corresponding Cohen-Macaulay module is

$$M = \left(x, z, \frac{z}{x}\right)_A \cong \left(x^2, z\right)_A.$$

It is easy to see that the corresponding matrix factorization is $M(\beta^+, \beta^-)$, where

$$\beta^+ = \begin{pmatrix} x^2 & z \\ z & y \end{pmatrix} \quad \text{and} \quad \beta^- = \begin{pmatrix} y & -z \\ -z & x^2 \end{pmatrix}.$$

To complete the proof of Theorem 5.7 it remains to prove Lemma 5.8.

Proof of Lemma 5.8. It is a straightforward verification that all listed canonical forms can not be further split and are pairwise non-isomorphic.

We prove using the induction on the size, that any matrix $\theta \in \text{Mat}_{m \times n}(k((t)))$ of full row rank can be decomposed into a direct sum of blocks listed in the formulation of Lemma 5.8. The case $m = 1$ is clear. Using the transformation rule (2) the matrix θ can be reduced to the following form:

$$\begin{pmatrix} t^{d_1} I_{s_1} & A_{1,2} & \dots & A_{1,l} & A_{1,l+1} \\ 0 & t^{d_2} I_{s_2} & \dots & A_{2,l} & A_{2,l+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & t^{d_l} I_{s_l} & A_{l,l+1} \end{pmatrix},$$

where $d_1 < d_2 < \dots < d_l$ and $A_{i,j} \in \text{Mat}_{s_i \times s_j}(t^{d_i+1}\bar{R})$ for $i \leq j \leq l$ and $\bar{R} = k[[t]]$.

- If all matrices $A_{1,j} = 0$ for $2 \leq j \leq l+1$, either (1) or (t) is a direct summand of θ and we can proceed by induction.
- Assume $A_{1,j} \neq 0$ and $A_{1,p} = 0$ for $2 \leq p < j \leq l$. It is easy to see that $A_{1,j}$ can be transformed to the form

$$A'_{1,j} = t^{d_1+1} \begin{pmatrix} I_{s'_1} & 0 \\ 0 & 0 \end{pmatrix},$$

where s'_1 can be also 0. If $s'_1 = 0$, we return to the previous step, otherwise we can split the matrix

$$\begin{pmatrix} t^{d_1} & t^{d_1+1} \\ 0 & t^{d_j} \end{pmatrix}$$

as a direct summand. If $d_j = d_2 = d_1 + 1$, this matrix splits into a direct sum of two 1×1 -matrices, otherwise it is equivalent to a matrix of the form

$$\begin{pmatrix} 1 & t \\ t^d & 0 \end{pmatrix}$$

for a proper $d \geq 1$.

- Finally, if $A_{1,j} = 0$ for $2 \leq j \leq l$ and $A_{1,l+1} \neq 0$, then we can split either (1), or (t) or $(t^{d_1} \ t^{d_1+1}) \sim (1 \ t)$ as a direct summand of θ . This concludes the induction step and proves Lemma 5.8, hence Theorem 5.7. □

Remark 5.9. In the notations of Theorem 5.7 the rank of the Cohen-Macaulay modules $M(\alpha^+, \alpha^-)$ and $M(\beta^+, \beta^-)$ is one, whereas $M(\gamma_m^+, \gamma_m^-)$ and $M(\delta_m^+, \delta_m^-)$ have ranks two.

Remark 5.10. Note that for $A = k[[x, y, z]]/(x^2y - z^2)$ the stable category $\underline{\mathbf{CM}}(A)$ has the following interesting property: it is a triangulated category with a shift functor \mathbb{T} such that $\mathbb{T}(M) \cong M$ for any object M of $\underline{\mathbf{CM}}(A)$. Moreover, the stable category $\underline{\mathbf{CM}}^{\text{lf}}(A)$ of Cohen-Macaulay A -modules, which are locally free on the punctured spectrum, is a Hom-finite triangulated category having the same property.

6. GEOMETRIC MCKAY CORRESPONDENCE FOR SIMPLE SURFACE SINGULARITIES

The main goal of this section is to give a geometric description of indecomposable Cohen-Macaulay modules on simple surface singularities. Throughout this section the base field k is equal to \mathbb{C} and we work either in the category of complex analytic spaces or in the category of algebraic schemes. Let us first recall some basic results on resolutions of singularities.

Let (X, o) be the germ of a normal surface singularity and $\pi : (\tilde{X}, E) \rightarrow (X, o)$ a resolution of singularities, i.e.

- \tilde{X} is a germ of a smooth surface;
- π is a proper morphism of germs of complex-analytic spaces;
- $\pi : \tilde{X} \setminus E \rightarrow X \setminus o$ is an isomorphism and $\tilde{X} \setminus E$ is dense in \tilde{X} . In particular, π is birationally an isomorphism.

In this case the *exceptional fiber* $E = \pi^{-1}(o)$ is a complex projective curve (possibly singular). Remind that E is always connected (it follows from Zariski's Main Theorem, see [48, Corollary III.11.4]).

Definition 6.1. A resolution of singularities $\pi : \tilde{X} \rightarrow X$ is called *minimal* if \tilde{X} does not contain contractible curves, i.e. smooth rational projective curves with self-intersection -1 .

Remark 6.2. A minimal resolution of singularities has the following universal property. If $\pi' : (\tilde{X}', E') \rightarrow (X, o)$ is any other resolution of singularities, then there exists a unique morphism of germs of complex-analytic spaces $f : (\tilde{X}', E') \rightarrow (\tilde{X}, E)$ such that the diagram

$$\begin{array}{ccc} \tilde{X}' & \xrightarrow{f} & \tilde{X} \\ & \searrow \pi' & \swarrow \pi \\ & X & \end{array}$$

is commutative, see [48, Section V.5], [43] and [68].

The main topological invariant of a normal surface singularity (X, o) is the intersection matrix of its minimal resolution.

Definition 6.3. Let $\pi : (\tilde{X}, E) \rightarrow (X, o)$ be a minimal resolution of a normal surface singularity (X, o) and $E = \cup_{i=1}^n E_i$ the decomposition of the exceptional divisor E into a union of irreducible components. Then $M(X) = (m_{i,j}) = (E_i \cdot E_j)_{i,j} \in \text{Mat}_{n \times n}(\mathbb{Z})$ is called the *intersection matrix*.

Proposition 6.4 (Mumford). *Let (X, o) be a normal surface singularity, then its intersection matrix $M(X)$ is non-degenerate and negatively definite.*

For a proof of this result, see for example [68, Theorem 4.4].

The key technique to resolve normal surface singularities is provided by the following construction. Assume for simplicity of notation that we work in the category of algebraic schemes over an algebraically closed field k . Consider the scheme $\tilde{\mathbb{A}}^3$ defined as follows:

$$\mathbb{A}^3 \times \mathbb{P}^2 \supseteq \tilde{\mathbb{A}}^3 := \left\{ ((x_1, x_2, x_3), (y_1 : y_2 : y_3)) \mid x_i y_j = x_j y_i, \quad 1 \leq i, j \leq 3 \right\},$$

where \mathbb{A}^3 is the three-dimensional affine space over k and \mathbb{P}^2 is the two-dimensional projective space over k . Note that $\tilde{\mathbb{A}}^3$ is an algebraic scheme of Krull dimension three (although not affine) and we have a morphism of schemes $\pi : \tilde{\mathbb{A}}^3 \rightarrow \mathbb{A}^3$ defined as the composition of the inclusion $\tilde{\mathbb{A}}^3 \hookrightarrow \mathbb{A}^3 \times \mathbb{P}^2$ and the canonical projection $\mathbb{A}^3 \times \mathbb{P}^2 \rightarrow \mathbb{A}^3$. Moreover, the following properties hold:

1. The morphism π is projective (hence proper).
2. Let $o := (0, 0, 0) \in \mathbb{A}^3$. Then $\pi^{-1}(o) = \left\{ ((0, 0, 0), (y_1 : y_2 : y_3)) \right\} \cong \mathbb{P}^2$ and for any other point $x \in \mathbb{A}^3 \setminus \{o\}$ the preimage $\pi^{-1}(x)$ is a single point. Hence, $\pi : \tilde{\mathbb{A}}^3 \setminus \pi^{-1}(o) \rightarrow \mathbb{A}^3 \setminus \{o\}$ is an isomorphism and π itself is a birational isomorphism.
3. The projective space \mathbb{P}^2 has three affine charts

$$U_i = \{(y_1 : y_2 : y_3) \mid y_i \neq 0\}, 1 \leq i \leq 3.$$

Moreover, we have isomorphisms $\phi_i : \tilde{\mathbb{A}}_i^3 := \tilde{\mathbb{A}}^3 \cap (\mathbb{A}^3 \times U_i) \xrightarrow{\cong} \mathbb{A}^3$, $1 \leq i \leq 3$. For example, take $i = 1$ and denote $\tilde{y}_2 = \frac{y_2}{y_1}$ and $\tilde{y}_3 = \frac{y_3}{y_1}$. Then

$$\tilde{\mathbb{A}}_1^3 = \left\{ ((x_1, x_2, x_3), (1 : \tilde{y}_2 : \tilde{y}_3)) \mid x_2 = x_1 \tilde{y}_2, x_3 = x_1 \tilde{y}_3 \right\}$$

and the isomorphism $\phi_1 : \tilde{\mathbb{A}}_1^3 \rightarrow \mathbb{A}^3$ is given by the formula

$$((x_1, x_2, x_3), (1 : \tilde{y}_2 : \tilde{y}_3)) \mapsto (x_1, \tilde{y}_2, \tilde{y}_3).$$

In particular, the morphism

$$\pi_1 : \text{Spec}(k[x_1, \tilde{y}_2, \tilde{y}_3]) = \mathbb{A}^3 \xrightarrow{\phi_1^{-1}} \tilde{\mathbb{A}}_1^3 \hookrightarrow \tilde{\mathbb{A}}^3 \xrightarrow{\pi} \mathbb{A}^3 = \text{Spec}(k[x_1, x_2, x_3])$$

is given by the formulae: $x_1 \mapsto x_1, x_2 \mapsto x_1 \tilde{y}_2$ and $x_3 \mapsto x_1 \tilde{y}_3$.

Let $f \in k[x, y, z]$ be a polynomial such that $X := V(f) \subseteq \mathbb{A}^3$ is a normal surface and $o \in X$ is a singular point. Let $\tilde{X} := \pi^{-1}(X \setminus \{o\}) \subseteq \tilde{\mathbb{A}}^3$. From the commutative diagram

$$\begin{array}{ccc} \pi^{-1}(X \setminus \{o\}) & \hookrightarrow & \tilde{\mathbb{A}}^3 \setminus \pi^{-1}(o) \\ \downarrow & & \downarrow \\ X \setminus \{o\} & \hookrightarrow & \mathbb{A}^3 \setminus \{o\} \end{array}$$

we obtain an induced morphism $\pi_X : \tilde{X} \rightarrow X$, which is proper and birational. Moreover, the exceptional fiber $E = \pi_X^{-1}(o) = \tilde{X} \cap \pi^{-1}(o) \subseteq \mathbb{P}^2$ is a closed subscheme of a projective plane, hence it is a projective curve. The constructed morphism $\pi_X : \tilde{X} \rightarrow X$ is called the *blowing-up* of the surface X at the singular point o .

We illustrate the technique of resolutions of surface singularities on two examples.

Example 6.5. Consider a surface singularity of type A_1 given by the equation $x^2 + y^2 + z^2 = 0$. Taking the chart $u \neq 0$ of the scheme $\tilde{\mathbb{A}}^3 = \left\{ ((x, y, z), (u : v : w)) \mid xv = yu, xw = zu, yw = zv \right\}$, we get the following morphisms of affine schemes:

$$\mathbb{A}^3 \xleftarrow{\pi} \tilde{\mathbb{A}}_1^3 := \left\{ ((x, y, z), (1 : v : w)) \mid xv = y, yw = z \right\} \xrightarrow{\phi_1} \mathbb{A}^3,$$

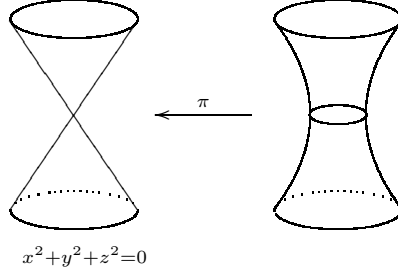
where $\pi\left(\left((x, y, z), (1 : v : w)\right)\right) = (x, y, z)$ and $\phi_1\left(\left((x, y, z), (1 : v : w)\right)\right) = (x, xv, xw)$. Then the morphism $\pi_1^* = (\pi \circ \phi_1^{-1})^* : k[x, y, z] \rightarrow k[x, v, w]$ maps $x^2 + y^2 + z^2$ to $x^2 + x^2v^2 + x^2w^2$. It is easy to see that

$$\phi_1(\tilde{X} \cap \tilde{\mathbb{A}}_1^3) = \phi_1(\overline{\pi^{-1}(X \setminus \{o\})} \cap \tilde{\mathbb{A}}_1^3) \subseteq \mathbb{A}^3 = \text{Spec}(k[x, v, w])$$

is given by the equation $1 + v^2 + w^2 = 0$, which is a smooth surface.

Since the fiber $\pi^{-1}(o)$ of the morphism $\pi : \tilde{\mathbb{A}}^3 \rightarrow \mathbb{A}^3$ in the local chart $\phi_1(\tilde{\mathbb{A}}^3)$ in the chart $\tilde{\mathbb{A}}_1^3$ is just $\mathbb{A}^2 = \{(0, v, w)\} = V(x)$, the exceptional curve $E = \tilde{X} \cap \pi^{-1}(o)$ is $V(x, 1 + v^2 + w^2) \cong \mathbb{A}^1$.

The description π in other charts $v \neq 0$ and $w \neq 0$ is completely symmetric. Hence, the blowing-up $\pi : \tilde{X} \rightarrow X$ is already a resolution of singularities and the exceptional fiber $E = \pi^{-1}(o)$ is isomorphic to \mathbb{P}^1 .



The next problem is to compute the self-intersection number of the exceptional divisor E . To do this, we use the following trick. Since the constructed morphism $\pi : \tilde{X} \rightarrow X$ is a birational isomorphism, it induces an isomorphism of the fields of rational functions $\pi^* : k(X) \rightarrow k(\tilde{X})$. Moreover, for any function $f \in k[X]$ we have: $\text{div}(\pi^*(f)) \cdot E = 0$. In our particular case, take f to be the class of y in the ring $k[x, y, z]/(x^2 + y^2 + z^2)$. Consider the chart $u \neq 0$ in $\tilde{\mathbb{A}}^3$ and the following commutative diagram

$$\begin{array}{ccc} \tilde{X} \cap \tilde{\mathbb{A}}_1^3 & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathbb{A}^3 & \xleftarrow{\phi_1} \tilde{\mathbb{A}}_i^3 \xrightarrow{\pi} & \mathbb{A}^3 \end{array}$$

where $\pi_1 = \pi \circ \phi_1^{-1} : \mathbb{A}^3 \rightarrow \mathbb{A}^3$ is given by $\pi_1(x, v, w) = (x, xv, xw)$. It is easy to see that $V(\pi^*(f)) \subseteq \tilde{X}$ is given by the equation $xv = 0$. Moreover, we have:

$$\text{div}(\pi^*(f)) = E + C_1 + C_2,$$

where $E = V(x)$ is the exceptional fiber and two other components are $C_1 = V(v, w - \sqrt{-1})$ and $C_2 = V(v, w + \sqrt{-1})$. Since E intersect C_1 and C_2 transversally exactly at one point, we get:

$$0 = \text{div}(\pi^*(f)) \cdot E = (E + C_1 + C_2) \cdot E = E^2 + C_1 \cdot E + C_2 \cdot E = E^2 + 2,$$

hence $E^2 = -2$.

In Example 6.5 a minimal resolution of an A_1 -singularity was achieved by a single blow-up. Of course, it does not reflect the situation one has in general. The following example shows various tricks and pitfalls one can meet by resolving normal surface singularities.

Example 6.6. Let $X \subseteq \mathbb{A}^3$ be the surface given by the equation $x^2 + y^3 + z^7 + y^2z^2 = 0$. At the point $o = (0, 0, 0)$ it has the so-called $T_{2,3,7}$ -singularity. Let us show how this normal singularity can be resolved.

Step 1. Recall that $\tilde{\mathbb{A}}^3 = \left\{ ((x, y, z), (u : v : w)) \mid xv = uy, yw = uz \right\}$. Let $q : \tilde{\mathbb{A}}^3 \rightarrow \mathbb{A}^3$ be the projection morphism $q((x, y, z), (u : v : w)) = (x, y, z)$ and $Y = \overline{\pi^{-1}(X \setminus \{o\})} \xrightarrow{q} X$ be the blowing-up of X at the point o . Consider the chart $w \neq 0$ and denote $\tilde{u} = \frac{u}{w}$, $\tilde{v} = \frac{v}{w}$. In these coordinates, the surface Y is given by the equation $\tilde{u}^2 + z\tilde{v}^3 + z^5 + z^2\tilde{v}^2 = 0$. At the point $(0, 0, 0)$ it has an isolated singularity. The exceptional divisor $E_1 = q^{-1}(o)$ is given in this chart by the equations $u = z = 0$. It is not difficult to check that in two other charts $u \neq 0$ and $v \neq 0$ the surface Y is smooth and that $E_1 \cong \mathbb{P}^1$.

Step 2. For the sake of simplicity of the notation, denote again $x = \tilde{u}$ and $y = \tilde{v}$. The surface singularity Y is defined by the equation

$$x^2 + zy^3 + z^2y^2 + z^5 = 0$$

and $o = (0, 0, 0)$ is its unique singular point. We also have to keep track of the equation of the exceptional fiber $E_1 = V(x, z)$. Consider again the blowing-up $q : \tilde{Y} \rightarrow Y$ of the surface Y at the point o obtained from the morphism $\tilde{\mathbb{A}}^3 = \left\{ ((x, y, z), (u : v : w)) \mid xv = uy, yw = uz \right\} \rightarrow \mathbb{A}^3$. This time, however, we have to look at two charts.

Case 1. Consider first the chart $w \neq 0$ and denote $u_1 = \frac{u}{w}$, $v_1 = \frac{v}{w}$. In these coordinates, the surface \tilde{Y} is given by the equation

$$u_1^2 + z^2v_1^3 + z^4v_1^2 + z^5 = 0.$$

Note that \tilde{Y} is not normal! Indeed, in this chart the singular locus of \tilde{Y} is given by the equations $z = u_1 = 0$. Moreover, the singular locus coincides with the exceptional fiber $E_2 = p^{-1}(o)$! Note also, that $q^{-1}(E_1)$ does not belong to this chart.

Case 2. Consider another chart $v \neq 0$ and denote $u_2 = \frac{u}{v}$, $w_2 = \frac{w}{v}$. In these coordinates, the surface \tilde{Y} is given by the equation

$$u_2^2 + y^2w_2 + y^2w_2^2 + y^3w_2^5 = 0.$$

In this chart, the exceptional fiber $E_2 = V(u_2, y)$ coincides with the singular locus of \tilde{Y} and $q^{-1}(E_1) = V(u_2, w_2)$.

Case 3. In the third chart $v \neq 0$, the surface \tilde{Y} is given by the following equation:

$$1 + x^2v_3^3w_3 + x^2v_3^2w_3^2 + x^3w_3^5 = 0.$$

It is easy to see that in this chart \tilde{Y} is smooth.

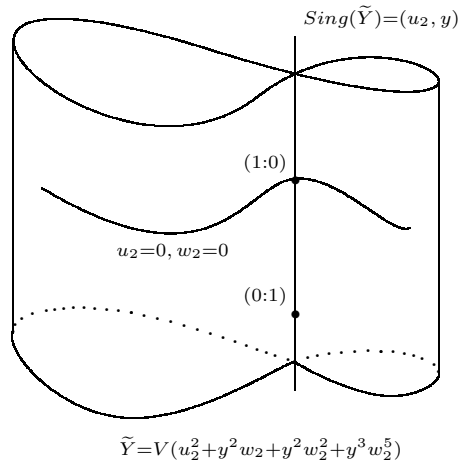
Summing everything up, we get that $\tilde{Y} \subset \tilde{\mathbb{A}}^3 \subseteq \mathbb{A}^3 \times \mathbb{P}^2$ is a surface with one-dimensional singular locus,

$$E_1 = \left\{ ((0, y, 0), (0 : 1 : 0)) \mid y \in \mathbb{A}^1 \right\}$$

is the strict transform of the first exceptional divisor and

$$E_2 = \left\{ ((0, 0, 0), (0 : v : w)) \mid (v : w) \in \mathbb{P}^2 \right\}$$

is the the second exceptional divisor, which coincides with the singular locus of \tilde{Y} .



Step 3. We resolve singularities of \tilde{Y} using the *normalization* morphism $n : \bar{Y} \rightarrow \tilde{Y}$.

Case 1. Consider the ring homomorphism $k[u_1, v_1, z] \rightarrow k[t_1, v, z]$ given by $u_1 \mapsto t_1 z, v_1 \mapsto v_1$ and $z \mapsto z$. This induces the following ring homomorphism of coordinate algebras:

$$k[u_1, v_1, z]/(u_1^2 + z^2 v_1^3 + z^4 v_1^2 + z^5) \longrightarrow k[t_1, v_1, z]/(t_1^2 + v_1^3 + v_1^2 + z).$$

Note that $\bar{Y} = V(t_1^2 + v_1^3 + v_1^2 + z) \subseteq \mathbb{A}_{(t_1, v_1, z)}^3$ is a smooth surface and the morphism of algebraic schemes $n : \bar{Y} \rightarrow \tilde{Y}$ is the normalization map. Next, the preimage of \tilde{E}_2 under n is given by the equations

$$t_1^2 + v_1^3 + v_1^2 = 0, \quad z = 0.$$

This is a nodal cubic curve, having an A_1 -singularity at the point $t_1 = v_1 = z = 0$. Note that $n(t_1, v_1, 0) = v_1$, hence $n : \tilde{E}_2 \rightarrow E_2$ is a ramified covering of order two.

Case 2. In a similar way, consider the morphism of affine spaces $n : \mathbb{A}_{(u_2, w_2, y)}^3 \rightarrow \mathbb{A}_{(t_2, w_2, y)}^3$ given by the formula $(u_2, w_2, y) \mapsto (u_2 y, w_2, y)$. This induces the following ring homomorphism of coordinate algebras:

$$k[u_2, w_2, y]/(u_2^2 + y^2 w_2 + y^2 w_2^2 + y^3 w_2^5) \longrightarrow k[t_2, w_2, y]/(t_2^2 + w_2 + w_2^2 + y w_2^5).$$

Again, $\bar{Y} = V(t_2^2 + w_2 + w_2^2 + y w_2^5) \subseteq \mathbb{A}_{(t_2, w_2, y)}^3$ is a smooth surface and the morphism of algebraic schemes $n : \bar{Y} \rightarrow \tilde{Y}$ is the normalization map. In this chart, the preimage \tilde{E}_2 of the exceptional divisor E_2 is given by the equation

$$t_2^2 + w_2 + w_2^2 = 0, \quad y = 0.$$

Moreover, the morphism n induces an isomorphism of E_1 on its preimage \tilde{E}_1 , given by the equation

$$t_2 = w_2 = 0.$$

Step 4. So far, we have constructed a smooth surface \bar{Y} and a birational isomorphism $\pi : \bar{Y} \rightarrow X$, given by a sequence of projective birational isomorphisms

$$\bar{Y} \xrightarrow{n} \tilde{Y} \xrightarrow{p} Y \xrightarrow{q} X.$$

Moreover, $\pi^{-1}(o) = \tilde{E}_1 \cup \tilde{E}_2$ is a reducible curve with two rational components intersecting transversally, where $\tilde{E}_1 \cong \mathbb{P}^1$ and \tilde{E}_2 is isomorphic to a plane projective nodal cubic curve.

Our next goal is to determine the self-intersection numbers of the divisors \tilde{E}_1 and \tilde{E}_2 . To do this, take the function

$$f = x \in k[X] = k[x, y, z]/(x^2 + y^3 + z^7 + y^2 z^2).$$

Then a straightforward computation shows that the function $\tilde{f} := \pi^*(f) \in k[\bar{Y}]$ is given by the following formulae.

Case 1. In the coordinates from Step 3 (Case 1) we have: $\tilde{f} = \pi^*(f) = z^3 t_1$. Note that the order of \tilde{f} in the local ring $k[x, y, z]_{\langle z \rangle} / (t_1^2 + v_1^3 + v_1^2 + z)$ is three, where $\langle z \rangle \subset k[x, y, z]$ is the prime ideal generated by z . Moreover, the exceptional fiber $\tilde{E}_2 = V(t_1^2 + v_1^3 + v_1^2, z)$ intersects the curve given by the equation $t_1 = 0$ at two points of $\mathbb{A}_{(t_1, v_1, z)}^3$. The first of them is $(0, 0, 0)$, which is the singular point of \tilde{E}_2 . The second is $(0, -1, 0)$. Note that the intersection of these two curves is transversal at the second point and has multiplicity two at the first one. Hence, in this chart

$$\operatorname{div}(\tilde{f}) = 3E_2 + C$$

and $C \cdot E_2 = 2 + 1 = 3$.

Case 2. In the coordinates from Step 3 (Case 2) we have: $\tilde{f} = \pi^*(f) = y^3 t_2 w_2$. Recall that the exceptional divisor \tilde{E}_1 in this chart is described as follows:

$$\tilde{E}_1 = V(t_2, w_2) \subseteq \bar{Y} = V(t_2^2 + w_2 + w_2^2 + y w_2^5) \subseteq \mathbb{A}_{(t_2, w_2, y)}^3.$$

Moreover, the second exceptional divisor \tilde{E}_2 is given by the formula

$$\tilde{E}_2 = V(y, t_2).$$

Since $t_2 = w_2(1 + w_2 + yw_2^4)$, the curve \tilde{E}_1 is locally given by the equation $t_2 = 0$, and \tilde{f} vanish on \tilde{E}_2 with multiplicity three. Note that the strict transform of f is given by the polynomial $g = 1 + w_2 + yw_2^4$. Observe that $V(g)$ and E_1 do not intersect, whereas

$$V(g) \cap E_1 = (0, -1, 0) \in \mathbb{A}_{(t_2, w_2, y)}^3.$$

However, it is the same point as the one found in Case 1! Summing everything up, we obtain:

- We have constructed a projective birational isomorphism $\pi : \bar{Y} \rightarrow X$, whose exceptional locus consists of the union of two rational curves \tilde{E}_1 and \tilde{E}_2 . Moreover, $\tilde{E}_1 \cong \mathbb{P}^1$ and \tilde{E}_2 is isomorphic to a plane nodal cubic curve. These curves intersect transversally at one smooth point.
- We have: $\text{div}(\tilde{f}) = \text{div}(\pi^*(f)) = 3\tilde{E}_1 + 3\tilde{E}_2 + C$.
- The curve C does not intersect \tilde{E}_1 . Moreover, it intersects \tilde{E}_2 with multiplicity two at its singular point and transversally at another point (which is smooth). All-together, this implies:

$$\begin{cases} C \cdot \tilde{E}_1 &= 0 \\ C \cdot \tilde{E}_2 &= 3 \\ \tilde{E}_1 \cdot \tilde{E}_2 &= 1. \end{cases}$$

These computations imply that

$$0 = \text{div}(\pi^*(f)) \cdot \tilde{E}_1 = 3\tilde{E}_1^2 + 3 = 0,$$

hence $\tilde{E}_1^2 = -1$. In a similar way,

$$0 = \text{div}(\pi^*(f)) \cdot \tilde{E}_2 = 3\tilde{E}_1^2 + 6 = 0,$$

hence $\tilde{E}_1^2 = -2$.

Step 5. We have constructed a resolution of singularities $\pi : \bar{Y} \rightarrow X$, whose exceptional divisor is $\tilde{E} = \tilde{E}_1 \cup \tilde{E}_2$. However, since $E_1 \cong \mathbb{P}^1$ and $\tilde{E}_1^2 = -1$, this resolution is not minimal! Hence, using Castelnuovo's theorem (see [48, Theorem V.5.7]) we can blow down \tilde{E}_1 and obtain a commutative diagram

$$\begin{array}{ccc} \bar{Y} & \xrightarrow{\phi} & \tilde{X} \\ & \searrow \pi & \swarrow \pi' \\ & X & \end{array}$$

where the surface \tilde{X} is smooth and $\phi : \bar{Y} \setminus \tilde{E}_1 \rightarrow \tilde{X} \setminus \phi(\tilde{E}_1)$ is an isomorphism. Note that the morphism ϕ induces a ring homomorphism of Chow groups $\phi^* : A^*(\tilde{X}) \rightarrow A^*(\bar{Y})$. Hence, if $D = \phi(\tilde{E}_2)$ is the exceptional divisor of the resolution of singularities $\pi' : \tilde{X} \rightarrow X$, then

$$D^2 = (\tilde{E}_1 + \tilde{E}_2)^2 = \tilde{E}_1^2 + 2\tilde{E}_1 \cdot \tilde{E}_2 + \tilde{E}_2^2 = -1.$$

Summing everything up, the singularity $(X, o) = (V(x^2 + y^3 + z^7 + y^2z^2), o) \subseteq (\mathbb{A}^3, o)$ has a minimal resolution $\pi' : (\tilde{X}, D) \rightarrow X$ such that D is isomorphic to the plane projective cubic curve $zy^2 = x^3 + x^2z$. Moreover, the self-intersection index of D is -1 . \square

Definition 6.7. Let (X, o) be the germ of a surface singularity and $\pi : (\tilde{X}, E) \rightarrow (X, o)$ its minimal resolution. A *fundamental cycle* is the minimal cycle $Z = m_1E_1 + m_2E_2 + \cdots + m_nE_n$, such that $m_i > 0$ and $(Z, E_i) \leq 0$ for all $1 \leq i \leq n$. By a result of Artin, see [3, Proposition 2], a fundamental cycle *exists* and is *unique*.

Definition 6.8. In the notations of Definition 6.7 the singularity (X, o) is called *rational* if $R^1\pi_*\mathcal{O}_{\tilde{X}} = 0$ (it is equivalent to $H^1(\mathcal{O}_{\tilde{X}}) = 0$). Note that since X is normal, we have $\pi_*\mathcal{O}_{\tilde{X}} = \mathcal{O}_X$.

Proposition 6.9. *Let (X, o) be a rational normal surface singularity, $\pi : (\tilde{X}, E) \rightarrow (X, o)$ its minimal resolution and E_1, E_2, \dots, E_n be the irreducible components of E . Then we have:*

- All components E_i are smooth and rational, i.e. $E_i \cong \mathbb{P}^1$;
- For all $i \neq j \neq l$ we have: $E_i \cap E_j \cap E_l = \emptyset$.
- For all $1 \leq i \neq j \leq n$ we have: $E_i \cdot E_j \in \{0, 1\}$, i.e. E is a configuration of projective lines intersecting transversally.
- Moreover E has no cycles, i.e. it is a tree of projective lines.

A proof of this Proposition can be found in [18, Lemma 1.3].

Remark 6.10. Note that there exist non-rational normal surface singularities such that E is a tree of projective lines, see [68, Proposition 3.5].

Knowing the intersection matrix $M(X)$ of a normal surface singularity (X, o) one can determine its local fundamental group. In particular, in the case of rational normal surface singularities we have:

Theorem 6.11 (Brieskorn). *The local fundamental group $G = \pi_1(X, o)$ of a rational normal surface singularity (X, o) can be presented by the following generators and relations:*

$$G = \left\langle g_1, g_2, \dots, g_n \mid g_i g_j^{m_{ij}} = g_j^{m_{ij}} g_i, 1 \leq i, j \leq n, g_1^{m_{i1}} g_2^{m_{i2}} \dots g_n^{m_{in}} = e, 1 \leq i \leq n \right\rangle,$$

where $M = M(X, o) = (m_{ij}) \in \mathrm{GL}_n(\mathbb{Z})$ is the intersection matrix of (X, o) .

For a proof of this theorem we refer to [18, Lemma 2.7] and [73, Chapter 1].

Lemma 6.12. *Let (X, o) be a rational singularity, $E = \cup_{i=1}^n E_i$. Then $\mathrm{Pic}(\tilde{X}) \cong H^2(\tilde{X}, \mathbb{Z}) \cong \mathbb{Z}^n$.*

Proof. Consider the exponential sequence $0 \rightarrow \mathbb{Z}_{\tilde{X}} \rightarrow \mathcal{O}_{\tilde{X}} \xrightarrow{\exp} \mathcal{O}_{\tilde{X}}^* \rightarrow 0$. Since $H^2(\mathcal{O}_{\tilde{X}}) = H^1(\mathcal{O}_{\tilde{X}}) = 0$, taking cohomology we get isomorphisms

$$\mathrm{Pic}(\tilde{X}) \cong H^1(\mathcal{O}_{\tilde{X}}^*) \xrightarrow{c_1} H^2(\tilde{X}, \mathbb{Z}) \xrightarrow{\phi} \mathbb{Z}^n$$

where $c_1(\mathcal{L})$ is the first Chern class of a line bundle \mathcal{L} and the isomorphism ϕ is defined by the rule

$$\phi(D) = (D \cdot E_1, D \cdot E_2, \dots, D \cdot E_n) \in \mathbb{Z}^n.$$

□

Remark 6.13. Lemma 6.12 essentially means that the divisors on a minimal resolution of a rational singularity can be described *topologically*.

At this point we would like to have some examples of rational normal surface singularities.

Lemma 6.14. *Let $G \subset \mathrm{GL}_2(\mathbb{C})$ be a finite small subgroup, $A = \mathbb{C}\{x_1, x_2\}^G$ and (X, o) the corresponding complex germ. Then the singularity (X, o) is rational.*

For a proof of this Lemma, see [3] and [18].

The following proposition says when a rational normal surface singularity is a complete intersection.

Theorem 6.15 (Wahl). *Let (X, o) be a complex rational normal surface singularity, which is a complete intersection and (A, \mathfrak{m}) be the corresponding local analytic algebra. Then $A \cong \mathbb{C}\{x_1, x_2\}^G$, where $G \subset \mathrm{SL}_2(\mathbb{C})$ is a finite subgroup.*

For a proof of this theorem, see [88, Theorem 2.1].

In other words, normal rational singularities, which are complete intersections, are exactly simple hypersurface singularities. Moreover, let $\Delta = \Delta_n$ be the Dynkin type of (X, o) , then

- For any irreducible component E_i of E we have: $E_i^2 = -2$.
- Moreover, the quadratic form on $\mathbb{Z}^n \times \mathbb{Z}^n \rightarrow \mathbb{Z}$ defined by the intersection matrix $M(X)$ is equal to $-q_\Delta$, where q_Δ is Tits quadratic form attached to Δ .
- The fundamental cycle $Z = m_1E_1 + m_2E_2 + \dots + m_nE_n$ is the *maximal root* of the quadratic form of $M(X)$. In particular, $Z^2 = -1$.

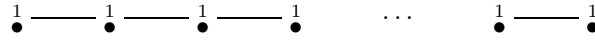
Lemma 6.16 (Artin). *More generally, the embedding dimension of a normal rational surface singularity $(X, o) = \text{Specan}(A, \mathfrak{m})$ can be computed by the formula*

$$\text{edim}(X, o) := \dim_{\mathbb{C}}(\mathfrak{m}/\mathfrak{m}^2) = -Z^2 + 1.$$

For a proof of this formula, see [3, Corollary 6].

Remark 6.17. The fundamental cycles of the simple surface singularities are described below. Bullets in the underlying diagrams denote irreducible components, they are connected by an edge if the corresponding components intersect. Integers drawn over bullets denote coefficients of these components in the formula for the fundamental cycle. The obtained graph is called the *dual intersection graph* of the isolated surface singularity. In all listed cases, if $\Delta = \Delta_n$ is the Dynkin type of the quotient singularity, then the number of irreducible components of the exceptional divisor of the minimal resolution is n .

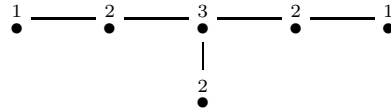
- A_n -singularity $x^2 + y^{n+1} + z^2$ ($n \geq 1$) has the following dual intersection graph and the fundamental cycle:



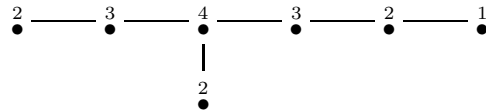
- In the case of D_n -singularities $x^2y + y^{n-1} + z^2$ ($n \geq 4$) we have



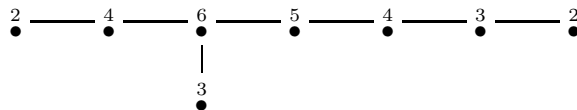
- E_6 -singularity $x^4 + y^3 + z^2$:



- E_7 -singularity $x^3y + y^3 + z^2$:



- Finally, in the case of E_8 -singularity $x^5 + y^3 + z^2$ we have:



Theorem 6.18 (Gonzalez-Springer–Verdier, Artin–Verdier). *Let $G \subset \mathrm{SL}_2(\mathbb{C})$ be a finite subgroup, $A = \mathbb{C}\{x_1, x_2\}^G$ and $(X, o) = \mathrm{Specan}(A)$ the corresponding quotient singularity, $\pi : (\tilde{X}, E) \rightarrow (X, o)$ its minimal resolution, $E = E_1 \cup E_2 \cup \dots \cup E_n$ the decomposition of E into a union of irreducible components and Z the fundamental cycle. Let M be an indecomposable Cohen-Macaulay A -module, \mathcal{M} the corresponding coherent sheaf on X and $\tilde{\mathcal{M}} = \pi^*\mathcal{M}/\mathrm{tors}(\pi^*\mathcal{M})$. Then the following properties hold:*

- $\tilde{\mathcal{M}}$ is a vector bundle on \tilde{X} ;
- Let $D = c_1(\tilde{\mathcal{M}})$. If $M \not\cong A$ then there exists $1 \leq i \leq n$ such that $D = E_i^*$, i.e. $D \cdot E_j = \delta_{i,j}$. Moreover, $\mathrm{rank}(M) = Z \cdot D$.

For a proof of this theorem, see [5, Theorem 1.11].

Remark 6.19. Theorem 6.18 implies that a Cohen-Macaulay M on a simple surface singularity (A, \mathfrak{m}) is determined by two discrete parameters: $\mathrm{rank}(M)$ and $D = c_1(\tilde{\mathcal{M}}) \in H^2(\tilde{X}, \mathbb{Z}) \cong \mathbb{Z}^n$, where n is the number of irreducible components of the exceptional divisor E . The cohomology class D satisfies the following property: $D \cdot E_i \geq 0$ for all $1 \leq i \leq n$.

Corollary 6.20. *Let $G \subset \mathrm{SL}_2(\mathbb{C})$ be a finite subgroup, $A = \mathbb{C}\{x_1, x_2\}^G$ and (X, o) the corresponding complex singularity. Let $\pi : (\tilde{X}, E) \rightarrow (X, o)$ be a minimal resolution of singularities. Combining Theorem 6.18 and Theorem 4.15, we obtain a bijection between the following three sets:*

- (1) *Isomorphism classes of indecomposable non-free Cohen-Macaulay A -modules;*
- (2) *Isomorphism classes of non-trivial irreducible representations of G ;*
- (3) *Irreducible components of the exceptional divisor E .*

Remark 6.21. A certain generalization of Theorem 6.18 in the case of arbitrary cyclic quotient singularities was obtained by Wunram [91]. The general case of Cohen-Macaulay modules on arbitrary quotient surface singularities was studied in a work of Esnault [36], see also [61, Chapter 4].

7. COHEN-MACAULAY MODULES OVER MINIMALLY ELLIPTIC SINGULARITIES AND VECTOR BUNDLES ON GENUS ONE CURVES

Two-dimensional Gorenstein quotient singularities are exactly the two-dimensional hypersurface singularities of modality zero [2]. This in particular means that a deformation of a simple surface singularity is again a simple surface singularity.

The next interesting class of surface singularities is formed by singularities of modality one. The most interesting among them are the so-called *minimally elliptic* singularities.

Definition 7.1 (Laufer, see Definition 3.2 in [69]). A normal surface singularity (X, o) is called *minimally elliptic* if it is Gorenstein and $H^1(\mathcal{O}_{\tilde{X}}) = k$, where $\pi : (\tilde{X}, E) \rightarrow (X, o)$ is its minimal resolution.

Theorem 7.2 (Laufer). *Let (X, o) be a minimally elliptic singularity, $\pi : (\tilde{X}, E) \rightarrow (X, o)$ its minimal resolution of singularities. Then E is a configuration of projective lines with transversal intersections and a tree as its dual graph, except the following cases:*

- (1) *E is an elliptic curve; then (X, o) is called simple elliptic.*
- (2) *E is a cycle of n projective lines (a plane nodal cubic curve $zy^2 = x^3 - x^2z$ for $n = 1$); then, following Hirzebruch [54], the corresponding singularity is called a cusp.*
- (3) *E is a cuspidal cubic curve $zy^2 = x^3$, a tacnode curve $(yz - x^2)(yz + x^2) = 0$ or three concurrent lines in a plane $xy(x - y) = 0$.*

For a proof of this theorem, see [69, Proposition 3.5].

Example 7.3. The surface singularity, given by the equation $x^2 + y^3 + z^7 = 0$ is minimally elliptic. In this case the exceptional divisor is isomorphic to a cuspidal plane cubic curve $zy^2 = x^3$.

The surface singularity $x^2 + y^3 + z^7 + xyz = 0$ is a cusp. Its resolution of singularities was described in Example 6.6. In particular, its exceptional divisor is a nodal plane cubic curve $zy^2 = x^3 + x^2z$ having self-intersection index -1 .

Other examples of minimally elliptic singularities of small multiplicities can be found in [69, Tables 1, 2 and 3].

Theorem 7.4 (Laufer). *Let (X, o) be a minimally elliptic singularity, $(\tilde{X}, E) \rightarrow (X, o)$ a minimal resolution of singularities and Z the fundamental cycle. Then*

$$\text{edim}(X, o) = \max\{-Z^2, 3\}.$$

For a proof of this theorem, see [69, Theorem 3.13].

Theorem 7.5. *Let (X, o) be a simple elliptic singularity, $(\tilde{X}, E) \rightarrow (X, o)$ its minimal resolution. Then the analytic type of the singularity (X, o) is determined by the analytic type of the elliptic curve E and its self-intersection number $E^2 = -b$, $b \geq 1$. Moreover, \tilde{X} is locally (in a neighborhood of E) isomorphic to the total space of the line bundle $|\mathcal{O}_E(-b[p_0])|$, where p_0 is a point corresponding to the zero element of E viewed as an algebraic group.*

For a proof of this result, see [82, Korollar 1.4].

Corollary 7.6. *All simple elliptic singularities are quasi-homogeneous.*

Example 7.7. Simple elliptic singularities of low multiplicities are given by the following equations, see [82, Satz 1.9].

- $\text{El}(E, 1) : z^2 = y(y - x^2)(y - ax^2);$
- $\text{El}(E, 2) : z^2 = xy(x - y)(x - ay);$
- $\text{El}(E, 3) : z^2 = y(y - x)(y - ax);$
- $\text{El}(E, 4) : z^2 = y(w - (a + 1)y + ax), \quad y^2 = xw;$

In all these cases $a \in \mathbb{C} \setminus \{0, 1\}$ and

$$j(E) = \frac{4}{27} \frac{(a^2 - a + 1)^3}{a^2(a - 1)^3}$$

is the j -invariant of the elliptic curve E . All other simple elliptic singularities are no longer complete intersections.

Remark 7.8. Following Saito [82], there is an alternative notation for those simple elliptic singularities which are complete intersections. Namely, let (X, o) be a simple elliptic singularity of type $\text{El}(E, i)$, $1 \leq i \leq 4$ and $\mathcal{X} \rightarrow \mathcal{B}$ its semi-universal deformation. For $t \in \mathcal{B} \setminus \{0\}$ the fiber \mathcal{B}_t (called *Milnor fiber*) is homotopy equivalent to a bouquet of spheres and $H^1(\mathcal{B}_t, \mathbb{Z}) \cong H^2(\mathcal{B}_t, \mathbb{Z}) \cong \mathbb{Z}^\mu$, where μ is the Milnor number of (X, o) . Moreover, the intersection form $H^1(\mathcal{B}_t, \mathbb{Z}) \times H^2(\mathcal{B}_t, \mathbb{Z}) \rightarrow \mathbb{Z}$ is non-negatively definite with radical of rank two and

- for $\text{El}(E, 1)$ it has Dynkin type \tilde{E}_8 ;
- for $\text{El}(E, 2)$ it has Dynkin type \tilde{E}_7 ;
- for $\text{El}(E, 3)$ it has Dynkin type \tilde{E}_6 ;
- for $\text{El}(E, 4)$ it has Dynkin type \tilde{D}_4 .

Proposition 7.9. *Let $(X, o) = \text{El}(E, b)$ be a simple elliptic singularity of analytic type (E, b) . Then its local fundamental group has the following presentation:*

$$\pi_1(X, o) = H_b = \langle \alpha, \beta, \gamma \mid [\alpha, \gamma] = 1, [\beta, \gamma] = 1, [\alpha, \beta] = \gamma^b \rangle.$$

In group theory, it is called discrete Heisenberg group.

For a proof of this proposition we refer to [73, Chapter 1] and [61, Proposition 6.2].

Theorem 7.10. *The analytic type of a cusp singularity is determined by its intersection matrix. Moreover, a cusp singularity is a complete intersection only in the following cases:*

- $T_{p,q,r}$ -singularities given by the equation $x^p + y^q + z^r - xyz$, $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$.
- $T_{p,q,r,t}$ -singularities given by two equations $x^p + y^q = uv$, $u^r + v^t = xy$, where $p, q, r, t \geq 2$ and $\max(p, q, r, t) \geq 3$.

This theorem follows from a general result of Wahl [88, Theorem 2.8] about syzygies of minimally elliptic singularities.

Inspired by results on the geometric McKay Correspondence for quotient surface singularities, Kahn has proven the following beautiful theorem on Cohen-Macaulay modules over minimally elliptic singularities.

Theorem 7.11 (Kahn). *Let (X, o) be a minimally elliptic singularity, (A, \mathfrak{m}) the corresponding analytic algebra, $\pi : (\tilde{X}, E) \rightarrow (X, o)$ its minimal resolution and Z the fundamental cycle. Then the functor*

$$\mathbb{F} : \text{CM}(A) \rightarrow \text{VB}(Z), \quad \mathbb{F}(M) = \pi^*(M)^{\vee\vee}|_Z$$

preserves iso-classes of objects, i.e. $\mathbb{F}(M) \cong \mathbb{F}(M')$ implies $M \cong M'$. Moreover, the Cohen-Macaulay modules correspond to the following vector bundles.

- The regular Cohen-Macaulay module A corresponds to the vector bundle \mathcal{O}_Z .
- Cohen-Macaulay modules without free direct summands correspond to vector bundles of the form $\mathcal{V} \oplus \mathcal{O}^n$ where
 - (1) \mathcal{V} is generically spanned by its global sections, i.e. the cokernel of the evaluation map $H^0(\mathcal{V}) \otimes_k \mathcal{O} \rightarrow \mathcal{V}$ is a skyscraper sheaf;
 - (2) we have: $H^1(\mathcal{V}) = 0$;
 - (3) finally, $n = h^0(\mathcal{V} \otimes \mathcal{O}_{\tilde{X}}(Z)|_Z)$.

This result is proven in [60, Theorem 2.1].

Remark 7.12. The functor \mathbb{F} from Theorem 7.11 is neither exact nor faithful. It is not known whether it is full or not.

Remark 7.13. The result of Kahn also remains valid in the case of complete rings over an algebraically closed field k of arbitrary characteristic, see [33].

Hence, a description of Cohen-Macaulay modules over minimally elliptic singularities reduces to the classification of indecomposable vector bundles (with some restrictions) on projective curves (maybe non-reduced) of arithmetic genus one. Note that if a minimally elliptic surface singularity is either simple elliptic or cusp then its fundamental cycle $Z = E$ and we deal with vector bundles on reduced curves.

Theorem 7.14 (Atiyah). *Let k be an algebraically closed field and E be an elliptic curve over k , \mathcal{V} an indecomposable vector bundle on E . Then \mathcal{V} is semi-stable and is uniquely determined by its rank r , degree d and a point x of the curve E . To be more precise, let \mathcal{V}_0 be the uniquely determined Jordan-Hölder factor of \mathcal{V} , then $\det(\mathcal{V}_0) = \mathcal{O}_E((d-1)[x] + [x-x_0])$, where x_0 is the point corresponding to the zero element of E viewed as an algebraic group.*

For a proof, see [6, Theorem 7].

Theorem 7.15 (Oda). *Let $E = E_\tau = \mathbb{C}/\langle 1, \tau \rangle$ be a complex torus, \mathcal{V} an indecomposable holomorphic vector bundle on E of rank r and degree d . If $h = \gcd(r, d)$, $r = r'h$ and $d = d'h$ then there exists a line bundle \mathcal{L} on the elliptic curve $E_{r'\tau}$ such that*

$$\mathcal{V} \cong \pi_*(\mathcal{L}) \otimes \mathcal{A}_h,$$

where $\pi : E_{r'\tau} \rightarrow E_\tau$ is an étale covering of degree r' and \mathcal{A}_h is the unipotent vector bundle of rank h , i.e. the vector bundle recursively defined by the following non-split sequences:

$$(3) \quad 0 \rightarrow \mathcal{A}_{h-1} \rightarrow \mathcal{A}_h \rightarrow \mathcal{O} \rightarrow 0, \quad h \geq 2, \quad \mathcal{A}_1 = \mathcal{O}.$$

Note that we automatically have: $\mathrm{Ext}_E^1(\mathcal{O}, \mathcal{A}_h) \cong H^1(\mathcal{A}_h) = \mathbb{C}$, hence the middle term of the short exact sequence (3) is uniquely determined.

For a proof of this theorem, see [75, Proposition 2.1].

Lemma 7.16. *Kahn's functor \mathbb{F} maps the fundamental module D to the Atiyah's bundle \mathcal{A}_2 .*

A proof goes along the same lines as in [60, Theorem 3.1]. Note that, since a simple elliptic singularity $(X, o) = \mathrm{Specan}(A)$ is quasi-homogeneous, we have: $D \cong (\Omega_A^1)^{**}$.

Remark 7.17. Since cusp singularities are not quasi-homogeneous, in that case $D \not\cong (\Omega_A^1)^{**}$. A description of the fundamental module D in this case is due to Behnke, see [11, Section 5].

Theorem 7.18 (Kahn). *Let (X, o) be a simple elliptic singularity, (A, \mathfrak{m}) the corresponding local ring, $\pi : (\tilde{X}, E) \rightarrow (X, o)$ its minimal resolution and $b = \max\{2, -E^2\}$ its multiplicity. Then the indecomposable Cohen-Macaulay A -modules are parameterized by two discrete parameters:*

$$\left\{ (r, d) \in \mathbb{Z}^2 \mid 1 \leq r \leq d \leq (b+1)r \right\}$$

and one continuous parameter $\lambda \in E$.

Recall that \tilde{X} can be locally written as the total space of a line bundle on E , i.e.

$$\tilde{X} = |\mathcal{O}_E(-b[p_0])|,$$

where $p_0 \in E$ is some fixed point, let $p : \tilde{X} \rightarrow E$ be the corresponding projection. Consider the commutative diagram

$$\begin{array}{ccc} & E & \\ & \nearrow s & \nwarrow p \\ \tilde{X} \setminus E & \xrightarrow{\quad c \quad} & \tilde{X} \\ \downarrow q & & \downarrow \pi \\ X \setminus \{0\} & \xrightarrow{\quad i \quad} & X \end{array}$$

Let $\mathcal{E}(r, d, \lambda)$ be the indecomposable vector bundle on E of rank r , degree d and continuous parameter λ , constructed in Theorem 7.14. Then

$$M(r, d, \lambda) = \left\{ \Gamma \left(i_* q_* s^* \mathcal{E}(r, d, \lambda) \right) \mid 1 \leq r \leq d \leq (b+1)r, \lambda \in E \right\}$$

is the complete list of indecomposable Cohen-Macaulay A -modules.

For a proof, see [61, Proposition 5.18].

Unfortunately, Kahn's description of Cohen-Macaulay modules over minimally elliptic singularities is not really explicit. In some cases, however, one can find families of matrix factorizations.

Example 7.19 (Etingof-Ginzburg). Consider the simple elliptic singularity of type \tilde{E}_6 given by the equation $w = x^3 + y^3 + z^3 + \tau xyz$, $\tau \in \mathbb{C}^*$. Let $E = E_\tau \subset \mathbb{P}^2$ be the corresponding elliptic curve and $(a : b : c) \in E$ a point such that its components a, b and c are non-zero. Then the matrix

$$\phi = \begin{pmatrix} ax & by & cz \\ cz & ay & bx \\ by & cz & ax \end{pmatrix}$$

and its adjoint

$$\phi' = \begin{pmatrix} a^2yz - bcx^2 & c^2xy - abz^2 & b^2xy - acy^2 \\ b^2xy - acz^2 & a^2yz - bcy^2 & c^2yz - abx^2 \\ c^2xz - aby^2 & b^2yz - acx^2 & a^2xy - bcz^2 \end{pmatrix}$$

satisfy the equality:

$$\phi \cdot \phi' = \phi' \cdot \phi = -(abc)w \cdot \text{id}.$$

In particular, the matrix factorization $M(\phi, \phi')$ defines a family of indecomposable Cohen-Macaulay modules of rank one [38, Example 3.6.5].

Other examples of matrix factorizations in the case of the simple elliptic singularity of type \tilde{E}_7 given by the equation $x^4 + y^4 + \tau x^2 y^2 + z^2 = 0$ were found by Knapp [65]. Some examples of matrix factorizations for $x^3 + y^3 + z^3 = 0$ corresponding to Cohen-Macaulay modules of higher ranks, were found by Laza, Pfister and Popescu [70].

Indecomposable vector bundles and torsion free sheaves on Kodaira cycles of projective lines were classified by Drozd and Greuel, see [29, Theorem 2.12]. In the case the ground field k is algebraically closed of characteristic zero, their classification can be presented in the following form, generalizing Oda's description, see [14, Theorem 19].

Theorem 7.20. *Let $E = E_n$ be a Kodaira cycle of n projective lines, \mathcal{V} an indecomposable holomorphic vector bundle on E of rank r and degree d . Then there exists an étale covering $\pi : E_{nt} \rightarrow E_n$ of degree t , a line bundle $\mathcal{L} \in \text{Pic}(E_{nt}) = \mathbb{Z}^{nt} \times k^*$ and a positive integer $h \geq 1$ such that $r = th$ and*

$$\mathcal{V} \cong \pi_*(\mathcal{L}) \otimes \mathcal{A}_h,$$

where \mathcal{A}_h is defined by short exact sequences (3) in the same way as in the case of elliptic curves.

Corollary 7.21. *Simple elliptic and cusp singularities as well as their quotients by a finite group of automorphisms (the so-called simple \mathbb{Q} -elliptic and \mathbb{Q} -cusp singularities) have tame Cohen-Macaulay representation type.*

The following result of Drozd and Greuel implies that the remaining minimally elliptic singularities are Cohen-Macaulay wild.

Theorem 7.22 (Drozd–Greuel). *Let E be a reduced projective curve.*

- (1) *If E is a chain of projective lines, then $\text{VB}(E)$ is of finite type.*
- (2) *If E is a smooth elliptic curve, then $\text{VB}(E)$ is tame of polynomial growth.*
- (3) *If E is a cycle of projective lines, $\text{VB}(E)$ is tame of exponential growth.*
- (4) *In all other cases, $\text{VB}(E)$ has wild representation type.*

For a proof of this theorem, see [29, Theorem 2.8].

Remark 7.23. By a result of Kawamata [62], simple \mathbb{Q} -elliptic and \mathbb{Q} -cusp singularities are precisely the *log-canonical* singularities. Moreover, results of Mumford [73] and Karras [58] imply they have solvable local fundamental group. Other way around, a classification of Karras, based on an earlier result of Wagreich [87] and completed by Kawamata's classification of rational log-canonical singularities implies that a normal surface singularity with an infinite solvable local fundamental group is either simple \mathbb{Q} -elliptic or \mathbb{Q} -cusp.

Remark 7.24. The local fundamental group of the E_8 -singularity is binary-icosahedral, which is known to be not solvable.

8. OTHER RESULTS ON COHEN-MACAULAY REPRESENTATION TYPE

In this section, we briefly mention some other results and conjectures related to our study of surface singularities of finite and tame Cohen-Macaulay representation types, see [30, 31] for the definition of Cohen-Macaulay tame and Cohen-Macaulay wild representation type.

The original motivation to classify the indecomposable Cohen-Macaulay modules over a Cohen-Macaulay local ring originates from the theory of integral representations of finite groups. For example, let $\mathbb{Z}_{(2)}$ be the ring of 2-adic integers and $G = \mathbb{Z}/8\mathbb{Z}$ be a cyclic group of order 8. Then the category of finitely generated torsion free $\mathbb{Z}_{(2)}[G]$ -modules is tame [92]. Moreover, the underlying classification problem is closely related to the study of Cohen-Macaulay modules over a minimally elliptic curve singularity of type T_{36} .

More generally, one can pose a question about the representation type of the category of *lattices* over a complete *order*. However, in this framework tameness results mainly concern the case of orders of Krull dimension one, see surveys of Dieterich [27] and Simson [85] for an overview. Complete two dimensional orders of finite lattice type were studied by Artin [4], Reiten and van den Bergh [79]. The following question is very natural.

Question 8.1. *Let (A, \mathfrak{m}) be a complete Cohen-Macaulay ring of Krull dimension two, and Λ be an A -order. When the category of lattices over Λ (i.e. the category of finitely generated left Λ -modules, which are Cohen-Macaulay over A) is of tame representation type?*

Some “trivial” examples of such orders are given by skew group orders $A * G$, where (A, \mathfrak{m}) is either a minimally elliptic or a cusp singularity and G a finite group of automorphisms of A . However, the general answer on question 8.1 is widely unknown.

Moreover, we also do not know the answer about the characterization of Cohen-Macaulay tame two-dimensional normal Noetherian local rings, which are not algebras over a field. It is an interesting problem to introduce an arithmetic notion of a minimally elliptic singularity and to generalize methods of the geometric McKay correspondence on the arithmetic case. Existence of resolutions of surface singularities in this general situation was proven by Lipman [71].

In a forthcoming paper of both authors we are going to show that the non-isolated surface singularities called *degenerate cusps* are Cohen-Macaulay tame [23]. This includes, for example, the complete intersections $k[[x, y, z]]/(xyz)$ and $k[[x, y, z, t]]/(xy, zt)$ as well as their deformations. A description of matrix factorizations corresponding to the Cohen-Macaulay modules of rank one and two over the completion of the affine cone of a nodal cubic curve $k[[x, y, z]]/(zy^2 + x^3 + x^2z)$ was obtained by Baciú [10].

This raises the question about a complete classification of two-dimension Cohen-Macaulay local rings of tame Cohen-Macaulay representation type.

Conjecture 8.2. *Let (A, \mathfrak{m}) be a normal surface singularity over a field $k = A/\mathfrak{m}$, which is algebraically closed of characteristic zero. Then A is of tame Cohen-Macaulay representation type if and only if it has the form $A = B^G$, where (B, \mathfrak{n}) is either a simple elliptic or a cusp singularity and G is a finite group of automorphisms of B . If $k = \mathbb{C}$, this is equivalent to the condition that the local fundamental group $\pi_1(X, o)$ is infinite and solvable, see [58].*

Recently, Drozd and Greuel have proven that a rational normal surface singularity of tame Cohen-Macaulay representation type is log-canonical [32]. In a work of Bondarenko [15] it was shown that a hypersurface singularity $A = k[[x, y, z]]/f$ with $f \in (x, y, z)^4$ is always of Cohen-Macaulay wild. However, the answer on the following question is still unknown.

Conjecture 8.3. *Let (A, \mathfrak{m}) be a local Cohen-Macaulay ring of Krull dimension two. Then the representation type of the category of Cohen-Macaulay A -modules $\text{CM}(A)$ is either finite, discrete, tame or wild.*

A positive solution of this conjecture was obtained by Drozd and Greuel in the case of the reduced curve singularities, see [30]. Knowing an answer on the following problem would considerably help to solve Conjecture 8.2 and Conjecture 8.3.

Conjecture 8.4. *Let $X \rightarrow T$ be a flat family of two-dimensional surface singularities. Then the set $B = \{t \in T \mid X_t \text{ is of wild Cohen-Macaulay representation type}\}$ is Zariski-closed in B .*

This conjecture essentially means that a Cohen-Macaulay tame surface singularity can not be locally deformed to a Cohen-Macaulay wild singularity. Such semi-continuity result is known in the case of the reduced curve singularities, see [64] and [31]. Note that the following proposition is true.

Proposition 8.5. *Let k be an algebraically closed field of characteristic zero, $X \rightarrow T$ be a flat family of normal two-dimensional surface singularities and $t_0 \in T$ be a closed point such that X_{t_0} has finite Cohen-Macaulay representation type. Then there exists an open neighborhood U of t_0 in B such that for all $t \in U$ the singularity X_t is of finite Cohen-Macaulay representation type.*

Proof. Indeed, since the singularity X_{t_0} has only finitely many indecomposable Cohen-Macaulay modules, it is a quotient singularity. However, by a result of Esnault and Viehweg [37] it is known that the quotient surface singularities deform to quotient surface singularities. This implies the claim. \square

Another approach to study Cohen-Macaulay modules is provided by the theory of cluster tilting.

Proposition 8.6 (see Theorem 4.1 and Theorem 7.6 in [22]). *Let A be a minimally elliptic curve singularity given by the equality*

$$A = T_{p,q}(\lambda) = k[[x, y]]/(x^p + y^q + \lambda x^2 y^2),$$

where $\frac{1}{p} + \frac{1}{q} \leq \frac{1}{2}$, $\lambda \in k^*$ and either both p and q are even, or $p = 3$ and q is even. Then there exists a Cohen-Macaulay A -module M such that

- It is rigid, i.e. $\text{Ext}_A^1(M, M) = 0$ and $M \not\cong 0$.
- If N is any indecomposable Cohen-Macaulay A -module such that $\text{Ext}_A^1(N, M) = 0$, then N is a direct summand of M .

Such an object M is called cluster-tilting. Moreover, if $\Gamma = \text{End}_{\underline{\text{CM}}(A)}(M)$ is the corresponding cluster-tilted algebra then the functor

$$\text{Hom}_{\underline{\text{CM}}(A)}(M, -) : \underline{\text{CM}}(A)/\tau(T) \longrightarrow \text{mod} - \Gamma$$

is an equivalence of categories, where $\underline{\text{CM}}(A)$ is the stable category of Cohen-Macaulay modules and τ is the Auslander-Reiten translation in $\underline{\text{CM}}(A)$.

Moreover, the cluster-tilted algebras Γ arising in Proposition 8.6 are finite-dimensional and symmetric. From works of Erdmann, Białkowski-Skowroński and Holm it follows they are tame, see [86] for an overview about tame self-injective algebras.

9. STABLE CATEGORY OF COHEN-MACAULAY MODULES AND COMPUTATIONS WITH `Singular`

Let (A, \mathfrak{m}) be a Gorenstein local ring. Since by the definition $K_A \cong A$, the exact category $\text{CM}(A)$ is Frobenius.

Definition 9.1. The stable category of Cohen-Macaulay modules $\underline{\text{CM}}(A) = \text{CM}(A)/\langle A \rangle$ is defined as follows

- $\text{Ob}(\underline{\text{CM}}(A)) = \text{Ob}(\text{CM}(A))$
- $\underline{\text{Hom}}_A(M, N) = \text{Hom}_{\underline{\text{CM}}(A)}(M, N) := \text{Hom}_A(M, N)/\mathfrak{P}(M, N)$, where $\mathfrak{P}(M, N)$ is the submodule of $\text{Hom}_A(M, N)$ generated by those morphisms which factors through a free A -module.

The following theorem can be considered as the *raison d'être* for the study of Cohen-Macaulay modules

Theorem 9.2 (Buchweitz). *Let (A, \mathfrak{m}) be a complete Gorenstein local ring, then the following holds,*

- *The functor $\Omega = \text{syz} : \underline{\text{CM}}(A) \longrightarrow \underline{\text{CM}}(A)$ is an auto-equivalence of $\underline{\text{CM}}(A)$.*
- *The category $\underline{\text{CM}}(A)$ has a structure of a triangulated category, where the shift functor is $T := \Omega^{-1}$.*
- *There is an equivalence of triangulated categories*

$$\underline{\text{CM}}(A) \rightarrow D^b(A - \text{mod}) / \text{Perf}(A)$$

induced by the fully faithful functor $\text{CM}(A) \rightarrow D^b(A - \text{mod})$

- *In particular, the canonical functor $\text{CM}(A) \rightarrow \underline{\text{CM}}(A)$ maps exact sequences into exact triangles.*

This theorem was proven for the first time in [20]. Recently, it was rediscovered in [76].

Theorem 9.3 (Eisenbud [35]). *Let (S, \mathfrak{n}) be a regular local ring, $f \in \mathfrak{n}$ and $A = S/f$. Then any Cohen-Macaulay module M without free direct summands has a 2-periodic minimal free resolution:*

$$\dots \xrightarrow{\mu} F \xrightarrow{\nu} F \xrightarrow{\mu} F \longrightarrow M \longrightarrow 0,$$

where $F \cong A^n$ is a free A -module. In these terms we can write $M = M(\mu, \nu)$, where $\mu, \nu \in \text{Mat}_n(\mathfrak{m})$. This implies that $\Omega^2 \cong \text{Id}$, hence $T^2 \cong \text{Id}_{\underline{\text{CM}}(A)}$. Moreover, we have an equivalence of triangulated categories:

$$\underline{\text{CM}}(A) = \text{Hot}_2(A),$$

where $\text{Hot}_2(A)$ is the homotopy category of (unbounded) minimal 2-periodic projective complexes.

Proposition 9.4. *Let (A, \mathfrak{m}) be a Cohen-Macaulay ring and M, N be Cohen-Macaulay modules. If either M or N is locally free on the punctured spectrum, then the A -module $\underline{\text{Hom}}_A(M, N)$ has finite length.*

Proof. Take any short exact sequence $0 \rightarrow N' \rightarrow A^n \rightarrow N \rightarrow 0$ and observe that if N is locally free on the punctured spectrum then N' is locally free on the punctured spectrum, too. From the exact sequence

$$\text{Hom}_A(M, A^n) \rightarrow \text{Hom}_A(M, N) \rightarrow \text{Ext}_A^1(M, N')$$

we obtain an embedding $\underline{\text{Hom}}_A(M, N) \hookrightarrow \text{Ext}_A^1(M, N')$. If either M or N is locally free on the punctured spectrum, then $\text{Ext}_A^1(M, N')$ has finite length, hence the claim. \square

It turns out that the converse statement is also true.

Theorem 9.5. *Let (A, \mathfrak{m}) be a Cohen-Macaulay local ring. Then it is an isolated singularity if and only for all Cohen-Macaulay modules M and N the module $\text{Ext}_A^1(M, N)$ has finite length.*

For a proof of this theorem, we refer to [93, Lemma 3.3].

Theorem 9.6 (Auslander). *Let (A, \mathfrak{m}) be an isolated Gorenstein singularity of Krull dimension d and $k = A/\mathfrak{m}$. Then for any pair of Cohen-Macaulay modules M and N we have a bifunctorial isomorphism of A -modules*

$$\text{Ext}_A^1(M, (\text{syz}^d \text{Tr}(N))^*) \cong \mathbb{D}(\underline{\text{Hom}}_A(M, N)),$$

where $X^ = \text{Hom}_A(X, A)$ and $\mathbb{D} = \text{Hom}_A(-, E(k))$ is the Matlis duality functor. Moreover, for any Cohen-Macaulay module N there is a functorial isomorphism $\text{syz}^d(\text{Tr}(M))^* \cong \text{syz}^{2-d}(M) : \underline{\text{CM}}(A) \rightarrow \underline{\text{CM}}(A)$.*

For a proof of this result, see [7, Proposition 8.8 in Chapter 1 and Proposition 1.3 in Chapter 3].

If (A, \mathfrak{m}) is a Gorenstein k -algebra, then this theorem can be restated as follows.

Corollary 9.7. *Let (A, \mathfrak{m}) be a Gorenstein k -algebra ($k = A/\mathfrak{m}$) of Krull dimension d , which is an isolated singularity. Then $\mathbb{S} = \text{syz}^{1-d}$ is the Serre functor in the stable category of Cohen-Macaulay modules $\underline{\text{CM}}(A)$. This means that we for any two Cohen-Macaulay modules M and N we have a bifunctorial isomorphism*

$$(4) \quad \underline{\text{Hom}}_A(M, N) \cong \underline{\text{Hom}}_A(N, \mathbb{S}(M))^*.$$

Remark 9.8. By [20, Proposition 10.1.5] the isomorphism (4) holds for an arbitrary Gorenstein ring d and a pair of Cohen-Macaulay modules M and N such that M is locally free on the punctured spectrum. This means, that the stable category of Cohen-Macaulay modules over a Gorenstein k -algebra A of Krull dimension d , which are locally free on the punctured spectrum, is a triangulated $(d - 1)$ -Calabi-Yau category.

The following lemma shows that the stable categories of Cohen-Macaulay modules over an isolated Cohen-Macaulay singularity and its completion are closely related.

Proposition 9.9. *Let (A, \mathfrak{m}) be an isolated Cohen-Macaulay singularity and \widehat{A} its completion. Then the canonical functor $\underline{\text{CM}}(A) \rightarrow \underline{\text{CM}}(\widehat{A})$ is fully faithful.*

Proof. Let M and N be two Cohen-Macaulay A -modules. Since they are automatically free on the punctured spectrum, the A -module $\underline{\text{Hom}}_A(M, N)$ is annihilated by some power of the maximal ideal: $\mathfrak{m}^t \cdot \underline{\text{Hom}}_A(M, N) = 0$ for $t \gg 0$. Hence it is isomorphic to its completion $\underline{\text{Hom}}_{\widehat{A}}(\widehat{M}, \widehat{N})$. \square

In order to compute the dimensions of Hom and Ext -spaces in the stable category of maximal Cohen-Macaulay modules, one can use the computer algebra system **Singular**, see [46]. Let

$$A = k[x_1, x_2, \dots, x_n]_{\langle x_1, x_2, \dots, x_n \rangle} / I$$

be a Cohen-Macaulay local ring, M and N be a pair of maximal Cohen-Macaulay modules.

Assume the vector space $\text{Ext}_A^i(M, N)$ ($i \geq 1$) is finite-dimensional over k . Since the functor $A - \text{mod} \rightarrow \widehat{A} - \text{mod}$ is exact, maps the maximal Cohen-Macaulay modules to maximal Cohen-Macaulay modules and the finite length modules to finite length modules, we can conclude that

$$\dim_k(\text{Ext}_A^i(M, N)) = \dim_k(\text{Ext}_{\widehat{A}}^i(\widehat{M}, \widehat{N})).$$

Moreover, if A is a hypersurface singularity then by Theorem 9.3 we have:

$$\underline{\text{Hom}}_A(M, N) \cong \text{Ext}_A^2(M, N).$$

Example 9.10. Let $A = k[[x, y, z]]/xyz$. Then the following modules

$$A^3 \xrightarrow{\begin{pmatrix} x & z^3 & 0 \\ 0 & y & x \\ y^2 & 0 & z \end{pmatrix}} A^3 \longrightarrow M \longrightarrow 0,$$

$$A^3 \xrightarrow{\begin{pmatrix} x & 0 & 0 \\ z^3 & y & 0 \\ y^2 & x & z \end{pmatrix}} A^3 \longrightarrow N \longrightarrow 0,$$

and

$$A^2 \xrightarrow{\begin{pmatrix} xy & -x^2 + y^3 \\ 0 & z \end{pmatrix}} A^2 \longrightarrow K \longrightarrow 0$$

are Cohen-Macaulay and locally free on the punctured spectrum of A . Let us compute certain Hom and Ext spaces between M , N and K .

```
> Singular (call the program ‘‘Singular’’)  
> LIB ‘‘homolog.lib’’; (call the library of homological algebra)  
> ring S = 0, (x,y,z), ds; (defines the ring  $S = \mathbb{Q}[x, y, z]_{\langle x, y, z \rangle}$ )
```

```

> ideal I = xyz; (defines the ideal xyz in S)
> qring A = std(I); (defines the ring  $\mathbb{Q}[x, y, z]_{\langle x, y, z \rangle} / I$ )
> module k = [x], [y], [z]; (defines the residue field k of A as an A-module)
> module M = [x, 0, y2], [z3, y, 0], [0, x, z];
> module N = [x, z3, y2], [0, y, x], [0, 0, z];
> module K = [xy, 0], [-x2 + y3, z];
> isCM(M);           (checks, whether M is Cohen-Macaulay)
> 1                  (yes, M is Cohen-Macaulay)
> isCM(N);           (checks, whether N is Cohen-Macaulay)
> 1                  (yes, N is Cohen-Macaulay)
> list l = Ext(1, k, N, 1);
// ** redefining l **
// dimension of Ext1: -1      ExtA1(k, N) = 0
> isCM(K);           (checks, whether K is Cohen-Macaulay)
> 1                  (yes, K is Cohen-Macaulay)
> list l = Ext(2, M, M, 1);
// ** redefining l **
// dimension of Ext2: 0      (kr. dim(ExtA2(M, M)) = 0)
// vdim of Ext2: 7      (dimk(ExtA2(M, M)) = dimk(HomA(M, M)) = 7)
> list l = Ext(2, N, K, 1);
// ** redefining l **
// dimension of Ext2: 0
// vdim of Ext2: 7      dimk ExtA2(M, M)  $\cong$  HomA(M, M) = 7

```

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