NON-COMMUTATIVE NODAL CURVES AND DERIVED TAME ALGEBRAS

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Abstract. In this paper, we develop a geometric approach to study derived tame finite dimensional associative algebras, based on the theory of non-commutative nodal curves.

Contents

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1. Introduction

Let $\Bbbk$ be an algebraically closed field such that $\mathsf{char}(\Bbbk) \ne 2$. For $\lambda \in \Bbbk \setminus \{0, 1\}$, let

$$Y_\lambda = V\bigl(zy^2 - x(x-z)(x-\lambda z)\bigr) \subset \mathbb{P}^2$$

be the corresponding elliptic curve and $Y_\lambda \xrightarrow{\imath} Y_\lambda$ be the involution, given by the rule $(x : y : z) \stackrel{\imath}{\mapsto} (x : -y : z)$. Let T_λ be the tubular canonical algebra of type $((2, 2, 2, 2); \lambda)$ [43], i.e. the path algebra of the following quiver

$$\begin{tikzcd}
 & & \circ \arrow[dll, "a_1"'] \arrow[dl, "a_2"] \arrow[dr, "a_3"'] \arrow[drr, "a_4"] & & \\
\circ \arrow[drr, "b_1"'] & \circ \arrow[dr, "b_2"] & & \circ \arrow[dl, "b_3"'] & \circ \arrow[dll, "b_4"] \\
 & & \circ & &
\end{tikzcd} \tag{1}$$

modulo the relations $b_1a_1 - b_2a_2 = b_3a_3$ and $b_1a_1 - \lambda b_2a_2 = b_4a_4$. According to Geigle and Lenzing [23, Example 5.8], there exists an exact equivalence of derived categories

$$D^b\bigl(\mathsf{Coh}^G(Y_\lambda)\bigr) \longrightarrow D^b(T_\lambda\text{-}\mathsf{mod}), \tag{2}$$

where $G = \langle \imath \rangle \cong \mathbb{Z}_2$ and $\mathsf{Coh}^G(Y_\lambda)$ is the category of G-equivariant coherent sheaves on Y_λ. It is well-known that $D^b\bigl(\mathsf{Coh}^G(Y_\lambda)\bigr)$ and $D^b(T_\lambda\text{-}\mathsf{mod})$ have *tame* representation type; see [3, 28, 37]. At this place one can ask the following natural

Question. Is there any link between $D^b\bigl(\mathsf{Coh}^G(Y_\lambda)\bigr)$ and $D^b(T_\lambda\text{-}\mathsf{mod})$ when the parameter $\lambda \in \Bbbk$ takes the "forbidden" value 0?

Let $E := Y_0 = V\bigl(zy^2 - x^2(x-z)\bigr) \subset \mathbb{P}^2$ be the plane nodal cubic and $T := T_0$ be the corresponding degenerate tubular algebra. Both derived categories $D^b\bigl(\mathsf{Coh}^G(E)\bigr)$ and $D^b(T\text{-}\mathsf{mod})$ are known to be representation tame. Moreover, it follows from our previous papers [11, 12] that the indecomposable objects in both categories can be described by very similar combinatorial patterns. However, since $\mathsf{gl.dim}\bigl(\mathsf{Coh}^G(E)\bigr) = \infty$ and $\mathsf{gl.dim}(T) = 2$, the derived categories $D^b\bigl(\mathsf{Coh}^G(E)\bigr)$ and $D^b(T\text{-}\mathsf{mod})$ can not be equivalent. Nevertheless, it turns out that the following result is true:

Proposition (see Remark 6.6). There exists a commutative diagram of triangulated categories and functors

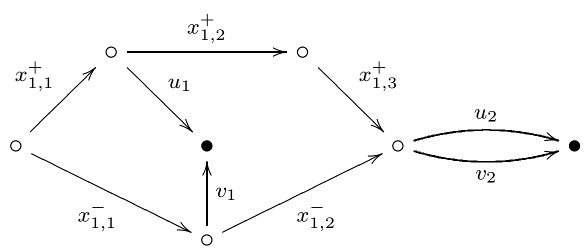
$$\begin{tikzcd}
D^b(T\text{-}\mathsf{mod}) \arrow[rr, two heads, "\mathsf{P}"] & & D^b\bigl(\mathsf{Coh}^G(E)\bigr) \\
 & \mathsf{Perf}^G(E) \arrow[ul, hook', "\mathsf{E}"] \arrow[ur, hook, "\mathsf{I}"'] &
\end{tikzcd} \tag{3}$$

where $\mathsf{Perf}^G(E)$ is the perfect derived category of $\mathsf{Coh}^G(E)$, I is the canonical inclusion functor, E is a fully faithful functor and P is an appropriate Verdier localization functor.

The main goal of this work is to extend the above result to a broader class of derived tame algebras. Let us start with a pair of tuples $\vec{p} = \big((p_1^+, p_1^-), \dots, (p_r^+, p_r^-)\big) \in \big(\mathbb{N}^2\big)^r$ and $\vec{q} = (q_1, \dots, q_s) \in \mathbb{N}^s$, where $r, s \in \mathbb{N}_0$ (either of this tuples is allowed to be empty). For any $1 \le i \le r$ and $1 \le j \le s$, consider the following sets: $\Xi_i^\pm := \big\{x_{i,1}^\pm, \dots, x_{i,p_i^\pm}^\pm\big\}$ and $\Xi_j^\circ := \big\{w_{j,1}, \dots, w_{j,q_j}\big\}$. Let $\approx$ be a symmetric relation (not necessarily an equivalence) on the set $\Xi := \big((\Xi_1^+ \cup \Xi_1^-) \cup \dots \cup (\Xi_r^+ \cup \Xi_r^-)\big) \cup \big(\Xi_1^\circ \cup \dots \cup \Xi_s^\circ\big)$ such that for any $\xi \in \Xi$, there exists at most one $\xi' \in \Xi$ such that $\xi \approx \xi'$. Then the datum $(\vec{p}, \vec{q}, \approx)$ defines a derived tame finite dimensional $\mathbb{k}$-algebra $\Lambda = \Lambda(\vec{p}, \vec{q}, \approx)$, obtained from the canonical algebras $\Gamma(p_1^+, p_1^-), \dots, \Gamma(p_r^+, p_r^-), \Gamma(2, 2, q_1), \dots, \Gamma(2, 2, q_s)$ by a certain "gluing/blowing-up process" determined by the relation $\approx$. In the case when $s = 0$ (i.e. when the tuple $\vec{q}$ is void), the algebra Λ is skew-gentle [24]. If additionally $\xi \not\approx \xi$ for all $\xi \in \Xi$, the algebra Λ is gentle [2]. Instead of defining $\Lambda(\vec{p}, \vec{q}, \approx)$ now, we refer to Definition 5.6 below and give here two examples explaining characteristic features of these class of algebras.

Example. Let $\vec{p} = (3, 2)$, $\vec{q}$ be void (i.e. $r = 1$ and $s = 0$) and $\approx$ be given by the rule $x_{1,1}^+ \approx x_{1,1}^-$ and $x_{1,3}^+ \approx x_{1,2}^-$. Then the corresponding gentle algebra $\Lambda(\vec{p}, \approx)$ is the path algebra of the following quiver

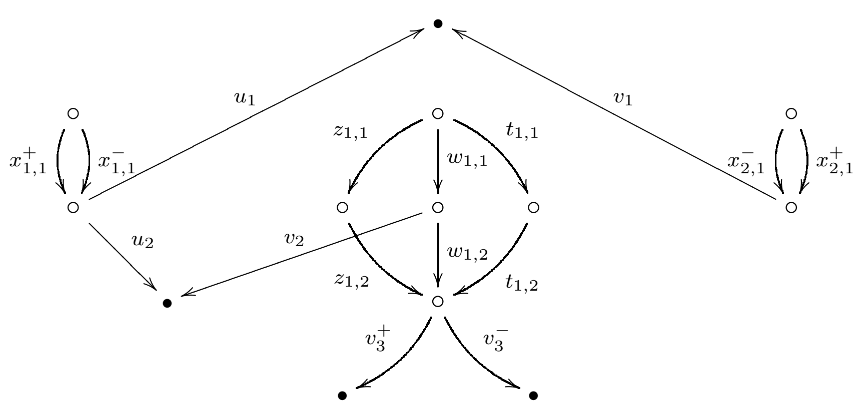
(4)



subject to the relations: $u_1 x_{1,1}^+ = 0 = v_1 x_{1,1}^-$ and $u_2 x_{1,3}^+ = 0 = v_2 x_{1,2}^-$.

Example. Let $\vec{p} = \big((1,1), (1,1)\big)$, $\vec{q} = (2)$ and $\approx$ be given by the rule: $x_{1,1}^+ \approx w_{1,1}$, $x_{1,1}^- \approx x_{2,1}^+$ and $w_{1,2} \approx w_{1,2}$. Then the corresponding algebra $\Lambda(\vec{p}, \vec{q}, \approx)$ is the path algebra of the following quiver

(5)



modulo the relations:

$$\begin{cases} z_{1,2}z_{1,1} + w_{1,2}w_{1,1} + t_{1,2}t_{1,1} = 0 \\ v_3^{\pm} w_{1,2} = 0 \\ u_1 x_{1,1}^{-} = 0 = v_1 x_{2,1}^{+} \\ u_2 x_{1,1}^{+} = 0 = v_2 w_{1,1}. \end{cases}$$

Let $(\vec{p}, \vec{q}, \approx)$ be a datum as in the definition of $\Lambda(\vec{p}, \vec{q}, \approx)$, additionally satisfying a certain *admissibility* condition. It turns out that it defines (uniquely up to Morita equivalence) a *tame non-commutative projective nodal curve* $\mathbb{X} = \mathbb{X}(\vec{p}, \vec{q}, \approx)$. Converesely, any tame non-commutative projective nodal curve is Morita equivalent to $\mathbb{X}(\vec{p}, \vec{q}, \approx)$ for an appropriate admissible datum $(\vec{p}, \vec{q}, \approx)$, see [21]. The main result of this paper is the following.

Theorem (see Corollary 5.5). Let $(\vec{p}, \vec{q}, \approx)$ be an admissible datum, $\mathbb{X}$ be the corresponding tame non-commutative nodal curve and Λ be the corresponding $\mathbb{k}$-algebra. Next, let $\mathbb{Y}$ be the *Auslander curve* of $\mathbb{X}$ (which is another tame non-commutative projective nodal curve) and $\widetilde{\mathbb{X}} \xrightarrow{\nu} \mathbb{X}$ be the *hereditary cover* of $\mathbb{X}$. Then there exists the following commutative diagram of triangulated categories and exact functors:

$$\begin{tikzcd}
 & D^b\big(\mathsf{Coh}(\widetilde{\mathbb{X}})\big) \arrow[dl, "\nu_*"'] \arrow[d, hook, "\widetilde{\mathsf{E}}"] & \\
D^b\big(\mathsf{Coh}(\mathbb{X})\big) & D^b\big(\mathsf{Coh}(\mathbb{Y})\big) \arrow[l, two heads, "\mathsf{P}"'] \arrow[r, "\mathsf{T}"] & D^b(\Lambda\text{-}\mathsf{mod}) \\
 & \mathsf{Perf}(\mathbb{X}) \arrow[ul, hook, "\mathsf{I}"] \arrow[u, hook, "\mathsf{E}"'] &
\end{tikzcd} \tag{6}$$

where T is an equivalence of triangulated categories, E and $\widetilde{\mathsf{E}}$ are fully faithful functors, I is the canonical inclusion, P is an appropriate Verdier localization functor and ν_* is induced by the forgetful functor $\mathsf{Coh}(\widetilde{\mathbb{X}}) \longrightarrow \mathsf{Coh}(\mathbb{X})$.

This theorem generalizes an earlier results of the authors [13], where $\mathbb{X}$ was a *commutative* tame nodal curve (i.e. a chain or a cycle of projective lines [20, 11]).

In [35], Lekili and Polishchuk proved a version of the homological mirror symmetry for punctured Riemann surfaces. According to their work, for any (in appropriate sense *graded*) compact Riemann surface Σ with finitely many punctures $x_1, \ldots, x_n \in \Sigma$, there exists either a *stacky* chain or a *stacky* cycle of projective lines $\mathbb{V}$ (actually, not uniquely determined) and equivalences of triangulated categories

$$\begin{tikzcd}
D^b\big(\mathsf{Coh}(\mathbb{V})\big) \arrow[r, "\cong"] & \mathsf{WFuk}(\mathring{\Sigma}) \\
\mathsf{Perf}(\mathbb{V}) \arrow[u, hook] \arrow[r, "\cong"] & \mathsf{Fuk}(\mathring{\Sigma}) \arrow[u, hook]
\end{tikzcd}$$

where $\mathsf{Fuk}(\mathring{\Sigma})$ (respectively, $\mathsf{WFuk}(\mathring{\Sigma})$) is the Fukaya category (respectively, the wrapped Fukaya category) of the punctured Riemann surface $\mathring{\Sigma} = \Sigma \setminus \{x_1, \dots, x_n\}$ (which is also viewed as a Riemann surface with boundary). The proof of [35, Theorem B] was based on properties of a partially wrapped Fukaya category $\mathsf{PWFuk}(\mathring{\Sigma}, \vec{m})$ introduced by Haiden, Katzarkov and Kontsevich [27], where $\vec{m} \in \mathbb{N}^n$ is the vector describing the numbers of marked points on the boundary components of $\mathring{\Sigma}$. The partially wrapped Fukaya category $\mathsf{PWFuk}(\mathring{\Sigma}, \vec{m})$ has the following two key features (see [27, Sections 3.4 and 3.5]):

- It is related with the wrapped Fukaya category via an appropriate localization functor $\mathsf{PWFuk}(\mathring{\Sigma}, \vec{m}) \twoheadrightarrow \mathsf{WFuk}(\mathring{\Sigma})$.
- There exists a (graded) gentle algebra $\Lambda = \Lambda(\mathring{\Sigma}, \vec{m})$ of finite global dimension such that we have an equivalence of triangulated categories $D^b(\Lambda) \longrightarrow \mathsf{PWFuk}(\mathring{\Sigma}, \vec{m})$, where $D^b(\Lambda)$ stands for the derived category of Λ viewed as a graded dg-algebra with trivial differential. In the when the grading of Λ is trivial, $D^b(\Lambda)$ is equivalent to the conventional derived category $D^b(\Lambda\text{-}\mathsf{mod})$.

Given a punctured Riemann surface $\mathring{\Sigma}$, Lekili and Polishchuk describe a stacky cycle/chain of projective lines $\mathbb{V}$ as well as a marking $\vec{m}$ of the boundary of $\mathring{\Sigma}$ and grading of $\mathring{\Sigma}$, for which there exists the following commutative diagram of categories and functors

$$
\begin{tikzcd}
D^b\big(\mathsf{Coh}(\mathbb{W})\big) \arrow[r] \arrow[d, two heads] & \mathsf{PWFuk}(\mathring{\Sigma}, \vec{m}) \arrow[d, two heads] \\
D^b\big(\mathsf{Coh}(\mathbb{V})\big) \arrow[r] & \mathsf{WFuk}(\mathring{\Sigma}) \\
\mathsf{Perf}(\mathbb{V}) \arrow[r] \arrow[u, hook] \arrow[uu, bend left=60] & \mathsf{Fuk}(\mathring{\Sigma}), \arrow[u, hook] \arrow[uu, bend right=60]
\end{tikzcd}
\tag{7}
$$

where all horizontal arrows are equivalences of triangulated categories. The first equivalence $D^b\big(\mathsf{Coh}(\mathbb{W})\big) \longrightarrow \mathsf{PWFuk}(\mathring{\Sigma}, \vec{m})$ is defined as the composition of two equivalences of triangulated categories

$$
D^b\big(\mathsf{Coh}(\mathbb{W})\big) \longrightarrow D^b(\Lambda\text{-}\mathsf{mod}) \longrightarrow \mathsf{PWFuk}(\mathring{\Sigma}, \vec{m}), \tag{8}
$$

both given by appropriate tilting complexes in $D^b\big(\mathsf{Coh}(\mathbb{W})\big)$ and $\mathsf{PWFuk}(\mathring{\Sigma}, \vec{m})$, respectively. We are going to explain how a stacky cycle/chain of projective lines $\mathbb{V}$ can be naturally viewed as a tame non-commutative nodal curve. In these terms, $\mathbb{W}$ is the Auslander curve of $\mathbb{V}$ and the left-hand side of (7) is a subpart of the diagram (6).

Let $\mathbb{Y}$ be the Auslander curve of an arbitrary tame non-commutative nodal curve $\mathbb{X}(\vec{p}, \approx)$ (in this notation, the vector $\vec{q}$ is void for the admissible datum $(\vec{p}, \vec{q}, \approx)$), which need not be a stacky cycle or chain of projective lines. Let $\Lambda = \Lambda(\vec{p}, \approx)$ be the corresponding gentle algebra. Then there exists a graded punctured marked Riemann surface $(\mathring{\Sigma}, \vec{m})$ as well as equivalences of triangulated categories as in diagram (8); see [27, 36] for further details. From this perspective, our work provides further examples for the homological mirror symmetry for partially wrapped Fukaya categories of marked punctured and appropriately graded Riemann surfaces.

At this place we want to stress that the introduced class of algebras $\Lambda(\vec{p}, \vec{q}, \approx)$ does not exhaust (even up to derived equivalence) all derived tame algebras which are derived equivalent to an appropriate non-commutative tame projective nodal curve. For example, in the paper [8, Theorem 2.1] it was observed that on a chain of projective lines there exists a tilting bundle whose endomorphism algebra is a gentle algebra of *infinite* global dimension. In this paper, we have found another class of gentle algebras which are derived equivalent to appropriate non-commutative tame projective nodal curves. For any $n \in \mathbb{N}$, let Υ_n be the path algebra of the following quiver

$$
\begin{tikzcd}
\bullet \arrow[d, bend right, "a_1^+"'] \arrow[d, bend left, "a_1^-"] & \bullet \arrow[d, bend right, "a_2^+"'] \arrow[d, bend left, "a_2^-"] & & \bullet \arrow[d, bend right, "a_n^+"'] \arrow[d, bend left, "a_n^-"] \\
\bullet \arrow[d, "b_1^+"'] \arrow[dr, "b_1^-"] & \bullet \arrow[d, "b_2^+"'] \arrow[dr, "b_2^-"] & \cdots & \bullet \arrow[d, "b_n^+"] \arrow[dlll, bend left=10, "b_n^-"] \\
\bullet \arrow[d, bend right, "c_1^+"'] \arrow[d, bend left, "c_1^-"] & \bullet \arrow[d, bend right, "c_2^+"'] \arrow[d, bend left, "c_2^-"] & \cdots & \bullet \arrow[d, bend right, "c_n^+"'] \arrow[d, bend left, "c_n^-"] \\
\bullet & \bullet & \cdots & \bullet
\end{tikzcd}
\tag{9}
$$

modulo the relations

$$b_i^{\pm} a_i^{\mp} = 0, c_i^- b_i^+ = 0 \quad \text{and} \quad c_{i+1}^+ b_i^- = 0 \quad \text{for} \quad 1 \le i \le n.$$

Since $\mathsf{HH}^3(\Upsilon_n) \neq 0$, the algebra Υ_n can not be derived equivalent to any gentle algebra of the form $\Lambda(\vec{p}, \approx)$. On the other hand, we prove that $D^b(\Upsilon_n\text{-}\mathsf{mod})$ is equivalent to the derived category of coherent sheaves on the so–called Zhelobenko non-commutative cycle of projective lines; see Theorem 6.9.

Acknowledgement. We are thankful to Christoph Geiß for giving us a hint of the construction of the derived equivalence from Proposition 6.5 as well as to Yanki Lekili for explaining us connections between gentle algebras and various versions of the Fukaya category of a Riemann surface. The first-named author is also indebted to Nicolo Sibilla for communicating him the statement of Proposition 6.7 as well as for helpful discussions. The work of the first-named author was partially supported by the DFG project Bu-1866/4-1. The results of this paper were mainly obtained during the stay of the second-named author at Max-Planck-Institut für Mathematik in Bonn.

2. Some algebraic prerequisitities

2.1. **Brief review of the theory of minors.** Throughout this subsection, let R be a commutative noetherian ring. For any R-algebra C, we denote by C° the opposite R-algebra, by C-mod (respectively, by mod-C) the category of finitely generated left (respectively, right) C-modules and by C-Mod (respectively, Mod-C) the category of all left (respectively, right) C-modules. For any C-module X we denote by $\mathsf{add}(X)$ the additive closure of X, i.e. the full subcategory of C-Mod consisting of all direct summands of all finite direct sums of X.

In what follows, B is an R-algebra, which is finitely generated as R-module.

Definition 2.1. Let P be a finitely generated projective left B-module. Then the R-algebra $A := \bigl(\mathsf{End}_B(P)\bigr)^\circ$ is called a *minor* of B; see [18, 15].

It is clear that P is a $(B$-$A)$-bimodule and we have an exact functor

$$\mathsf{G} = \mathsf{Hom}_B(P, -\,) : B\text{-}\mathsf{Mod} \xrightarrow{\mathsf{G}} A\text{-}\mathsf{Mod}.$$

In the case P is a *projective generator* of the category B-Mod (meaning that for any object X of B-Mod there exists an epimorphism $P^n \twoheadrightarrow X$ for some $n \in \mathbb{N}$), Morita theorem asserts that the functor G is an equivalence of categories.

It is not difficult to prove the following result.

Lemma 2.2. *Consider the dual (right) B-module $P^\vee := \mathsf{Hom}_B(P, B)$. The following statements hold.*

- *The canonical morphism $P \longrightarrow P^{\vee\vee} = \mathsf{Hom}_B(P^\vee, B)$ is an isomorphism of left B-modules. Moreover, the canonical morphism of R-algebras*
$$\mathsf{End}_B(P) \longrightarrow \bigl(\mathsf{End}_B(P^\vee)\bigr)^\circ$$
is an isomorphism too.
- *For any object X of B-Mod, the canonical morphism of left A-modules*
$$P^\vee \otimes_B X \longrightarrow \mathsf{Hom}_B(P, X), \quad l \otimes x \mapsto \bigl(y \mapsto l(y)\cdot x\bigr)$$
is an isomorphism, i.e. we have an isomorphism of functors $\mathsf{G} \cong -\otimes_B P^\vee$. As a consequence, $P^\vee$ is a flat (actually, even projective) right B-module.
- *In particular, the canonical morphism*
$$P^\vee \otimes_B P \longrightarrow \mathsf{Hom}_B(P, P), \quad l \otimes y \mapsto \bigl(x \mapsto l(y)\cdot x\bigr)$$
is an isomorphism of $\bigl(A$-$A\bigr)$-bimodules.

Using Lemma 2.2, one can deduce the following results.

Theorem 2.3. *Consider the functors $\mathsf{F} := P \otimes_A -$ and $\mathsf{H} := \mathsf{Hom}_A(P^\vee, -\,)$ from A-Mod to B-Mod. Then the following statements hold.*

- *The functors $(\mathsf{F}, \mathsf{G}, \mathsf{H})$ form an adjoint triple, i.e. (F, G) and (G, H) form adjoint pairs.*

- *The functors* F *and* H *are fully faithful, whereas* G *is essentially surjective.*
- *Let* $I_P := \mathrm{Im}\bigl(P \otimes_A P^\vee \xrightarrow{\mathrm{ev}} B\bigr)$*. Then* I_P *is a* $(B\text{-}B)$*-bimodule and*

$$\mathsf{ker}(\mathsf{G}) := \bigl\{X \in B\text{-}\mathsf{Mod} \,\big|\, I_P X = 0\bigr\}.$$

In other words, the kernel of the exact functor G *can be identified with the essential image of the (fully faithful) restriction functor* $\bar{B}\text{-}\mathsf{Mod} \longrightarrow B\text{-}\mathsf{Mod}$*, where* $\bar{B} = B/I_P$*. Moreover, the functor* G *induces an equivalence of categories*

$$B\text{-}\mathsf{Mod}/\mathsf{ker}(\mathsf{G}) \longrightarrow A\text{-}\mathsf{Mod}.$$

- *The same results remain true, when we consider* G*,* F *and* H *as functors between the categories* $B\text{-}\mathsf{mod}$ *and* $A\text{-}\mathsf{mod}$ *of finitely generated modules.*
- *The essential image of the functor* $\mathsf{F} : A\text{-}\mathsf{mod} \longrightarrow B\text{-}\mathsf{mod}$ *is the category*

$$P\text{-}\mathsf{mod} := \Bigl\{X \in B\text{-}\mathsf{mod} \,\Big|\, \text{there exists } P_1 \longrightarrow P_0 \longrightarrow X \longrightarrow 0 \text{ with } P_0, P_1 \in \mathsf{add}(P)\Bigr\}.$$

It turns out that the relation between B-Mod, A-Mod and $\bar{B}$-Mod becomes even more transparent, when we pass to the setting of derived categories.

Theorem 2.4. *Let* $\mathsf{DG} : D(B\text{-}\mathsf{Mod}) \longrightarrow D(A\text{-}\mathsf{Mod})$ *be the derived functor of (an exact) functor* G*,* $\mathsf{LF} : D(A\text{-}\mathsf{Mod}) \longrightarrow D(B\text{-}\mathsf{Mod})$ *be the left derived functor of (a right exact functor)* F *and* $\mathsf{RH} : D(A\text{-}\mathsf{Mod}) \longrightarrow D(B\text{-}\mathsf{Mod})$ *be the right derived functor of (a left exact functor)* H*. Then the following results hold.*

- $(\mathsf{LF}, \mathsf{DG}, \mathsf{RH})$ *is an adjoint triple of functors.*
- *The functors* LF *and* RH *are fully faithful, whereas* DG *is essentially surjective.*
- *The essential image of* LF *is equal to the left orthogonal category*

$$^{\perp}\bar{B} := \bigl\{X^\bullet \in \mathrm{Ob}\bigl(D(B\text{-}\mathsf{Mod})\bigr) \;\big|\; \mathsf{Hom}\bigl(X^\bullet, \bar{B}[i]\bigr) = 0 \text{ for all } i \in \mathbb{Z}\bigr\}$$

of $\bar{B}$ *(viewed as a left* B*-module). Similarly, the essential image of* RH *is equal to the right orthogonal category* $\bar{B}^{\perp}$*.*
- *We have a recollement diagram*

$$\begin{tikzcd} D_{\bar{B}}(B\text{-}\mathsf{Mod}) \arrow[r, "\mathsf{I}"] & D(B\text{-}\mathsf{Mod}) \arrow[l, bend right, "\mathsf{I}^*"'] \arrow[l, bend left, "\mathsf{I}^!"] \arrow[r, "\mathsf{DG}"] & D(A\text{-}\mathsf{Mod}) \arrow[l, bend right, "\mathsf{LF}"'] \arrow[l, bend left, "\mathsf{RH}"] \end{tikzcd}, \tag{10}$$

where $D_{\bar{B}}(B\text{-}\mathsf{Mod})$ *is the full subcategory of* $D(B\text{-}\mathsf{Mod})$ *consisting of those complexes whose cohomologies belong to* $\bar{B}\text{-}\mathsf{Mod}$*.*
- *Assume that the* $(B\text{-}B)$*-bimodule* I_P *is flat viewed as a right* B*-module.*
  - *Then the functor* $D\bigl(\bar{B}\text{-}\mathsf{Mod}\bigr) \longrightarrow D_{\bar{B}}(B\text{-}\mathsf{Mod})$ *is an equivalence of triangulated categories.*
  - *We have:* $\mathsf{gl.dim} B \leq \mathsf{max}\bigl\{\mathsf{gl.dim} \bar{B} + 2, \mathsf{gl.dim} A\bigr\}$.

– *Finally, suppose that* $\mathsf{gl.dim}\bar{B} < \infty$ *and* $\mathsf{gl.dim}A < \infty$. *Then we have a recollement diagram*

$$D^b(\bar{B}\text{-}\mathsf{Mod}) \xrightarrow{\mathsf{I}} D^b(B\text{-}\mathsf{Mod}) \xrightarrow{\mathsf{DG}} D^b(A\text{-}\mathsf{Mod})\ . \tag{11}$$

with left adjoints I^*, LF above and right adjoints $\mathsf{I}^!$, RH below.

Remark 2.5. In the case $P = Be$ for an idempotent $e \in B$, most of the results from this subsection are due to Cline, Parshall and Scott [17]. The "abelian" theory of minors attached to an arbitrary finitely generated projective B-module P was for the first time suggested in [18]. Detailed proofs of Theorems 2.3 and 2.4 can also be found in [15].

2.2. **Generalities on orders.** From now on, let R be an excellent reduced ring of *Krull dimension one* and

$$K := \mathsf{Quot}(R) \cong K_1 \times \cdots \times K_r \tag{12}$$

be the corresponding total ring of fractions, where $K_1, \dots, K_r$ are fields.

Definition 2.6. An R-algebra A is an R-*order* if the following conditions hold.

- A is a finitely generated torsion free R-module.
- $A_K := K \otimes_R A$ is a semisimple K-algebra having finite length as a K-module.

Lemma 2.7. *Let R be as above, $R' \subseteq R$ be a finite ring extension and A be an R-algebra. Then A is an R-order if and only if A is an R'-order. Moreover, if $K := \mathsf{Quot}(R)$ then we have: $A_K \cong A_{K'}$.*

Proof. It is clear that A is finitely generated and torsion free over R if and only if it is finitely generated and torsion free over R'. Next, note that the ring extension $R' \subseteq R$ induces a finite ring extension $K' \subseteq K$ of the corresponding total rings of fractions. Moreover, Chinese remainder theorem implies that the multiplication map $K' \otimes_{R'} R \longrightarrow K$ is an isomorphism. Therefore, for any finitely generated R-module M, the natural map $K' \otimes_{R'} M \longrightarrow K \otimes_R M$ is an isomorphism of K'-modules. In particular, we get an isomorphism of K'-algebras $A_{K'} \longrightarrow A_K$. $\square$

Definition 2.8. Let A be a ring.

- A is a *one-dimensional order* (or just an *order*) provided its center $R = Z(A)$ is a reduced excellent ring of Krull dimension one, and A is an Rorder.
- Let $K := \mathsf{Quot}(R)$. Then $A_K := K \otimes_R A$ is called the *rational envelope* of A.
- A ring $\widetilde{A}$ is called an *overorder* of A if $A \subseteq \widetilde{A} \subseteq A_K$ and $\widetilde{A}$ is finitely generated as (a left) A-module.

Note that for any overorder $\widetilde{A}$ of A, the map $K \otimes_R \widetilde{A} \longrightarrow A_K$ is automatically an isomorphism. Hence, $A_K = \widetilde{A}_K$ and $\widetilde{A}$ is an order over R.

Theorem 2.9. *Let H be a hereditary order (i.e.* $\mathsf{gl.dim}H = 1$*) and $R = Z(H)$ be the center of H. Then the following results are true.*

- *We have: $R \cong R_1 \times \cdots \times R_r$, where R_i is a Dedekind domain for all $1 \le i \le r$.*

- *Let K_i be the quotient field of R_i. Then we have: $H_K \cong \Upsilon_1 \times \dots \times \Upsilon_r$, where Υ_i is a finite dimensional* central *simple K_i-algebra. Moreover, we have a decomposition $H = H_1 \times \dots \times H_r$, where each factor H_i is a hereditary order whose rational envelope is Υ_i.*
- *If $\widetilde{H}$ is an overorder of H then $\widetilde{H}$ is hereditary too.*
- *If H' is a minor of H then H' is a hereditary order too.*
- *Assume that R is semilocal. Let J be the radical of H and $\widehat{H} = \varprojlim_k \bigl(H/J^k\bigr)$ be the radical completions of H. Then $\widehat{H}$ is a hereditary order.*

Proofs of all these results can be for instance found in [39, 40].

Theorem 2.10. *Let $\Bbbk$ be an algebraically closed field and K be either $\Bbbk(\!(w)\!)$ or the function field of an algebraic curve over $\Bbbk$. Let Υ be a finite dimensional* central *simple algebra over K. Then $\Upsilon \cong \mathsf{Mat}_t(K)$ for some $t \in \mathbb{N}$.*

Proof. This is a restatement of the fact that the Brauer group of the field K is trivial; see for instance [26, Proposition 6.2.3, Theorem 6.2.8 and Theorem 6.2.11]. □

The following result is well-known; see [40].

Lemma 2.11. *Let R be a discrete valuation ring, $\mathfrak{m}$ be its maximal ideal, $\Bbbk := R/\mathfrak{m}$ the corresponding residue field and K the field of fractions of R. For any sequence of natural numbers $\vec{p} = \bigl(p_1, \dots, p_r\bigr)$, consider the R-algebra*

$$H(R,\vec{p}) := \left(\begin{array}{ccc|ccc|c|ccc}
R & \dots & R & \mathfrak{m} & \dots & \mathfrak{m} & \dots & \mathfrak{m} & \dots & \mathfrak{m} \\
\vdots & \ddots & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\
R & \dots & R & \mathfrak{m} & \dots & \mathfrak{m} & \dots & \mathfrak{m} & \dots & \mathfrak{m} \\
\hline
R & \dots & R & R & \dots & R & \dots & \mathfrak{m} & \dots & \mathfrak{m} \\
\vdots & & \vdots & \vdots & \ddots & \vdots & & \vdots & & \vdots \\
R & \dots & R & R & \dots & R & \dots & \mathfrak{m} & \dots & \mathfrak{m} \\
\hline
\vdots & & \vdots & \vdots & & \vdots & \ddots & \vdots & & \vdots \\
\hline
R & \dots & R & R & \dots & R & \dots & R & \dots & R \\
\vdots & & \vdots & \vdots & & \vdots & & \vdots & \ddots & \vdots \\
R & \dots & R & R & \dots & R & \dots & R & \dots & R
\end{array}\right) \subset \mathsf{Mat}_p(K), \tag{13}$$

where the size of the i-th diagonal block is $(p_i \times p_i)$ for each $1 \le i \le r$ and $p := |\vec{p}| = p_1 + \dots + p_r$. Then $H(R,\vec{p})$ is a hereditary R-order.

In what follows, $H(R,\vec{p})$ will be called *standard* hereditary R-order of type $\vec{p}$. For any $M \in H(R,\vec{p})$ and any pair $1 \le i,j \le r$, we denote by $M^{(i,j)}$ the corresponding block of M, which is a matrix of size $p_i \times p_j$. In particular, $M^{(i,i)}(0) \in \mathsf{Mat}_{p_i}(\Bbbk)$. In the simplest case when $\vec{p} = \vec{p}_r := (\underbrace{1, \dots, 1}_{r \text{ times}})$, we shall use the notation $H_r(R) := H\bigl(R, \vec{p}_r\bigr)$.

Theorem 2.12. *Let $\Bbbk$ be an algebraically closed field, $R = \Bbbk[\![w]\!]$ and $K = \Bbbk(\!(w)\!)$. Then the following results are true.*

- *Assume that H is a hereditary R-order in the central simple K-algebra $\mathsf{Mat}_p(K)$. Then there exists $S \in \mathsf{Mat}_p(K)$ such that $H = \mathsf{Ad}_S\bigl(H(R,\vec{p})\bigr) := S \cdot H(R,\vec{p}) \cdot S^{-1}$ for some tuple $\vec{p} = (p_1, \dots, p_r)$ such that $p = |\vec{p}|$. Moreover, such a tuple $\vec{p}$ is uniquely determined up to a permutation.*
- *Let H be a hereditary R-order. Then we have: $H \cong H_1 \times \dots \times H_s$, where each H_i is some standard R-order for any $1 \le i \le s$.*
- *For any vector $\vec{p}$, the orders $H(R,\vec{p})$ and $H_r(R)$ are Morita equivalent.*

Proofs of these results can be found in [40].

3. Nodal orders

Nodal orders are appropriate non-commutative generalizations of the commutative nodal ring $D := \Bbbk[\![u,v]\!]/(uv)$.

3.1. Definition and basic properties of nodal orders.

Definition 3.1. An order A is called *nodal* if its center is a semilocal excellent ring and there exists a *hereditary* overorder $H \supseteq A$ such that the following conditions are satisfied.

- $J := \mathsf{rad}(A) = \mathsf{rad}(H)$.
- For any finitely generated simple left A-module U we have: $l_A(H \otimes_A U) \le 2$.

Remark 3.2. In what follows, hereditary orders will be considered as special cases of nodal orders. Next, it is clear that an order A is nodal if and only if its radical completion $\widehat{A}$ is nodal. Moreover, it is not difficult to show that for a nodal order A, the hereditary overorder H from Definition 3.1 is in fact *uniquely determined* and admits the following intrinsic description:

$$H = \bigl\{x \in A_K \,\big|\, xJ \subseteq J\bigr\} \cong \mathsf{End}_A(J),$$

where J is viewed as a right A-module and A_K is the rational envelope of A. For a nodal order A, the order H will be called the *hereditary cover* of A.

Remark 3.3. The classical commutative nodal ring $D = \Bbbk[\![u,v]\!]/(uv)$ is a nodal order in the sense of Definition 3.1. Indeed, we have an embedding $D \subsetneq \widetilde{D} := \Bbbk[\![u]\!] \times \Bbbk[\![v]\!]$ and $\mathsf{rad}(D) = \mathsf{rad}(\widetilde{D}) = (u,v)$. Moreover, $\dim_{\Bbbk}\bigl(\widetilde{D} \otimes_D \Bbbk\bigr) = 2$. Thus $\widetilde{D}$ is the hereditary cover of D.

Nodal orders were introduced by the second-named author in [19]. In that work it was shown that the category of finite length modules over a (non-hereditary) nodal order is representation tame and conversely an order of nonwild representation type is automatically nodal. In our previous joint work [10] we proved that even the derived category of a nodal order has tame representation type.

Theorem 3.4. *Let A be a nodal order. Then the following statements are true.*

- *Any overorder of A is nodal.*

- *Any minor of A is nodal. In particular, if A' is Morita equivalent to A then A' is a nodal order too.*
- *Let G be a finite group acting on A. If $|G|$ is invertible in A then the skew group product $A * G$ is a nodal order.*

Proofs of the above statements can be found or easily deduced from the results of [19].

3.2. **Combinatorics of nodal orders.** Let $\Bbbk$ be an algebraically closed field. It turns out that nodal orders over the discrete valuation ring $\Bbbk[\![x]\!]$ can be completely classified.

Definition 3.5. Let Ω be a finite set and $\approx$ be a symmetric but not necessarily reflexive relation on Ω such that for any $\omega \in \Omega$ there exists *at most one* $\omega' \in \Omega$ (possibly, $\omega' = \omega$) such that $\omega \approx \omega'$ (note that $\approx$ is automatically transitive). We say that $\omega \in \Omega$ is

- *simple* if $\omega \not\approx \omega'$ for all $\omega' \in \Omega$;
- *reflexive* if $\omega \approx \omega$;
- *tied* if there exists $\omega' \neq \omega \in \Omega$ such that $\omega \approx \omega'$.

It is clear that any element of Ω is either simple, or reflexive, or tied with respect to the relation $\approx$.

Given $(\Omega, \approx)$ as above, we define two new sets $\Omega^\ddagger$ and $\widetilde{\Omega}^\ddagger$ by the following procedures.

- We get $\Omega^\ddagger$ from Ω by replacing each reflexive element $\omega \in \Omega$ by a pair of new simple elements ω_+ and ω_-. The tied elements of $\Omega^\ddagger$ are the same as for Ω.
- The set $\widetilde{\Omega}^\ddagger$ is obtained from $\Omega^\ddagger$ by replacing each pair of tied elements $\{\omega', \omega''\}$ by a single element $\overline{\{\omega', \omega''\}}$.

A map $\Omega^\ddagger \xrightarrow{\mathsf{wt}^\ddagger} \mathbb{N}$ is called a *weight function* provided $\mathsf{wt}^\ddagger(\omega') = \mathsf{wt}^\ddagger(\omega'')$ for all $\omega' \approx \omega''$ in $\Omega^\ddagger$. It is clear that $\mathsf{wt}^\ddagger$ descends to a map $\widetilde{\Omega}^\ddagger \xrightarrow{\mathsf{wt}^\ddagger} \mathbb{N}$. A weight function $\Omega^\ddagger \xrightarrow{\mathsf{wt}^\ddagger} \mathbb{N}$ determines a map (also called weight function) $\Omega \xrightarrow{\mathsf{wt}} \mathbb{N}$ given by the rule $\mathsf{wt}(\omega) := \mathsf{wt}^\ddagger(\omega_+) + \mathsf{wt}^\ddagger(\omega_-)$ for any reflexive point $\omega \in \Omega$. Abusing the notation, we shall drop the symbol $\ddagger$ in the notation of $\mathsf{wt}^\ddagger$ and write wt for all weight functions introduced above.

Remark 3.6. The simplest weight function $\Omega^\ddagger \xrightarrow{\mathsf{wt}_\circ} \mathbb{N}$ is given by the rule $\mathsf{wt}_\circ(\omega) = 1$ for all $\omega \in \Omega^\ddagger$.

Let $(\Omega, \approx)$ be as in Definition 3.5 and $\Omega \xrightarrow{\sigma} \Omega$ be a permutation. Then we have a decomposition

$$\Omega = \Omega_1 \sqcup \dots \sqcup \Omega_t, \tag{14}$$

where $\sigma(\Omega_i) = \Omega_i$ and the restricted permutation $\sigma\big|_{\Omega_i}$ is cyclic for any $1 \le i \le t$. In a similar way, we get a decomposition $\Omega^\ddagger = \Omega_1^\ddagger \sqcup \dots \sqcup \Omega_t^\ddagger$.

Definition 3.7. Let Ω be a finite set and $\Omega \xrightarrow{\sigma} \Omega$ a permutation. A *marking* m of (Ω, σ) is a choice of an element $\omega_i \in \Omega_i$ for any $1 \le i \le t$, where Ω_i are given by (14).

Note that a choice of marking m makes each set Ω_i totally ordered:

$$\omega_i < \sigma(\omega_i) < \dots < \sigma^{l_i - 1}(\omega_i), \tag{15}$$

where $l_i := |\Omega_i|$. Let $\approx$ be a relation on Ω as in Definition 3.5 and $\Omega^\ddagger \xrightarrow{\mathsf{wt}} \mathbb{N}$ be a weight function. Then for any $1 \le i \le t$, we get a vector

$$\vec{p_i} := \bigl(\mathsf{wt}(\omega_i), \mathsf{wt}(\sigma(\omega_i)), \dots, \mathsf{wt}(\sigma^{l_i-1}(\omega_i))\bigr) \in \mathbb{N}^{l_i}. \tag{16}$$

Definition 3.8. Let $(\Omega, \approx, \sigma)$ be a datum as in Definition 3.7, m be a marking of (Ω, σ) and $\Omega^\ddagger \xrightarrow{\mathsf{wt}} \mathbb{N}$ be a weight function. For any $1 \le i \le t$, let $H_i := H(R, \vec{p_i})$ be the corresponding standard hereditary order (13). Then we put:

$$H = H\bigl(R, (\Omega, \sigma, \approx, \mathsf{m}, \mathsf{wt})\bigr) := H_1 \times \dots \times H_t. \tag{17}$$

It is clear that H is a hereditary order, whose rational envelope is the semisimple algebra

$$\Lambda := \mathsf{Mat}_{s_1}(K) \times \dots \times \mathsf{Mat}_{s_t}(K), \tag{18}$$

where K is the fraction field of R and $s_i = \bigl|\vec{p_i}\bigr|$ for $1 \le i \le t$.

Remark 3.9. According to the definition (13) of a standard hereditary order, any matrix belonging to H_i is endowed with a division into vertical and horizontal stripes labelled by the elements of the set Ω_i. Moreover, for any reflexive element $\omega \in \Omega_i$, the corresponding vertical and horizontal stripes have further subdivisions labelled by the elements $\omega_\pm \in \Omega_i^\ddagger$.

Definition 3.10. Let R be a discrete valuation ring and $\bigl(\Omega, \sigma, \approx, \mathsf{m}, \mathsf{wt}\bigr)$ be a datum as above. Then we have a ring $A = A\bigl(R, (\Omega, \sigma, \approx, \mathsf{m}, \mathsf{wt})\bigr) \subseteq H$ defined as follows:

$$A := \left\{ (X_1, \dots X_t) \in H \;\middle|\; \begin{array}{c} X_{i'}^{(\omega',\omega')}(0) = X_{i''}^{(\omega'',\omega'')}(0) \;\text{ for all }\; \begin{array}{c} 1 \le i', i'' \le t \\ \omega' \in \Omega_{i'},\, \omega'' \in \Omega_{i''} \\ \omega' \approx \omega'',\; \omega' \ne \omega'' \end{array} \\ \hline \\ X_i^{(\omega_+,\omega_-)}(0) = 0 \text{ and } X_i^{(\omega_-,\omega_+)}(0) = 0 \quad \text{for all} \; \begin{array}{l} 1 \le i \le t \\ \omega \in \Omega_i \\ \omega \approx \omega \end{array} \end{array} \right\}.$$

The proof of the following results is straightforward.

Theorem 3.11. *Let R be a discrete valuation ring, $\bigl(\Omega, \sigma, \approx, \mathsf{m}, \mathsf{wt}\bigr)$ a datum as in Definition 3.7 and $A = A\bigl(R, (\Omega, \sigma, \approx, \mathsf{m}, \mathsf{wt})\bigr)$ the corresponding ring from Definition 3.10. Then the following statements hold.*

- *The ring A is connected if and only if there exists a surjection*
$$\{1, \dots, m\} \xrightarrow{\tau} \{1, \dots, t\}$$
for some m as well as elements $\upsilon_i \in \Omega_{\tau(i)}$ for $1 \le i \le m-1$ such that each $\upsilon_i \approx \upsilon_i'$ for some $\upsilon_i' \in \Omega_{\tau(i+1)}$. If A is connected and $R = \Bbbk[\![w]\!]$ then the center of A is isomorphic to $\Bbbk[\![w_1, \dots, w_t]\!]/(w_i w_j, 1 \le i \ne j \le t)$.
- *The ring A is an order whose rational envelope is the semisimple algebra Λ given by (18). The order H defined by (17) is an overorder of A.*

- *Let* $\widetilde{\mathsf{m}}$ *be any other marking of* (Ω, σ) *and* $\widetilde{A} = A\bigl(R, (\Omega, \sigma, \approx, \widetilde{\mathsf{m}}, \mathsf{wt})\bigr)$ *be the corresponding order. Then there exists* $S \in \Lambda$ *such that* $\widetilde{A} = \mathsf{Ad}_S(A)$*, i.e. the orders* $\widetilde{A}$ *and* A *are conjugate in* Λ*. It means that the order* $A\bigl(R, (\Omega, \sigma, \approx, \mathsf{m}, \mathsf{wt})\bigr)$ *does not depend (up to a conjugation) on the choice of marking of* (Ω, σ)*, so in what follows it will be denoted by* $A\bigl(R, (\Omega, \sigma, \approx, \mathsf{wt})\bigr)$*.*
- *The orders* $A\bigl(R, (\Omega, \sigma, \approx, \mathsf{wt})\bigr)$ *and* $A\bigl(R, (\Omega, \sigma, \approx, \mathsf{wt}_\circ)\bigr)$ *are Morita equivalent.*
- *Let*

$$J := \left\{ (X_1, \dots X_t) \in A \;\middle|\; X_i^{(\omega, \omega)}(0) = 0 \text{ for all } \begin{array}{c} 1 \le i \le t \\ \omega \in \Omega_i \end{array} \right\}. \tag{19}$$

Then we have: $J = \mathsf{rad}(A) = \mathsf{rad}(H)$*. Moreover, the natural map*

$$H \longrightarrow \mathsf{End}_A(J), \quad X \mapsto (Y \mapsto XY)$$

is an isomorphism, where J *is viewed as a right* A*-module.*
- *For any* $\omega \in \Omega$*, let* $\bar{H}_\omega := \mathsf{Mat}_{m_\omega}(\mathbb{k})$*, where* $m_\omega = \mathsf{wt}(\omega)$*. Similarly, for any* $\gamma \in \widetilde{\Omega}^\ddagger$*, let* $\bar{A}_\gamma := \mathsf{Mat}_{m_\gamma}(\mathbb{k})$*, where* $m_\gamma = \mathsf{wt}(\gamma)$*. Then we have a commutative diagram:*

$$\begin{tikzcd} A/J \arrow[r, "\cong"] \arrow[d, hook] & \prod\limits_{\gamma \in \widetilde{\Omega}^\ddagger} \bar{A}_\gamma \arrow[d, hook, "\imath"] \\ H/J \arrow[r, "\cong"] & \prod\limits_{\omega \in \Omega} \bar{H}_\omega \end{tikzcd} \tag{20}$$

where the components of the embedding $\imath$ *are defined as follows.*
  - *Let* $\omega \in \Omega$ *be a simple element and* γ *be the corresponding element of* $\widetilde{\Omega}^\ddagger$*. Then the corresponding component* $\bar{A}_\gamma \longrightarrow \bar{H}_\omega$ *of* $\imath$ *is the identity map.*
  - *Let* $\omega \in \Omega$ *be reflexive. Then the corresponding component of the map* $\imath$ *is given by the rule:*

$$\bar{A}_{\omega_+} \times \bar{A}_{\omega_-} \longrightarrow \bar{H}_\omega, \quad (X, Y) \mapsto \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}.$$

  - *Finally, let* $\omega', \omega'' \in \Omega$ *be a pair of tied elements and* $\gamma := \overline{\{\omega', \omega''\}}$ *be the corresponding element of* $\widetilde{\Omega}^\ddagger$*. Then the corresponding component of* $\imath$ *is the diagonal embedding*

$$\bar{A}_\gamma \longrightarrow \bar{H}_{\omega'} \times \bar{H}_{\omega''}, \quad X \mapsto (X, X).$$

- *The order* $A\bigl(R, (\Omega, \sigma, \approx, \mathsf{wt})\bigr)$ *is nodal and* $H\bigl(R, (\Omega, \sigma, \approx, \mathsf{wt})\bigr)$ *is its hereditary cover.*
- *Let* $(\Omega, \sigma, \approx, \mathsf{wt}), (\Omega', \sigma', \approx', \mathsf{wt}')$ *be two data as in Definition 3.7. If* R *is complete then we have:*

$$A\bigl(R, (\Omega, \sigma, \approx, \mathsf{wt})\bigr) \cong A\bigl(R, (\Omega', \sigma', \approx', \mathsf{wt}')\bigr)$$

if and only if there exists a bijection $\Omega \xrightarrow{\varphi} \Omega'$ *such that the diagram*

$$\begin{tikzcd} \Omega \arrow[r, "\sigma"] \arrow[dd, "\varphi"'] & \Omega \arrow[dd, "\varphi"] \arrow[dr, "\mathsf{wt}"] & \\ & & \mathbb{N} \\ \Omega' \arrow[r, "\sigma'"] & \Omega \arrow[ur, "\mathsf{wt}'"'] & \end{tikzcd}$$

is commutative and $\omega_1 \approx \omega_2$ *in* Ω *if and only if* $\varphi(\omega_1) \approx \varphi(\omega_2)$ *in* Ω'.

Remark 3.12. For the weight function $\Omega^{\ddagger} \xrightarrow{\mathsf{wt}_\circ} \mathbb{N}$, given by the rule $\mathsf{wt}_\circ(\omega) = 1$ for all $\omega \in \Omega^{\ddagger}$, we shall write $A\big(R, (\Omega, \sigma, \approx)\big) := A(R, \Omega, \sigma, \approx, \mathsf{wt}_\circ)$ or simply $A(\Omega, \sigma, \approx)$ in the case when it is clear which ring R is meant.

In the examples below we take $R = \mathbb{k}[\![w]\!]$.

Example 3.13. Let $\Omega = \{1, 2\}$, σ is identity and $1 \approx 2$. Then we have:

$$A(\Omega, \sigma, \approx) \cong D := \mathbb{k}[\![x, y]\!]/(xy).$$

As already mentioned, the hereditary cover of D is $\widetilde{D} := \mathbb{k}[\![x]\!] \times \mathbb{k}[\![y]\!]$.

Example 3.14. Let $\Omega = \{1, 2, 3\}$, $\sigma = \left(\begin{smallmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{smallmatrix}\right)$ and $1 \approx 2$. Then we have:

$$H = H(\Omega, \sigma, \approx) = \mathbb{k}[\![x]\!] \times \begin{pmatrix} \mathbb{k}[\![y]\!] & (y) \\ \mathbb{k}[\![y]\!] & \mathbb{k}[\![y]\!] \end{pmatrix}$$

and

$$A = A(\Omega, \sigma, \approx) = \Big\{ (X, Y) \in H \;\Big|\; X(0) = Y^{(11)}(0) \Big\}. \tag{21}$$

Note that $A \cong \mathsf{End}_D(\mathbb{k}[\![x]\!] \oplus D)$, where $D = \mathbb{k}[\![x, y]\!]/(xy)$. Alternatively, one can identify A with the arrow completion of the path algebra of the following quiver with relations:

$$\begin{tikzcd} \bullet \arrow[loop left, "a"] \arrow[r, bend left, "b"] & \bullet \arrow[l, bend left, "c"] \end{tikzcd} \qquad ba = 0,\; ac = 0, \tag{22}$$

The order given by (21) will be called *Zhelobenko order*, since it appeared for the first time in a work of Zhelobenko [50] dedicated to the study of admissible finite length representations of the Lie group $\mathsf{SL}_2(\mathbb{C})$.

Example 3.15. Let $\Omega = \{1, 2, 3, 4\}$, $\sigma = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{smallmatrix}\right)$ and $2 \approx 3$. Then we have:

$$A = A(\Omega, \sigma, \approx) \cong \begin{pmatrix} D & \widetilde{D} \\ I & \widetilde{D} \end{pmatrix} \cong \mathsf{End}_D(D \oplus I), \tag{23}$$

where $D = \Bbbk[\![x,y]\!]/(xy)$, $I = (x,y)$ and $\widetilde{D} = \Bbbk[\![x]\!] \times \Bbbk[\![y]\!]$. The order A is isomorphic to the arrow completion of the following quiver with relations:

$$\begin{tikzcd} - \arrow[r, bend left, "u_-"] & \star \arrow[l, bend left, "v_-"] \arrow[r, bend right, "v_+"'] & + \arrow[l, bend right, "u_+"'] \end{tikzcd} \qquad v_\pm u_\mp = 0. \tag{24}$$

Following the terminology of our previous work [13], A will be called the Auslander order of D, or just *Auslander order*.

Example 3.16. Let $\Omega = \{1\}$ with $1 \approx 1$ (of course, $\sigma = \mathrm{id}$ in this case). Then we have:

$$A = A(\Omega, \sigma, \approx) \cong \begin{pmatrix} \Bbbk[\![w]\!] & (w) \\ (w) & \Bbbk[\![w]\!] \end{pmatrix}. \tag{25}$$

The hereditary cover H of A ist just the matrix algebra $\mathsf{Mat}_2\big(\Bbbk[\![w]\!]\big)$.

Example 3.17. Let $\Omega = \{1, 2\}$, $\sigma = \left(\begin{smallmatrix} 1 & 2 \\ 2 & 1 \end{smallmatrix}\right)$ and $2 \approx 2$. Then we have:

$$A = A(\Omega, \sigma, \approx) \cong \begin{pmatrix} \Bbbk[\![w]\!] & (w) & (w) \\ \Bbbk[\![w]\!] & \Bbbk[\![w]\!] & (w) \\ \Bbbk[\![w]\!] & (w) & \Bbbk[\![w]\!] \end{pmatrix}. \tag{26}$$

Note that A is isomorphic to the arrow completion of the following quiver with relations:

$$\begin{tikzcd} - \arrow[r, bend left, "a_-"] & \star \arrow[l, bend left, "b_-"] \arrow[r, bend right, "b_+"'] & + \arrow[l, bend right, "a_+"'] \end{tikzcd} \qquad a_+ b_+ = a_- b_-. \tag{27}$$

The order (26) appeared for the first time in the 1970 ICM talk of I. Gelfand [25] in the context of the study of admissible finite length representations of the Lie group $\mathsf{SL}_2(\mathbb{R})$. In what follows, it will be called *Gelfand order*. The hereditary cover of A is

$$H = H(\Omega, \sigma, \approx) \cong \begin{pmatrix} \Bbbk[\![w]\!] & (w) & (w) \\ \Bbbk[\![w]\!] & \Bbbk[\![w]\!] & \Bbbk[\![w]\!] \\ \Bbbk[\![w]\!] & \Bbbk[\![w]\!] & \Bbbk[\![w]\!] \end{pmatrix}.$$

Theorem 3.18. *Let $\Bbbk$ be an algebraically closed field, $R = \Bbbk[\![w]\!], K = \Bbbk(\!(w)\!)$ and*

$$\Lambda = \mathsf{Mat}_{s_1}(K) \times \dots \times \mathsf{Mat}_{s_t}(K)$$

for some $s_1, \dots s_t \in \mathbb{N}$. If A is a nodal order whose rational envelope is Λ then there exists a datum $(\Omega, \sigma, \approx, \mathsf{wt})$ and $S \in \Lambda$ such that $A = \mathsf{Ad}_S\big(A(\Omega, \sigma, \approx, \mathsf{wt})\big)$, where $A(\Omega, \sigma, \approx, \mathsf{wt})$ is the nodal order from Definition 3.10.

Proof. Let $A \subset \Lambda$ be a nodal order, whose rational envelope is Λ and $A^\diamond$ be a basic order Morita equivalent to A. Then $A^\diamond$ is also nodal, see Theorem 3.4. Let $H^\diamond$ be the hereditary cover of $A_\diamond$. Then $H^\diamond \cong H(R, \vec{p}_1) \times \dots \times H(R, \vec{p}_r)$, where each component $H(R, \vec{p}_i)$ is a standard hereditary order given by (13). Let $J^\diamond = \mathsf{rad}(A^\diamond) = \mathsf{rad}(H^\diamond)$ be the common radical of $A^\diamond$ and $H^\diamond$, $\bar{A}^\diamond = A^\diamond/J^\diamond$ and $\bar{H}^\diamond = H^\diamond/J^\diamond$. Since $A^\diamond$ is basic and $\bar{A}^\diamond$ is semisimple, $\bar{A}^\diamond_\gamma$ is isomorphic to a product of several copies of the field $\Bbbk$. By the definition of nodal orders, the embedding of semisimple algebras $\bar{A}^\diamond \stackrel{\imath}{\hookrightarrow} \bar{H}^\diamond$ has the following property: for any simple $\bar{A}^\diamond$-module U we have: $l_{\bar{A}^\diamond}\big(\bar{H}^\diamond \otimes_{\bar{A}^\diamond} U\big) \le 2$. From this property one can easily deduce that

- Each simple component of $\bar{H}^\diamond$ is either $\mathbb{k}$ or $\mathsf{Mat}_2(\mathbb{k})$. In other words, for any $1 \le i \le r$, each entry of the vector $\vec{p}_i$ is either 1 or 2.
- The embedding $\bar{A}^\diamond \stackrel{\imath}{\hookrightarrow} \bar{H}^\diamond$ splits into the product of the following components:

$$\mathbb{k} \xrightarrow{\mathsf{id}} \mathbb{k}, \quad \mathbb{k} \xrightarrow{\mathsf{diag}} \mathbb{k} \times \mathbb{k} \quad \text{or} \quad (\mathbb{k} \times \mathbb{k}) \xrightarrow{\mathsf{diag}} \mathsf{Mat}_2(\mathbb{k}).$$

Let $1 \le i \le r$ and $\vec{p}_i = \big(p_{i,1}, \dots, p_{i,k_i}\big)$. Then we put:

$$\Omega_i := \big\{(i,1), \dots, (i,k_i)\big\}, \quad \Omega := \Omega_1 \sqcup \dots \sqcup \Omega_r \quad \text{and} \quad \sigma(i,j) := (i, j+1 \ \mathsf{mod}\ k_i).$$

Note that the set Ω parameterizes simple components of the algebra $\bar{H}^\diamond$. If $p_{i,j} = 2$ for $(i,j) \in \Omega_i$, we put: $(i,j) \approx (i,j)$. If $(i,j), (i',j') \in \Omega$ are such that the corresponding component of the embedding $\imath$ is $\mathbb{k} \xrightarrow{\mathsf{diag}} \mathbb{k} \times \mathbb{k}$, we put: $(i,j) \approx (i',j')$.

Note that the following diagram of $\mathbb{k}$-algebras

$$\begin{tikzcd}
A^\diamond \arrow[r] \arrow[d, hook] & \bar{A}^\diamond \arrow[d, hook, "\imath"] \\
H^\diamond \arrow[r, two heads, "\pi"] & \bar{H}^\diamond
\end{tikzcd}$$

is a pull-back diagram, i.e. $A^\diamond \cong \pi^{-1}\big(\mathrm{Im}(\imath)\big)$. From this it is not difficult to deduce that

$$A^\diamond = A(\Omega, \sigma, \approx, \mathsf{wt}_\circ),$$

where $\mathsf{wt}_\circ$ is the trivial weight function.

Since the order A is Morita equivalent to $A^\diamond$, there exists a projective left $A^\diamond$-module P such that $A \cong \big(\mathsf{End}_{A^\diamond}(P)\big)^\circ$. Recall that the isomorphism classes of indecomposable $A^\diamond$-modules are parameterized by the elements of the set $\widetilde{\Omega}^\ddagger$. Let $P \cong \bigoplus\limits_{\gamma \in \widetilde{\Omega}^\ddagger} P_\gamma^{\oplus m_\gamma}$ be a decomposition of P into a direct sum of indecomposable modules. Then we get a weight function $\widetilde{\Omega}^\ddagger \xrightarrow{\mathsf{wt}} \mathbb{N}$, $\gamma \mapsto m_\gamma$. It is easy to see that $A^\diamond \cong A(\Omega, \sigma, \approx, \mathsf{wt})$. $\square$

3.3. **Skew group products of $\mathbb{k}[\![u,v]\!]/(uv)$ with a finite group.** Let $\mathbb{k}$ be an algebraically closed field of characteristic zero, $\zeta \in \mathbb{k}$ be a primitive n-th root of 1, $G := \big\langle \rho \,\big|\, \rho^n = e \big\rangle$ be a cyclic group of order $n \in \mathbb{N}_{\ge 2}$ and $\mathbb{k}[G]$ be the corresponding group algebra. The following results is well-known.

Lemma 3.19. *For $1 \le k \le n$, let $\xi_k := \zeta^k$ and $\varepsilon_k := \frac{1}{n} \sum\limits_{j=0}^{n-1} \xi_k^j \rho^j \in \mathbb{k}[G]$. Then we have:*

$$\left\{ \begin{array}{ccc} 1 & = & \varepsilon_1 + \dots + \varepsilon_n \\ \varepsilon_k \cdot \varepsilon_l & = & \delta_{kl} \varepsilon_k \end{array} \right.$$

In other words, $\{\varepsilon_1, \dots, \varepsilon_n\}$ is a complete set of primitive idempotents of $\mathbb{k}[G]$.

Proposition 3.20. *Consider the action of the cyclic group G on the polynomial algebra $\mathbb{k}[u]$ given by the rule: $\rho \circ u := \zeta u$. Then the skew product $\mathbb{k}[u] * G$ is isomorphic to the path algebra of the cyclic quiver $\vec{C}_n$*

$$
\begin{tikzcd}
 & \overset{2}{\circ} \arrow[dl, "a_1"'] & & \circ \arrow[ll, "a_2"'] & \\
\overset{1}{\circ} \arrow[dr, "a_n"'] & & \vdots & & \circ \arrow[ul] \\
 & \overset{n}{\circ} \arrow[rr] & & \circ \arrow[ur] &
\end{tikzcd}
\tag{28}
$$

Proof. An isomorphism $\mathbb{k}[u] * G \longrightarrow \mathbb{k}\big[\vec{C}_n\big]$ is given by the rule:

$$
\begin{cases} \varepsilon_k & \mapsto & e_k \\ \varepsilon_k u \varepsilon_{k+1} & \mapsto & a_k, \end{cases}
$$

where $e_k \in \mathbb{k}\big[\vec{C}_n\big]$ is the trivial path corresponding to the vertex k. □

Corollary 3.21. *Let $R := \mathbb{k}[\![u]\!]$. Then the skew group product $R * G$ is isomorphic to the arrow completion of the path algebra of (28). Note that the latter algebra can in its turn be identified with the algebra of matrices*

$$
T_n(R) := \begin{pmatrix} R & R & \dots & R \\ \mathfrak{m} & R & \dots & R \\ \vdots & \vdots & \ddots & \vdots \\ \mathfrak{m} & \mathfrak{m} & \dots & R \end{pmatrix}, \tag{29}
$$

where the primitive idempotent e_k corresponding to the vertex k of $\vec{C}_n$ is sent to the k-th diagonal matrix unit of $T_n(R)$.

Remark 3.22. Let us notice that, strictly speaking, $R * G$ depends on the choice of an n-th primitive root of unity ζ. On the other hand, the primitive idempotent

$$
\varepsilon = \varepsilon_n := \frac{1}{n}(e + \rho + \dots + \rho^{n-1}) \in \mathbb{k}[G] \cong R * G / \operatorname{rad}(R * G)
$$

does not depend on the choice of ζ. Therefore, identifying the skew group product $R * G$ with the completed path algebra of a cyclic quiver, we shall always choose a labeling of vertices of $\vec{C}_n$ such that the idempotent ε is identified with the trivial path corresponding to the vertex labeled by n.

Note also that the orders $T_n(R)$ and $H_n(R)$ are isomorphic.

Let $0 < c < n$ be such that $\mathsf{gcd}(n, c) = 1$. Then we have a permutation

$$
\big\{\bar{1}, \dots, \bar{n}\big\} \xrightarrow{\tau_c} \big\{\bar{1}, \dots, \bar{n}\big\} \quad \bar{k} \mapsto \overline{c \cdot k},
$$

where $\bar{k}$ denotes the remainder of k modulo n.

Proposition 3.23. *For any $0 < c < n$ such that $\mathsf{gcd}(n, c) = 1$, consider the action of the cyclic group $G = \langle \rho \,|\, \rho^n = e \rangle$ on the nodal algebra $D = \mathbb{k}[\![u, v]\!]/(uv)$, given by the rule*

$$\left\{ \begin{array}{lcl} \rho \circ u & = & \zeta\, u \\ \rho \circ v & = & \zeta^c v. \end{array} \right. \tag{30}$$

Then the nodal order $A := D * G$ has the following description:

$$A \cong \Big\{ (U, V) \in T_n(R) \times T_n(R) \,\Big|\, U^{\tau_c(k)\tau_c(k)}(0) = V^{kk}(0) \text{ for } 1 \le k \le n \Big\}. \tag{31}$$

Proof. Let $\widetilde{D} = \mathbb{k}[\![u]\!] \times \mathbb{k}[\![v]\!]$ be the hereditary cover of D. Then

$$H := \widetilde{D} * G \cong \big(\mathbb{k}[\![u]\!] * G\big) \times \big(\mathbb{k}[\![v]\!] * G\big) \cong T_n(R) \times T_n(R)$$

is the hereditary cover of A. For any $1 \le k \le n$, let

$$\widetilde{\varepsilon}_k := \varepsilon_{\tau_c k} = \frac{1}{n}\big(1 + \zeta^{ck}\rho + \zeta^{2ck}\rho^2 + \cdots + \zeta^{(n-1)ck}\rho^{n-1}\big) \in \mathbb{k}[G].$$

We consider the elements ε_k and $\widetilde{\varepsilon}_k$ as elements, respectively, of $\mathbb{k}[[u]] * G$ and $\mathbb{k}[[v]] * G$. It is convenient, taking into account that the actions of G on $\mathbb{k}[\![u]\!]$ and on $\mathbb{k}[\![v]\!]$ are actually different.

We have the following commutative diagram of algebras and algebra homomorphisms:

$$\begin{tikzcd} A \arrow[r, two heads] \arrow[d, hook] & A/\operatorname{\mathsf{rad}}(A) \arrow[r, "\cong"] \arrow[d, hook] & \mathbb{k}[G] \arrow[d, hook, "\mathsf{diag}"] \\ H \arrow[r, two heads] & H/\operatorname{\mathsf{rad}}(H) \arrow[r, "\cong"] & \mathbb{k}[G] \times \mathbb{k}[G] \end{tikzcd}$$

Viewing ε_k and $\widetilde{\varepsilon}_k$ as elements of $\big(\mathbb{k}[\![u]\!] * G\big)/\big((u) * G\big) \cong \mathbb{k}[G] \cong \big(\mathbb{k}[\![v]\!] * G\big)/\big((v) * G\big)$, we get: $\mathsf{diag}\big(\widetilde{\varepsilon}_k\big) = \big(\widetilde{\varepsilon}_k, \widetilde{\varepsilon}_k\big) = \big(\varepsilon_{\sigma_c(k)}, \widetilde{\varepsilon}_k\big)$. Taking into account the rules for the isomorphisms

$$\mathbb{k}[\![u]\!] * G \xrightarrow{\cong} T_n(R) \xleftarrow{\cong} \mathbb{k}[\![v]\!] * G,$$

we get the description (31) of the nodal order A. $\square$

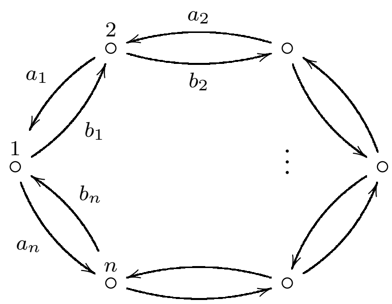
Remark 3.24. In the terms of Theorem 3.11, the order A given by (31) has the following description.

- Let $\Omega = \big\{1, \dots, n, \tilde{1}, \dots, \tilde{n}\big\}$.
- The relation $\approx$ is given by the rule: $\tilde{k} \approx \tau_c(k)$ for $1 \le k \le n$.
- The permutation $\Omega \xrightarrow{\sigma} \Omega$ is given by the formula

$$\sigma = \left(\begin{array}{cccc|cccc} 1 & 2 & \dots & n & \tilde{1} & \tilde{2} & \dots & \tilde{n} \\ n & 1 & \dots & n-1 & \tilde{n} & \tilde{1} & \dots & \widetilde{n-1} \end{array} \right).$$

Then we have: $A \cong A(\Omega, \sigma, \approx, \mathsf{wt}_\circ)$. It is not difficult to derive the quiver description of the order A. Of the major interest is the case $c = n - 1$. Then A is isomorphic to the arrow completion of the path algebra of the following quiver

(32)



modulo the relations $a_k b_k = 0 = b_k a_k$ for all $1 \le k \le n$.

Lemma 3.25. *Consider the action of the cyclic group $G = \langle \tau \,|\, \tau^2 = e \rangle$ on the nodal algebra $D = \mathbb{k}[\![u, v]\!]/(uv)$, given by the rule $\tau(u) = v$. Then the nodal order $A := D * G$ has the following description:*

$$A \cong \begin{pmatrix} \mathbb{k}[\![w]\!] & (w) \\ (w) & \mathbb{k}[\![w]\!] \end{pmatrix}. \tag{33}$$

Proof. Consider the elements $e_{\pm} := \dfrac{1 \pm \tau}{2} \in A$. Then the following statements are true.

- $e_{\pm}^2 = e_{\pm}$, $e_{\pm} e_{\mp} = 0$ and $1 = e_+ + e_-$. Moreover, $\tau \cdot e_{\pm} = \pm e_{\pm}$.
- For any $s, t \in \{+, -\}$ we have: $e_s A e_t = e_s D e_t$.

Therefore we have the Peirce decomposition:

$$A \cong \begin{pmatrix} e_+ A e_+ & e_+ A e_- \\ e_- A e_+ & e_- A e_- \end{pmatrix} = \begin{pmatrix} e_+ D e_+ & e_+ D e_- \\ e_- D e_+ & e_- D e_- \end{pmatrix}. \tag{34}$$

For any $m \in \mathbb{N}_0$, set $w_{\pm}^{(m)} := \dfrac{1}{2^m}\bigl(u^m \pm v^m\bigr)$ (within this notation, $w_-^{(0)} = 0$). It is easy to see that $w_{s_1}^{(m_1)} \cdot w_{s_2}^{(m_2)} = w_{s_1 \cdot s_2}^{(m_1 + m_2)}$ for any $m_1, m_2 \in \mathbb{N}_0$ and $s_1, s_2 \in \bigl\{+, -\bigr\}$. Moreover, one can check that

$$e_{\pm} D e_{\pm} = \Bigl\langle w_+^{(m)} e_{\pm} \,\Big|\, m \in \mathbb{N}_0 \Bigr\rangle_{\mathbb{k}} \quad \text{and} \quad e_{\pm} D e_{\mp} = \Bigl\langle w_-^{(m)} e_{\mp} \,\Big|\, m \in \mathbb{N} \Bigr\rangle_{\mathbb{k}}.$$

One can check that the linear map

$$A \cong \begin{pmatrix} e_+ D e_+ & e_+ D e_- \\ e_- D e_+ & e_- D e_- \end{pmatrix} \longrightarrow \begin{pmatrix} \mathbb{k}[\![w]\!] & (w) \\ (w) & \mathbb{k}[\![w]\!] \end{pmatrix}$$

given by the following rules:

$$\left\{ \begin{array}{lcl} w_+^{(m)} e_+ & \mapsto & w^m e_{11} \\ w_+^{(m)} e_- & \mapsto & w^m e_{22} \end{array} \right. \qquad m \in \mathbb{N}_0$$

and

$$\left\{ \begin{array}{lcl} w_-^{(m)} e_- & \mapsto & w^m e_{12} \\ w_-^{(m)} e_+ & \mapsto & w^m e_{21} \end{array} \right. \qquad m \in \mathbb{N}$$

is an algebra isomorphism we are looking for. $\square$

Proposition 3.26. *For any $n \in \mathbb{N}$, let $G := \langle \rho, \tau \,|\, \rho^n = e = \tau^2, \tau\rho\tau = \rho^{-1} \rangle$ be the dihedral group and $\zeta \in \Bbbk$ be a primitive n-th root of 1. Consider the action of G on the nodal ring $D = \Bbbk[\![u, v]\!]/(uv)$ given by the rules:*

$$\begin{cases} \rho \circ u &= \zeta\, u \\ \rho \circ v &= \zeta^{-1} v \\ \tau \circ u &= v \\ \tau \circ v &= u. \end{cases}$$

Then the nodal order $A := D * G$ has the following description.

- *If $n = 2l + 1$ for $l \in \mathbb{N}_0$ then $A \cong A(\Omega, \sigma, \approx, \mathsf{wt}_\circ)$, where*

$$\Omega := \{1, 2, \dots, n\}, \quad \sigma = \begin{pmatrix} 1 & 2 & \dots & n \\ n & 1 & \dots & n-1 \end{pmatrix}$$

  and $(2k-1) \approx 2k$ for $1 \le k \le l$, whereas $(2l+1) \approx (2l+1)$.
- *If $n = 2l + 2$ for $l \in \mathbb{N}_0$ then $A \cong A(\Omega, \sigma, \approx, \mathsf{wt}_\circ)$, where*

$$\Omega := \{0, 1, 2, \dots, 2l+1\}, \quad \sigma = \begin{pmatrix} 1 & 2 & \dots & n \\ n & 1 & \dots & n-1 \end{pmatrix}$$

  and $(2k-1) \approx 2k$ for $1 \le k \le l$, whereas $0 \approx 0$ and $(2l+1) \approx (2l+1)$.

Sketch of the proof. The cyclic group $K := \langle \rho \rangle$ is a normal subgroup of G of index two. Let $L = \langle \tau \rangle$. Then we have a commutative diagram of $\Bbbk$-algebras and algebra homomorphisms

$$\begin{tikzcd} A * G \arrow[r, "\cong"] \arrow[d, hook] & (A * K) * L \arrow[d, hook] \\ H * G \arrow[r, "\cong"] & (H * K) * L, \end{tikzcd}$$

where the action of L on $H * K = \big(\Bbbk[\![u]\!] * K\big) \times \big(\Bbbk[\![v]\!] * K\big)$ is given by the rule

$$\tau \circ \big(u^{k_1}\rho^{l_1}, v^{k_2}\rho^{l_2}\big) = \big(u^{k_2}\rho^{-l_2}, v^{k_1}\rho^{-l_1}\big)$$

for any $k_1, k_2, l_1, l_2 \in \mathbb{N}_0$. The nodal ring $A * K$ is described in Remark 3.24. In the terms of the quiver presentation (32) we have:

$$\begin{cases} \tau \circ a_k &= b_{\bar{k}-1} \\ \tau \circ b_k &= a_{\bar{k}-1} \\ \tau \circ e_k &= e_{\bar{k}} \end{cases}$$

where $\bar{k} = (n-k)$ for $1 \le k \le n$ and all indices are taken modulo n. The remaining part is a lengthy computation analogous to the one made in the course of the proof of Lemma 3.25 which we leave for the interested reader.

3.4. **Auslander order of a nodal order.**

Definition 3.27. Let A be a nodal order and H be its hereditary cover. Then

$$C := \bigl\{a \in A \,\big|\, ah \in A \text{ for all } h \in H\bigr\} \tag{35}$$

is called *conductor ideal* of A.

Remark 3.28. It follows from the definition that $C = ACH$ and the canonical morphism

$$C \longrightarrow \mathsf{Hom}_A(H, A), \quad c \mapsto \bigl(h \mapsto ch\bigr) \tag{36}$$

is a bijection (here, we view both H and A as left A-modules).

Proposition 3.29. *Let R be a discrete valuation ring and $A = A\bigl(R, (\Omega, \sigma, \approx, \mathsf{wt})\bigr)$ be the nodal order from the Definition 3.10. Then we have:*

$$C = \left\{ (X_1, \dots X_t) \in A \;\middle|\; X_{i'}^{(\omega', \omega')}(0) = 0 = X_{i''}^{(\omega'', \omega'')}(0) \text{ for all } \begin{array}{l} 1 \le i', i'' \le t \\ \omega' \in \Omega_{i'},\ \omega'' \in \Omega_{i''} \\ \omega' \approx \omega'' \end{array} \right\}$$

and $C = \bigl\{a \in A \,\big|\, ha \in A \text{ for all } h \in H\bigr\}$. In particular, C is a two-sided *ideal both in H and A containing the common radical $J = \mathsf{rad}(A) = \mathsf{rad}(H)$ of A and H.*

Corollary 3.30. *Let $\Bbbk$ be an algebraically closed field, R be the local ring of an affine curve over $\Bbbk$ at a smooth point, H be a hereditary R-order, A be a nodal order whose hereditary cover is H and C be the corresponding conductor ideal. Then C is a two sided ideal in H.*

Proof. We have to show that the canonical map of R-modules $C \longrightarrow HCH$ is surjective. For this, it is sufficient to prove the corresponding statement for the radical completions of A and H. However, the structure of nodal orders over $\widehat{R} \cong \Bbbk[\![w]\!]$ is known; see Theorem 3.18. Hence, the statement follows from Proposition 3.29. $\square$

Lemma 3.31. *Let $A = A\bigl(R, (\Omega, \sigma, \approx, \mathsf{wt})\bigr)$ be a nodal order, H be its hereditary cover, C be the conductor ideal, $\bar{A} := A/C$ and $\bar{H} := H/C$. Let $\Omega_\circ$ be the subset of Ω whose elements are reflexive or tied elements of Ω and $\widetilde{\Omega}^\ddagger_\circ$ be the subset of $\widetilde{\Omega}^\ddagger$ defined in a similar way. Then the following diagram*

$$\begin{tikzcd} \bar{A} \arrow[r, "\cong"] \arrow[d, hook] & \prod\limits_{\gamma \in \widetilde{\Omega}^\ddagger_\circ} \bar{A}_\gamma \arrow[d, hook, "\imath"] \\ \bar{H} \arrow[r, "\cong"] & \prod\limits_{\omega \in \Omega_\circ} \bar{H}_\omega \end{tikzcd} \tag{37}$$

is commutative, where the components of the embedding $\imath$ are described in the same way as in diagram (20)*.*

Definition 3.32. Let A be a nodal order, H be its hereditary cover and C be the corresponding conductor ideal. The order

$$B := \begin{pmatrix} A & H \\ C & H \end{pmatrix} \tag{38}$$

is called *Auslander order* of A.

Example 3.33. The Auslander order of the commutative nodal ring $\mathbb{k}[\![u,v]\!]/(uv)$ is

$$\begin{pmatrix} \mathbb{k}[\![u,v]\!]/(uv) & \mathbb{k}[\![u]\!] \times \mathbb{k}[\![v]\!] \\ (u,v) & \mathbb{k}[\![u]\!] \times \mathbb{k}[\![v]\!] \end{pmatrix},$$

i.e. the order from Example 3.15.

Example 3.34. Let $R = \mathbb{k}[\![w]\!]$, $\mathfrak{m} = (w)$ and $A = \begin{pmatrix} R & \mathfrak{m} \\ \mathfrak{m} & R \end{pmatrix}$ be the nodal order from Example 3.16. The hereditary cover H of A ist just the matrix algebra $\mathsf{Mat}_2(R)$, whereas the corresponding conductor ideal $C = \mathsf{Mat}_2(\mathfrak{m})$. Therefore, the Auslander order of A is

$$B = \begin{pmatrix} R & \mathfrak{m} & R & R \\ \mathfrak{m} & R & R & R \\ \mathfrak{m} & \mathfrak{m} & R & R \\ \mathfrak{m} & \mathfrak{m} & R & R \end{pmatrix}.$$

It is easy to see that B is Morita equivalent to the Gelfand order (26).

Let A be an arbitrary nodal order and B be the Auslander order of A. For the idempotents $e := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $f := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in B$, consider the corresponding projective left B-modules $P := Be = \begin{pmatrix} A \\ C \end{pmatrix}$ and $Q := Bf = \begin{pmatrix} H \\ H \end{pmatrix}$. Note that

$$A = eBe \cong \bigl(\mathsf{End}_B(P)\bigr)^\circ \quad \text{and} \quad H = fBf \cong \bigl(\mathsf{End}_B(Q)\bigr)^\circ.$$

The action of B on the projective left B-modules $\begin{pmatrix} A \\ C \end{pmatrix}$ and $\begin{pmatrix} H \\ H \end{pmatrix}$ is given by the matrix multiplication, whereas the isomorphisms $A \cong \bigl(\mathsf{End}_B(P)\bigr)^\circ$, respectively $H \cong \bigl(\mathsf{End}_B(Q)\bigr)^\circ$, are compatible with the canonical right actions on P, respectively Q. The nodal order A as well as its hereditary cover H are *minors* of the Auslander order B in the sense of Definition 2.1. Let

$$P^\vee := \mathsf{Hom}_B(P, B) \cong eB \quad \text{and} \quad Q^\vee := \mathsf{Hom}_B(Q, B) \cong fB.$$

In the terms of Subsection 2.1, we have the following functors.

- Since P is a projective left A-module, we get an exact functor

  $$\mathsf{G} := \mathsf{Hom}_B(P, -) \simeq P^\vee \otimes_B -$$

  from B-Mod to A-Mod. Of course, it restricts to an exact functor

  $$B\text{-}\mathsf{mod} \xrightarrow{\mathsf{G}} A\text{-}\mathsf{mod}$$

between the corresponding categories of finitely generated modules.
- Similarly, we have an exact functor $\widetilde{\mathsf{G}} = \mathsf{Hom}_B(Q, -) \simeq Q^\vee \otimes_B -$ from B-Mod to H-Mod, as well as its restriction on the full subcategories of the corresponding finitely generated modules.
- We have functors $\mathsf{F} := P \otimes_A -$ and $\mathsf{H} := \mathsf{Hom}_A(P^\vee, -)$ from A-Mod to B-Mod.
- Similarly, we have functors $\widetilde{\mathsf{F}} := Q \otimes_H -$ and $\widetilde{\mathsf{H}} := \mathsf{Hom}_H(Q^\vee, -)$ from H-Mod to B-Mod.

Additionally to Theorem 2.3, the following result is true.

Proposition 3.35. *The functor $\widetilde{\mathsf{F}}$ is exact, maps projective modules to projective modules and has the following explicit description: if $\widetilde{N}$ is a left H-module, then*

$$\widetilde{\mathsf{F}}(\widetilde{N}) = \begin{pmatrix} \widetilde{N} \\ \widetilde{N} \end{pmatrix} \cong \widetilde{N} \oplus \widetilde{N},$$

where for $b = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \in B$ and $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \begin{pmatrix} \widetilde{N} \\ \widetilde{N} \end{pmatrix}$, the element bz is given by the matrix multiplication.

Remark 3.36. Since the order H is hereditary, the conductor ideal C is projective viewed as a left H-module. The functor $\widetilde{\mathsf{F}}$ transforms projective modules into projective modules, hence the left B-module $\begin{pmatrix} C \\ C \end{pmatrix}$ is projective, too. Consider the left B-modules S and T, given by the projective resolutions

$$0 \longrightarrow \begin{pmatrix} C \\ C \end{pmatrix} \longrightarrow \begin{pmatrix} A \\ C \end{pmatrix} \longrightarrow S \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow \begin{pmatrix} C \\ C \end{pmatrix} \longrightarrow \begin{pmatrix} H \\ H \end{pmatrix} \longrightarrow T \longrightarrow 0.$$

Obviously, both S and T have finite length viewed as B-modules. Moreover, S is isomorphic to $\bar{A}$ viewed as an A-module and T is isomorphic to $\bar{H}$ viewed as an H-module.

Theorem 3.37. *The following results are true.*
- $(\mathsf{LF}, \mathsf{DG}, \mathsf{RH})$ *and* $(\mathsf{L}\widetilde{\mathsf{F}}, \mathsf{D}\widetilde{\mathsf{G}}, \mathsf{R}\widetilde{\mathsf{H}})$ *are triples pairs of functors.*
- *The functors* LF, $\mathsf{L}\widetilde{\mathsf{F}}$, RH *and* $\mathsf{R}\widetilde{\mathsf{H}}$ *are fully faithful, whereas the functors* DG *and* $\mathsf{D}\widetilde{\mathsf{G}}$ *are essentially surjective.*
- *The essential image of* $\mathsf{L}\widetilde{\mathsf{F}}$ *is equal to the left orthogonal category*

  $$^{\perp}S := \bigl\{X^\bullet \in \mathrm{Ob}\bigl(D(B\text{-}\mathsf{mod})\bigr) \bigm| \mathsf{Hom}\bigl(X^\bullet, S[i]\bigr) = 0 \text{ for all } i \in \mathbb{Z}\bigr\}$$

  of S, whereas the essential image of $\mathsf{R}\widetilde{\mathsf{H}}$ *is equal to the right orthogonal category $S^\perp$. Similarly, the essential image of* LF *is equal to $^{\perp}T$ and the essential image of* RH *is equal to $T^\perp$.*
- *We have a recollement diagram*

$$D(\bar{A}\text{-}\mathsf{mod}) \underset{\mathsf{I}^!}{\overset{\mathsf{I}^*}{\leftleftarrows}} \xrightarrow{\ \mathsf{I}\ } D(B\text{-}\mathsf{mod}) \underset{\mathsf{R}\widetilde{\mathsf{H}}}{\overset{\mathsf{L}\widetilde{\mathsf{F}}}{\leftleftarrows}} \xrightarrow{\ \mathsf{D}\widetilde{\mathsf{G}}\ } D(H\text{-}\mathsf{mod}) \ , \tag{39}$$

where $\mathsf{I}(\bar{A}) := S$*, the functor* I^* *is left adjoint to* I *and* $\mathsf{I}^!$ *is right adjoint to* I.

- *Similarly, we have another recollement diagram*

(40)
$$\begin{tikzcd} D_T(B\text{-}\mathsf{mod}) \arrow[r, "\mathsf{J}" description] & D(B\text{-}\mathsf{mod}) \arrow[l, bend right, "\mathsf{J}^*"'] \arrow[l, bend left, "\mathsf{J}^!"] \arrow[r, "\mathsf{DG}" description] & D(A\text{-}\mathsf{mod}) \arrow[l, bend right, "\mathsf{LF}"'] \arrow[l, bend left, "\mathsf{RH}"] \end{tikzcd},$$

where $D_T(B\text{-}\mathsf{mod})$ *is the full subcategory of the derived category* $D(B\text{-}\mathsf{mod})$ *consisting of those complexes whose cohomologies belong to* $\mathsf{Add}(T)$ *and* J *is the canonical inclusion functor.*

- *We have:* $\mathsf{gl.dim}B = 2$.

Proof. These results are specializations of Theorem 2.4. The statements about both recollement diagrams (39) and (40) follow from the description of the kernels of the functors $\mathsf{D}\widetilde{\mathsf{G}}$ and DG. Namely, consider the two-sided ideal

$$I_Q := \mathrm{Im}\big(Q \otimes_A Q^\vee \xrightarrow{\mathrm{ev}} B\big) = \begin{pmatrix} C & H \\ C & H \end{pmatrix}$$

in the algebra B. As one can easily see, I_Q is projective viewed as a right B-module. Moreover, $B/I_Q \cong A/C =: \bar{A}$ is semisimple. Hence, Theorem 2.4 gives the first recollement diagram (39). Analogously, for

$$I_P := \mathrm{Im}\big(P \otimes_H P^\vee \xrightarrow{\mathrm{ev}} B\big) = \begin{pmatrix} A & H \\ C & C \end{pmatrix}$$

the algebra $B/I_P \cong H/C =: \bar{H}$ is again semisimple. However, this time I_P is not projective viewed as a right B-module.

To show the last statement, note that according to Theorem 2.4 we have: $\mathsf{gl.dim}B \leq 2$. Since A is a non-hereditary minor of B, the order B itself can not be hereditary; see Theorem 2.9. Hence $\mathsf{gl.dim}B = 2$, as claimed. $\square$

Corollary 3.38. *We have a recollement diagram*

(41)
$$\begin{tikzcd} D^b(\bar{A}\text{-}\mathsf{mod}) \arrow[r, "\mathsf{I}" description] & D^b(B\text{-}\mathsf{mod}) \arrow[l, bend right, "\mathsf{I}^*"'] \arrow[l, bend left, "\mathsf{I}^!"] \arrow[r, "\mathsf{D}\widetilde{\mathsf{G}}" description] & D^b(H\text{-}\mathsf{mod}) \arrow[l, bend right, "\mathsf{L}\widetilde{\mathsf{F}}"'] \arrow[l, bend left, "\mathsf{R}\widetilde{\mathsf{H}}"] \end{tikzcd}.$$

As a consequence, we have a semiorthogonal decomposition

(42)
$$D^b(B\text{-}\mathsf{mod}) = \Big\langle \mathsf{Im}(\mathsf{I}),\, \mathsf{Im}(\mathsf{L}\widetilde{\mathsf{F}}) \Big\rangle = \Big\langle D^b\big(\bar{A}\text{-}\mathsf{mod}\big),\, D^b(H\text{-}\mathsf{mod})\Big\rangle.$$

Moreover, we have the following commutative diagram of categories and functors:

(43)
$$\begin{tikzcd} D^b(H\text{-}\mathsf{mod}) \arrow[r, hook, "\mathsf{L}\widetilde{\mathsf{F}}"] \arrow[dr, "\mathsf{P}"'] & D^b(B\text{-}\mathsf{mod}) \arrow[d, "\mathsf{DG}"] & \mathsf{Perf}(A) \arrow[l, hook', "\mathsf{LF}"'] \arrow[dl, hook', "\mathsf{E}"] \\ & D^b(A\text{-}\mathsf{mod}) & \end{tikzcd}$$

where $\mathsf{Perf}(A)$ *is the perfect derived category of* A*,* E *is the canonical inclusion functor and* P *is the derived functor of the restriction functor* $H\text{-}\mathsf{mod} \longrightarrow A\text{-}\mathsf{mod}$*.*

Proof. The recollement diagram (41) is just the restriction of the recollement diagram (39) on the corresponding full subcategories of compact objects. The isomorphism $\mathsf{E} \simeq \mathsf{DG} \circ \mathsf{LF}$ follows from the fact that the adjunction unit $\mathsf{Id}_{D(A\text{-}\mathsf{mod})} \longrightarrow \mathsf{DG} \circ \mathsf{LF}$ is an isomorphism of functors (already on the level of unbounded derived categories). Next, since the functor $\widetilde{\mathsf{F}}$ is exact, we have: $\mathsf{DG} \circ \mathsf{L}\widetilde{\mathsf{F}} \simeq \mathsf{D}(\mathsf{G} \circ \widetilde{\mathsf{F}})$. For any H-module $\widetilde{N}$ we have:

$$(\mathsf{G} \circ \widetilde{\mathsf{F}})(\widetilde{N}) = \mathsf{Hom}_B\big(P, \widetilde{\mathsf{F}}(\widetilde{N})\big) = \mathsf{Hom}_B\left(Be, \begin{pmatrix} \widetilde{N} \\ \widetilde{N} \end{pmatrix}\right) \cong e \cdot \begin{pmatrix} \widetilde{N} \\ \widetilde{N} \end{pmatrix} \cong \widetilde{N}.$$

Hence, $\mathsf{G} \circ \widetilde{\mathsf{F}}$ is isomorphic to the restriction functor $H\text{-}\mathsf{mod} \longrightarrow A\text{-}\mathsf{mod}$, what finishes a proof of the second statement. $\square$

Proposition 3.39. *Let A be a nodal order. Then the corresponding Auslander order B is nodal too.*

Proof. As usual, let H be the hereditary cover of A and $J = \mathsf{rad}(A) = \mathsf{rad}(H)$ be the common radical of A and H. Consider the following orders:

$$\widetilde{B} := \begin{pmatrix} A & H \\ J & H \end{pmatrix} \quad \text{and} \quad \widetilde{H} := \begin{pmatrix} H & H \\ J & H \end{pmatrix}.$$

It is not difficult to show that

$$\widetilde{J} := \mathsf{rad}(\widetilde{H}) = \begin{pmatrix} J & H \\ J & J \end{pmatrix} = \mathsf{rad}(\widetilde{B}).$$

Since $\widetilde{J}$ is projecive as H-module, H is hereditary. Then the commutative diagram

$$\begin{tikzcd} \widetilde{B}/\widetilde{J} \arrow[r, "\cong"] \arrow[d, hook] & A/J \times H/J \arrow[d, hook] \\ \widetilde{H}/\widetilde{J} \arrow[r, "\cong"] & H/J \times H/J \end{tikzcd}$$

implies that $\widetilde{B}$ is a nodal order and $\widetilde{H}$ is its hereditary cover. Since the conductor ideal C contains the radical J, the Auslander order B is an overorder of $\widetilde{B}$. It follows from Theorem 3.4 that the order B is nodal, too. $\square$

In what follows, we shall need the following result about the finite length B-module S. Assume that $R = \mathbb{k}[\![w]\!]$ and $A = A\big(R, (\Omega, \sigma, \approx, \mathsf{wt})\big)$. Recall that

$$\bar{A} := A/C \cong \prod_{\gamma \in \widetilde{\Omega}^{\ddagger}_{\circ}} \bar{A}_\gamma \cong \prod_{\gamma \in \widetilde{\Omega}^{\ddagger}_{\circ}} \mathsf{Mat}_{\mathsf{wt}(\gamma)}(\mathbb{k}).$$

It is clear that the set $\widetilde{\Omega}^{\ddagger}_{\circ}$ also parameterizes the isomorphism classes of the simple $\bar{A}$-modules. For any $\gamma \in \widetilde{\Omega}^{\ddagger}_{\circ}$, let S_γ be the simple left B-module which corresponds to the (unique, up to an isomorphism) simple $\bar{A}_\gamma$-module and P_γ be its projective cover. Then we have: $S \cong \bigoplus\limits_{\gamma \in \widetilde{\Omega}^{\ddagger}_{\circ}} S_\gamma^{\oplus \mathsf{wt}(\widetilde{\Omega}^{\ddagger}_{\circ})}$. Our next goal is to describe a minimal projective resolution of S_γ. For any $\omega \in \Omega$, let $\widetilde{Q}_\omega$ be the corresponding indecomposable projective left H-module and

$$Q_\omega := \widetilde{\mathsf{F}}\bigl(\widetilde{Q}_\omega\bigr) = \begin{pmatrix} \widetilde{Q}_\omega \\ \widetilde{Q}_\omega \end{pmatrix}$$

be the corresponding indecomposable projective left B-module.

Lemma 3.40. *The following statements hold.*

- *Let $\omega \in \Omega$ be a reflexive element and $\gamma = \omega_\pm$ be one of the corresponding elements of $\widetilde{\Omega}^{\ddagger}_{\circ}$. Then*

$$0 \longrightarrow Q_{\sigma(\omega)} \longrightarrow P_\gamma \longrightarrow S_\gamma \longrightarrow 0 \tag{44}$$

  is a minimal projective resolution of the simple B-module S_γ. In particular, for any $\delta \in \Omega$ we have:

$$\mathsf{Ext}^1_B\bigl(S_\gamma, Q_\delta\bigr) \cong \begin{cases} \Bbbk & \text{if } \delta = \sigma(\omega) \\ 0 & \text{otherwise.} \end{cases} \tag{45}$$

- *Let $\omega', \omega'' \in \Omega$ be a pair of tied elements and $\gamma = \overline{\{\omega', \omega''\}}$ be the corresponding element of $\widetilde{\Omega}^{\ddagger}_{\circ}$. Then a minimal projective resolution of the simple B-module S_γ has the following form:*

$$0 \longrightarrow Q_{\sigma(\omega')} \oplus Q_{\sigma(\omega'')} \longrightarrow P_\gamma \longrightarrow S_\gamma \longrightarrow 0. \tag{46}$$

  In particular, for any $\delta \in \Omega$ we have:

$$\mathsf{Ext}^1_B\bigl(S_\gamma, Q_\delta\bigr) \cong \begin{cases} \Bbbk & \text{if } \delta = \sigma(\omega) \text{ for } \omega \in \{\omega', \omega''\} \\ 0 & \text{otherwise.} \end{cases} \tag{47}$$

Proof. It is not difficult to show that $\mathsf{rad}(P_\gamma) = Q_{\sigma(\omega)}$ for $\gamma = \omega_\pm$ if $\omega \in \Omega$ is reflexive and $\mathsf{rad}(P_\gamma) = Q_{\sigma(\omega')} \oplus Q_{\sigma(\omega'')}$ if $\gamma = \overline{\{\omega', \omega''\}}$ for $\omega', \omega'' \in \Omega$ tied. The formulae (45) and (47) follow from the fact that $\mathsf{rad}(Q_\omega) = Q_{\sigma(\omega)}$ for any $\omega \in \Omega$. $\square$

4. Non-commutative nodal curves: global theory

In this section, we are going to explain the construction as well as main properties of non-commutative nodal curves of tame representation type.

4.1. **The idea of a non-commutative nodal curve.** Before going to technicalities and details, let us consider the following example. Let $\Bbbk$ be a field, $S = \Bbbk[x]$, $J = (x^2 - 1)$ and $K = \Bbbk(x)$. Consider the hereditary order

$$H = \begin{pmatrix} S & J & J \\ S & S & J \\ S & S & S \end{pmatrix} \subset \Upsilon := \mathsf{Mat}_3(K).$$

Next, consider the order

$$A := \left\{ X \in H \;\middle|\; \begin{array}{l} X_{11}(1) = X_{22}(1) \\ X_{33}(1) = X_{33}(-1) \\ X_{21}(-1) = 0 \end{array} \right\} \subset H.$$

Let $Z = Z(A)$ be the center of A. Then we have:

$$Z = \bigl\{p \in S \bigm| p(1) = p(-1)\bigr\} = \mathbb{k}\bigl[x^2-1, x(x^2-1)\bigr] \cong \mathbb{k}[u,v]/(v^2-u^3-u^2).$$

The multiplication maps $K \otimes_Z A \longrightarrow \Upsilon$ and $K \otimes_Z H \longrightarrow \Upsilon$ are isomorphisms. In other words, A and H are Z-*orders* in the *central simple* K-algebra Υ.

Let $E_\circ = V(v^2-u^3-u^2) \subset \mathbb{A}^2$ be an affine plane nodal cubic and $s = (0,0) \in E_\circ$ be its unique singular point. For any $x \in E_\circ \setminus \{s\}$ we have: $A_x = \mathsf{Mat}_3\bigl(O_x\bigr)$, where O_x is the local ring of $E_\circ$ at the point x. On the other hand,

$$\widehat{A}_s = A\bigl(R, (\Omega, \sigma, \approx, \mathsf{wt}_\circ)\bigr)$$

is a nodal order, where $R = \mathbb{k}[\![t]\!]$,

$$\Omega := \bigl\{1,2,3,4,5\bigr\}, \quad \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 5 & 4 \end{pmatrix}$$

and $1 \approx 2$, $3 \approx 5$ and $4 \approx 4$.

Let $E = \overline{V(v^2-u^3-u^2)} \subset \mathbb{P}^2$ be the projective closure of $E_\circ$. Then the Z-order A can be extended to a sheaf of orders $\mathcal{A}$ on the projective curve E in such a way that the stalk of $\mathcal{A}$ at the "infinite point" $(0:1:0)$ of E is the maximal order (see for instance [7]). The ringed space $\mathbb{E} = (E, \mathcal{A})$ is a typical example of a projective non-commutative nodal curve of tame representation type.

Let $H' = \begin{pmatrix} S & I & J \\ S & S & J \\ S & S & S \end{pmatrix}$, where $J \subset I := (x-1) \subset S$. Then H' is a hereditary order, too. Moreover, H' is the hereditary cover of the order A (the notion of the hereditary cover can be defined locally). As above, we can extend H' to a sheaf of orders $\mathcal{H}$ on the projective curve E in such a way that the stalk of $\mathcal{H}$ at the infinite point is the maximal order. Thus we get a non-commutative curve $\widetilde{\mathbb{E}} = (E, \mathcal{H})$. Note that the center of H is S. Therefore, one can actually construct a sheaf of hededitary orders $\widetilde{\mathcal{H}}$ on $\mathbb{P}^1$ such that $\mathcal{H} = \nu_*(\widetilde{\mathcal{H}})$, where $\mathbb{P}^1 \xrightarrow{\nu} E$ is the normalization map. The functor ν_* provides an equivalence between the categories of coherent sheaves on $\widetilde{\mathbb{E}}$ and $(\mathbb{P}^1, \widetilde{\mathcal{H}})$. In what follows, we shall consider $\widetilde{\mathbb{E}}$ as the hereditary cover of the non-commutative nodal curve $\mathbb{E}$ what can be viewed as an appropriate non-commutative generalization of the normalization of a singular commutative nodal curve.

Definition 4.1. Let X be a reduced quasi-projective curve over a field $\mathbb{k}$ and $\mathcal{A}$ be a sheaf of orders on X. Then the ringed space $\mathbb{X} = (X, \mathcal{A})$ is called a *non-commutative curve*. We say that $\mathbb{X}$ is projective if the commutative curve X is projective. If for any point $x \in X$ the corresponding stalk $\mathcal{A}_x$ is a nodal order then $\mathbb{X}$ is a *non-commutative nodal curve*.

Theorem 4.2. *Let $\Bbbk$ be an algebraically closed field, X be a quasi-projective curve over $\Bbbk$, $\mathcal{A}$ and $\mathcal{B}$ be two sheaves of orders on X such that*

$$Z(\widehat{\mathcal{A}}_x) = \widehat{O}_x = Z(\widehat{\mathcal{B}}_x)$$

for any $x \in X$, where $\widehat{O}_x$ is the completion of the local ring of X at the point x and $Z(\widehat{\mathcal{A}}_x)$ (respectively, $Z(\widehat{\mathcal{B}}_x)$) is the center of $\widehat{\mathcal{A}}_x$ (respectively, of $\widehat{\mathcal{B}}_x$). Assume that for any $x \in X$ there exists a Morita equivalence $\widehat{\mathcal{A}}_x\text{-}\mathsf{mod} \xrightarrow{\Phi_x} \widehat{\mathcal{B}}_x\text{-}\mathsf{mod}$ such that the following diagram

$$\begin{tikzcd}
\widehat{O}_x \arrow[r, "\mathsf{id}"] \arrow[d, "\cong"'] & \widehat{O}_x \arrow[d, "\cong"] \\
Z\big(\widehat{\mathcal{A}}_x\text{-}\mathsf{mod}\big) \arrow[r, "\bar{\Phi}_x"] & Z\big(\widehat{\mathcal{B}}_x\text{-}\mathsf{mod}\big)
\end{tikzcd}$$

is commutative, where $\bar{\Phi}_x$ denotes the induced map of centers. Then the non-commutative curves $\mathbb{X} = (X, \mathcal{A})$ and $\mathbb{Y} = (X, \mathcal{B})$ are Morita equivalent (i.e. the categories of quasi-coherent sheaves $\mathsf{QCoh}(\mathbb{X})$ and $\mathsf{QCoh}(\mathbb{Y})$ are equivalent).

For a proof of this result, see [15, Theorem 7.5].

4.2. **Construction of non-commutative nodal curves.** Let $\Bbbk$ be an algebraically closed field and $(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$ be a smooth quasi-projective curve over $\Bbbk$.

- Let $\widetilde{X} \xrightarrow{l} \mathbb{N}$ be a function such that $l(\tilde{x}) = 1$ for all but finitely many points $\tilde{x} \in \widetilde{X}$ (such function will be called a *length function*).
- For any $\tilde{x} \in \widetilde{X}$ we put:

$$\Pi_{\tilde{x}} := \big\{(\tilde{x}, 1), \dots, (\tilde{x}, l(\tilde{x}))\big\} \quad \text{and} \quad \Pi := \bigcup_{\tilde{x} \in \widetilde{X}} \Pi_{\tilde{x}}. \tag{48}$$

- For any function $\Pi \xrightarrow{\mathsf{wt}} \mathbb{N}$ and any point $\tilde{x} \in \widetilde{X}$ we denote:

$$\vec{p}(\tilde{x}) := \big(\mathsf{wt}(\tilde{x}, 1), \dots, \mathsf{wt}(\tilde{x}, l(\tilde{x}))\big) \quad \text{and} \quad m(\tilde{x}) := \big|\vec{p}(\tilde{x})\big| := \sum_{i=1}^{l(\tilde{x})} \mathsf{wt}(\tilde{x}, i).$$

- We say that wt is a *weight function* compatible with the given length function l if $m(\tilde{x}') = m(\tilde{x}'')$ for any pair of points $\tilde{x}', \tilde{x}'' \in \widetilde{X}$ belonging to the same irreducible component of $\widetilde{X}$.
- For $\tilde{x} \in \widetilde{X}$, let $O_{\tilde{x}}$ be the stalk of the structure sheaf $\mathcal{O}_{\widetilde{X}}$ at the point $\tilde{x}$. Let

$$\widetilde{H}_{\tilde{x}} := H\big(O_{\tilde{x}}, \vec{p}(\tilde{x})\big) \subseteq \mathsf{Mat}_{m(\tilde{x})}(O_{\tilde{x}})$$

  be the standard hereditary order defined by (13).

Definition 4.3. Assume (for simplicity of notation) that $\widetilde{X}$ is connected, $\widetilde{X} \xrightarrow{l} \mathbb{N}$ be a length function and wt be a weight function. Let $m = m(\tilde{x})$ for some (hence for any) point $\tilde{x} \in \widetilde{X}$. Then we define the sheaf of hereditary orders $\widetilde{\mathcal{H}} = \widetilde{\mathcal{H}}(l, \mathsf{wt})$ on the curve $\widetilde{X}$ as the subsheaf of the sheaf of maximal orders $Mat_m(\mathcal{O}_{\widetilde{X}})$ such that $\widetilde{\mathcal{H}}_{\tilde{x}} = \widetilde{H}_{\tilde{x}}$ for all $\tilde{x} \in \widetilde{X}$. The corresponding ringed space $\widetilde{\mathbb{X}} := (\widetilde{X}, \widetilde{\mathcal{H}})$ is a *non-commutative hereditary curve* over the field $\Bbbk$ defined by the datum $(\widetilde{X}, l, \mathsf{wt})$.

Theorem 4.4. *Let $\widetilde{X}$ be a smooth quasi-projective curve over $\Bbbk$ and $\widetilde{X} \xrightarrow{l} \mathbb{N}$be a length function. Let $\mathsf{wt}, \mathsf{wt}' : \Pi \longrightarrow \mathbb{N}$ be two weight functions compatible with l and $\widetilde{\mathbb{X}}$ and $\widetilde{\mathbb{X}}'$ be the corresponding non-commutative hereditary curves. Then the categories $\mathsf{QCoh}(\widetilde{\mathbb{X}})$ and $\mathsf{QCoh}(\widetilde{\mathbb{X}}')$ (respectively, $\mathsf{Coh}(\widetilde{\mathbb{X}})$ and $\mathsf{Coh}(\widetilde{\mathbb{X}}')$) are equivalent. Let $\widetilde{Y}$ be a smooth quasi-projective curve over $\Bbbk$, $\widetilde{Y} \xrightarrow{t} \mathbb{N}$ be a length function and $\widetilde{\mathbb{Y}}$ be the corresponding non-commutative hereditary curve. Then $\widetilde{\mathbb{X}}$ and $\widetilde{\mathbb{Y}}$ are Morita equivalent if and only if there exists an isomorphism (of commutative curves) $\widetilde{X} \xrightarrow{f} \widetilde{Y}$ such that the following diagram*

$$\begin{tikzcd}
\widetilde{X} \arrow[rr, "f"] \arrow[dr, "l"'] & & \widetilde{Y} \arrow[dl, "t"] \\
& \mathbb{N} &
\end{tikzcd}$$

is commutative. In other words, the Morita type of a non-commutative hereditary curve does not depend on the choice of a weight function wt *and is determined by the underlying commutative curve $\widetilde{X}$ and length function l.*

Comment to the proof. This result is a special case of Theorem 4.2.

Remark 4.5. Let $\widetilde{X} = \mathbb{P}^1$, $\widetilde{X} \xrightarrow{l} \mathbb{N}$ be a length function, $\Pi \xrightarrow{\mathsf{wt}} \mathbb{N}$ be a weight functions compatible with l and $\mathbb{X}$ be the corresponding hereditary curve. Then $\mathbb{X}$ can be identified with an appropriate *weighted projective line* of Geigle and Lenzing [23] in the sense that the categories of (quasi-)coherent sheaves on both objects are equivalent (see, for instance, the paper [41]).

Let us choose homogeneous coordinates on $\mathbb{P}^1$ and put: $\tilde{o}^+ := (0:1)$, $\tilde{o}^- := (1:0)$ and $\tilde{o} := (1:1)$. In what follows, we shall use the following notation.

- $\mathbb{P}^1(n_+, n_-)$ is the weighted projective line corresponding to the length function given by the rule:
$$l(\tilde{x}) = \begin{cases} n_\pm & \text{if } \tilde{x} = \tilde{o}^\pm \\ 1 & \text{otherwise.} \end{cases}$$
- $\mathbb{P}^1(n_+, n_-, n)$ is the weighted projective line corresponding to the length function given by the rule:
$$l(\tilde{x}) = \begin{cases} n_\pm & \text{if } \tilde{x} = \tilde{o}^\pm \\ n & \text{if } \tilde{x} = \tilde{o} \\ 1 & \text{otherwise,} \end{cases}$$
where we additionally assume that $n_\pm \geq 2$.

Definition 4.6. Let $\widetilde{X}$ be a smooth quasi-projective curve over $\Bbbk$ and $\widetilde{X} \xrightarrow{l} \mathbb{N}$ be a length function. Let $\approx$ be a relation on the set Π defined by (48) such that

- For any $\omega \in \Pi$ there exists at most one $\omega' \in \Pi$ such that $\omega \approx \omega'$ (such elements ω, ω' will be called *special*).
- There are only finitely many special elements in Π.

Non-special elements of Π will be called *simple*. The set of special elements of Π will be denoted by $\Pi_\circ$. An element $\omega \in \Pi_\circ$ is called *reflexive* if $\omega \approx \omega$ and *tied* if $\omega \approx \omega'$ for some $\omega \neq \omega'$.

Similarly to Definition 3.5 we define the set $\Pi^\ddagger$ by replacing each reflexive element $\omega \in \Pi$ by two new simple elements ω_+ and ω_-. The pairs of tied elements of $\Pi^\ddagger$ are the same as for Π.

Let $\Pi^\ddagger \xrightarrow{\mathsf{wt}^\ddagger} \mathbb{N}$ be a function such that $\mathsf{wt}^\ddagger(\omega') = \mathsf{wt}^\ddagger(\omega'')$ for all $\omega' \approx \omega''$ in $\Pi^\ddagger$. Then we define the map $\Pi \xrightarrow{\mathsf{wt}} \mathbb{N}$ by the following rule:

$$\mathsf{wt}(\omega) := \begin{cases} \mathsf{wt}^\ddagger(\omega_+) + \mathsf{wt}^\ddagger(\omega_-) & \text{if } \omega \in \Pi \text{ is reflexive} \\ \mathsf{wt}^\ddagger(\omega) & \text{otherwise.} \end{cases}$$

We call such a relation $\approx$ on the set Π *admissible* if there exists a function $\Pi^\ddagger \xrightarrow{\mathsf{wt}^\ddagger} \mathbb{N}$ for which the corresponding function $\Pi \xrightarrow{\mathsf{wt}} \mathbb{N}$ is a weight function compatible with the length function l. Abusing the notation, we shall drop the symbol $\ddagger$ in the notation of $\mathsf{wt}^\ddagger$ and write wt for all weight functions introduced above.

We say that two points $\tilde{x}' \neq \tilde{x}'' \in \widetilde{X}$ are *tied* if there are $\omega' \in \Omega_{\tilde{x}'}$ and $\omega'' \in \Omega_{\tilde{x}''}$ such that $\omega' \approx \omega''$. Let

$$\widetilde{Z} := \Bigl\{ \tilde{x} \in \widetilde{X} \,\Big|\, \text{there exists } \tilde{y} \in \widetilde{X} \setminus \{\tilde{x}\} \text{ such that } \tilde{x} \text{ and } \tilde{y} \text{ are tied} \Bigr\} \tag{49}$$

be the set of tied points of $\widetilde{X}$. Taking the transitive closure, we get an equivalence relation $\sim$ on $\widetilde{Z}$. We put: $Z := \widetilde{Z}/\sim$. In what follows, we shall also consider $\widetilde{Z}$ as a reduced subscheme of $\widetilde{X}$, Z as a reduced scheme over $\Bbbk$ and the projection map $\widetilde{Z} \xrightarrow{\tilde{\nu}} Z$ as a morphism of schemes.

Given an admissible datum $(\widetilde{X}, l, \approx)$, we define a quasi-projective curve X requiring the following diagram of algebraic schemes

$$\begin{tikzcd} \widetilde{Z} \arrow[r, hook, "\tilde{\eta}"] \arrow[d, "\tilde{\nu}"'] & \widetilde{X} \arrow[d, "\nu"] \\ Z \arrow[r, hook, "\eta"] & X. \end{tikzcd} \tag{50}$$

to be cartesian. In other words, the curve X is obtained from $\widetilde{X}$ by gluing transversally the equivalent points. It is clear that X is singular provided $\widetilde{Z}$ is non-empty and that $\widetilde{X} \xrightarrow{\nu} X$ is the normalization map. It always exists, as follows from [47].

We put: $\mathcal{H} := \nu_*\bigl(\widetilde{\mathcal{H}}\bigr)$. For any $x \in X$ (respectively, $\tilde{x} \in \widetilde{X}$) let $\widehat{H}_x$ (respectively, $\widehat{H}_{\tilde{x}}$) be the radical completion of $\mathcal{H}_x$ (respectively, $\widetilde{\mathcal{H}}_{\tilde{x}}$). Note that in the notation of (13) we have:

$$\widehat{H}_{\tilde{x}} := H\bigl(\widehat{O}_{\tilde{x}}, \vec{p}(\tilde{x})\bigr),$$

where $\widehat{O}_{\tilde{x}}$ is the completion of the local ring $O_{\tilde{x}}$. It is clear that $\widehat{H}_{\tilde{x}}$ is also an order over the local ring $\widehat{O}_x$, which is the completion of the local ring of the structure sheaf of X at the point x.

Assume now that $x \in Z$ and $\nu^{-1}(x) = \{\tilde{x}_1, \dots, \tilde{x}_r\}$. Then we have a canonical isomorphism

$$\widehat{H}_x \xrightarrow{\cong} \widehat{H}_{\tilde{x}_1} \times \cdots \times \widehat{H}_{\tilde{x}_r}.$$

Next, we put: $\Omega_x := \Omega_{\tilde{x}_1} \sqcup \cdots \sqcup \Omega_{\tilde{x}_r}$ Then we have a permutation $\Omega_x \xrightarrow{\sigma_x} \Omega_x$ given by the rule $\sigma_x(\tilde{x}, i) := \bigl(\tilde{x}, i+1 \ \mathsf{mod}\ l(\tilde{x})\bigr)$ for any $\tilde{x} \in \{\tilde{x}_1, \dots, \tilde{x}_r\}$. In the terms of Definition 3.10 we put:

$$\widehat{A}_x := A(\Omega_x, \sigma_x, \approx, \mathsf{wt}) \subset \widehat{H}_x. \tag{51}$$

Then $\widehat{A}_x$ is a nodal order and $\widehat{H}_x$ is its hereditary cover. Moreover, the center of $\widehat{A}_x$ contains the local ring $\widehat{O}_x$.

Definition 4.7. We define the sheaf of orders $\mathcal{A}$ on the curve X to be the subsheaf of $\mathcal{H}$ satisfying the following conditions on the stalks:

$$\widehat{\mathcal{A}}_x := \begin{cases} \widehat{\mathcal{H}}_x & \text{if} \quad x \notin Z \\ \widehat{A}_x & \text{if} \quad x \in Z. \end{cases} \tag{52}$$

We call the ringed space $\mathbb{X} = (X, \mathcal{A})$ the *non-commutative nodal curve* attached to the datum $(\widetilde{X}, l, \approx, \mathsf{wt})$. The ringed space $\widetilde{\mathbb{X}} = (X, \mathcal{H})$ will be called the *hereditary cover* of $\mathbb{X}$.

Note that for $\widetilde{\mathbb{X}}' := (\widetilde{X}, \widetilde{\mathcal{H}})$ we have a natural morphism of ringed spaces $\widetilde{\mathbb{X}}' \xrightarrow{\nu} \mathbb{X}$, which induces an equivalence of categories $\mathsf{Coh}(\widetilde{\mathbb{X}}') \longrightarrow \mathsf{Coh}(\widetilde{\mathbb{X}})$.

Theorem 4.8. *Let $(\widetilde{X}, l, \approx)$ be an admissible datum, $\Pi^{\ddagger} \xrightarrow{\mathsf{wt}} \mathbb{N}$ be any compatible weight function and $\mathbb{X}$ be the corresponding non-commutative nodal curve. Then the following results hold.*

- *Let $\Pi^{\ddagger} \xrightarrow{\mathsf{wt}'} \mathbb{N}$ be any other compatible weight function and $\mathbb{X}'$ be the corresponding non-commutative nodal curve. Then the categories of quasi-coherent sheaves $\mathsf{QCoh}(\mathbb{X})$ and $\mathsf{QCoh}(\mathbb{X}')$ are equivalent.*
  That is why we often do not mention the weight wt and say that $\mathbb{X}$ is attached to the admissible datum $(\widetilde{X}, l, \approx)$.
- *Let $\approx'$ be another equivalence relation on Π and $\Pi \xrightarrow{\mathsf{wt}'} \mathbb{N}$ be a weight function compatible with $\approx'$. Suppose that for any $\tilde{x} \in \widetilde{X}$ there exists a cyclic permutation $\Pi_{\tilde{x}} \xrightarrow{f_x} \Pi_{\tilde{x}}$ such that the diagram*

$$\begin{tikzcd} \Pi_{\tilde{x}} \arrow[rr, "f_x"] \arrow[dr, "\mathsf{wt}"'] & & \Pi_{\tilde{x}} \arrow[dl, "\mathsf{wt}'"] \\ & \mathbb{N} & \end{tikzcd}$$

is commutative. Then the categories $\mathsf{QCoh}(\mathbb{X})$ *and* $\mathsf{QCoh}(\mathbb{X}')$ *are equivalent.*

Comment to the proof. This result is a consequence of Theorem 4.2 and Theorem 3.11.

Example 4.9. Let $(\widetilde{X}, l, \approx)$ be such that for any $\tilde{x} \in \widetilde{X}$ with $l(\tilde{x}) \geq 2$, the set $\Pi_{\tilde{x}}$ contains a non-tied element. Then the datum $(\widetilde{X}, l, \approx)$ is admissible.

Example 4.10. Let $\widetilde{X}$ be any curve and $\tilde{x}_1 \neq \tilde{x}_2 \in \widetilde{X}$ be two distinct points. Define a weight function $\mathbb{P}^1 \xrightarrow{l} \mathbb{N}$ by the rule:

$$l(\tilde{x}) = \begin{cases} 2 & \text{if } \tilde{x} = \tilde{x}_1 \\ 1 & \text{otherwise.} \end{cases}$$

Let $\approx$ be given by the rule: $(\tilde{x}_1, 1) \approx (\tilde{x}_2, 1)$. Then the datum $(\widetilde{X}, l, \approx)$ is not admissible.

4.3. **Non-commutative nodal curves of tame representation type.** In this subsection we recall, following the paper [21], the description of those non-commutative projective nodal curves $\mathbb{X} = (X, \mathcal{A})$ for which the category $\mathsf{VB}(\mathbb{X})$ of vector bundles (i.e. of locally projective coherent $\mathcal{A}$-modules) has tame representation type. Let $\widetilde{X} = \widetilde{X}_1 \sqcup \cdots \sqcup \widetilde{X}_r$ be a disjoint union of r projective lines. We choose homogeneous coordinates on each component $\widetilde{X}_k$ and define points $\tilde{o}_k, \tilde{o}_k^{\pm}, \in \widetilde{X}_k$ setting: $\tilde{o}_k := (1:1), \tilde{o}_k^{+} := (0:1)$ and $\tilde{o}_k^{-} := (1:0)$. Assume that $(\widetilde{X}, l, \approx)$ is an admissible datum defining a non-commutative nodal curve $\mathbb{X}$. For each $1 \leq k \leq r$, let $\Sigma_k \subset \widetilde{X}_k$ be the corresponding set of special points.

- If $\bigl|\Sigma_k\bigr| = 2$, we may without loss of generality assume that $\Sigma_k = \bigl\{\tilde{o}_k^{+}, \tilde{o}_k^{-}\bigr\}$.
- Similarly, if $\bigl|\Sigma_k\bigr| = 3$, we assume that $\Sigma_k = \bigl\{\tilde{o}_k, \tilde{o}_k^{+}, \tilde{o}_k^{-}\bigr\}$.

The following result is proved in [21].

Theorem 4.11. *Let* $\mathbb{X}$ *be a non-commutative projective curve. Then* $\mathsf{VB}(\mathbb{X})$ *has tame representation type if and only if the following conditions are satisfied.*

- $\mathbb{X}$ *is Morita equivalent to a commutative elliptic curve, i.e.* $\widetilde{X}$ *is an elliptic curve, while* l *and* $\approx$ *are trivial.*
- $\mathbb{X}$ *is the rational non-commutative nodal curve attached to an admissible datum* $(\widetilde{X}, l, \approx)$ *such that* $\widetilde{X} = \widetilde{X}_1 \sqcup \cdots \sqcup \widetilde{X}_r$ *is a disjoint union of* r *projective lines, whereas* $(l, \approx)$ *satisfies the following conditions:*
  - *For any* $1 \leq k \leq r$ *we have:* $\bigl|\Sigma_k\bigr| \leq 3$.
  - *If* $\bigl|\Sigma_k\bigr| = 3$ *then we additionally have: the set* $\Pi_{\tilde{x}}$ *contains special elements for precisely one point* $\tilde{x} \in \Sigma_k$ *(say, for* $\tilde{x} = \tilde{o}$*), whereas for the remaining two points of* $\Pi_{\tilde{x}}$ *(say, for* $\tilde{o}^{\pm}$*) we have:* $l(\tilde{o}^{\pm}) = 2$*).*

Definition 4.12. Consider the pair $\bigl(\vec{p}, \vec{q}\bigr)$, where

$$\vec{p} = \bigl((p_1^{+}, p_1^{-}), \dots, (p_t^{+}, p_t^{-})\bigr) \in \bigl(\mathbb{N}^2\bigr)^t \quad \text{and} \quad \vec{q} = (q_1, \dots, q_s) \in \mathbb{N}^s$$

for some $t, s \in \mathbb{N}_0$ (either of this tuples may be empty). Let

$$\widetilde{X} := \widetilde{X}_1 \sqcup \cdots \sqcup \widetilde{X}_t \sqcup \widetilde{X}_{t+1} \sqcup \cdots \sqcup \widetilde{X}_{t+s}$$

be a disjoint union of $t+s$ projective lines. We define the weight function $\widetilde{X} \xrightarrow{l} \mathbb{N}$ by the following rules

- For each $1 \le k \le t$ we put: $l(\tilde{o}_k^{\pm}) = p_k^{\pm}$.
- For each $1 \le k \le s$ we put: $l(\tilde{o}_{t+k}^{\pm}) = 2$ and $l(\tilde{o}_{t+k}) = q_k$.

Let $\approx$ be a relation on the set $\bigl(\Pi_{\tilde{o}_1^+} \sqcup \Pi_{\tilde{o}_1^-}\bigr) \sqcup \dots \sqcup \bigl(\Pi_{\tilde{o}_t^+} \sqcup \Pi_{\tilde{o}_t^-}\bigr) \sqcup \bigl(\Pi_{\tilde{o}_{t+1}} \sqcup \dots \sqcup \Pi_{\tilde{o}_{t+s}}\bigr)$. satisfying the conditions of Definition 4.6. If $\bigl(\vec{p}, \vec{q}, \approx\bigr)$ is admissible and wt is a compatible weight, we denote by $\mathbb{X}\bigl(\vec{p}, \vec{q}, \approx, \mathsf{wt}\bigr)$ the corresponding non-commutative nodal rational projective curve. Since the weight wt does not imply the derived category, we often omit it and write $\mathbb{X}\bigl(\vec{p}, \vec{q}, \approx\bigr)$.

One can rephrase Theorem 4.11 in the following way.

Theorem 4.13. *The category $\mathsf{VB}(\mathbb{X})$ of vector bundles on a non-commutative projective curve $\mathbb{X}$ is representation tame if and only if $\mathbb{X}$ is either a commutative elliptic curve or a non-commutative nodal curve $\mathbb{X}\bigl(\vec{p}, \vec{q}, \approx\bigr)$, where $(\vec{p}, \vec{q}, \approx)$ is an admissible datum as in Definition 4.12.*

Remark 4.14. Let $\vec{p} = \bigl((2,2),(2,2)\bigr)$, $\vec{q}$ be void and $\approx$ be given by the following rule: $(\tilde{o}_k^+, 1) \approx (\tilde{o}_k^-, 1)$ for $k = 1,2$ and $(\tilde{o}_1^{\pm}, 2) \approx (\tilde{o}_2^{\pm}, 2)$. Then the central curve X of the corresponding non-commutative nodal curve $\mathbb{X}(\vec{p}, \approx)$ is given by the following Cartesian diagram:

$$\begin{tikzcd}
\mathsf{Spec}(\Bbbk) \sqcup \mathsf{Spec}(\Bbbk) \arrow[r, "\imath"] \arrow[d] & E \sqcup E \arrow[d] \\
\mathsf{Spec}(\Bbbk) \arrow[r] & X
\end{tikzcd}$$

where $E \cong \overline{V(v^2 - u^3 - u^2)} \subset \mathbb{P}^2$ is a plane nodal cubic and the image of $\imath$ of the singular set of $E \sqcup E$. Note that the arithmetic genus of X is *two*. In fact, the central curve of a tame non-commutative nodal curve can have arbitrary high arithmetic genus.

4.4. **Remarks on stacky cycles of projective lines.** In this subsection, let $\Bbbk$ be an algebraically closed field of characteristic zero.

Example 4.15. Let E be a plane nodal cubic and $\mathbb{P}^1 \xrightarrow{\nu} E$ be its normalization. Let us choose homogeneous coordinates $(z : w)$ on $\mathbb{P}^1$ in such a way that $\nu^{-1}(s) = \bigl\{\tilde{o}^+, \tilde{o}^-\bigr\}$, where s is the singular point of E and $\tilde{o}^+ = (0:1)$ and $\tilde{o}^- = (1:0)$. Consider the action of the cyclic group $G := \bigl\langle \rho \bigm| \rho^n = e \bigr\rangle$ on $\mathbb{P}^1$ given by the formula $(z : w) \overset{\rho}{\mapsto} (\zeta z : w)$, where ζ is a primitive n-th root of unitity. It is clear that the action of G on $\mathbb{P}^1$ descends to an action of G on E such that $E' := E/G \cong E$. Let $\mathcal{A} := \mathcal{O}_{E'} * G$ be the sheaf on E' defined by the following rule:

$$U \mapsto \Gamma\bigl(\pi^{-1}(U), \mathcal{O}_E\bigr) * G \quad \text{for any open } \; U \subseteq E',$$

where $E \xrightarrow{\pi} E'$ is the projection map. Then $\mathcal{A}$ is a sheaf of nodal orders on the projective curve E'. For any $x \in E' \setminus \{s\}$, the order $\mathcal{A}_x$ is maximal, whereas $\widehat{\mathcal{A}}_s$ is the nodal order given by (32).

The following result is obvious.

Lemma 4.16. *The category* $\mathsf{Coh}^G(E)$ *of* G*-equivariant sheaves on* E *is equivalent to the category* $\mathsf{Coh}(\mathbb{E})$*, where* $\mathbb{E} = (E', \mathcal{A})$*.*

Remark 4.17. The non-commutative nodal curve $\mathbb{E}$ admits the following description. Consider the length function $\mathbb{P}^1 \xrightarrow{l} \mathbb{N}$ given by the rule:

$$l(\tilde{x}) = \left\{ \begin{array}{ll} n & \text{if } \tilde{x} \in \{\tilde{o}^+, \tilde{o}^-\} \\ 1 & \text{otherwise.} \end{array} \right.$$

We then define the relation $\approx$ on the set Π setting $(\tilde{o}^+, k) \approx (\tilde{o}^-, n-k)$ for $1 \le k \le n$, where we replace 0 by n. It is easy to see that the datum $(\mathbb{P}^1, l, \approx)$ is admissible. Using Theorem 4.2 one can conclude that $\mathbb{E}$ and the non-commutative nodal curve corresponding to the datum $(\mathbb{P}^1, l, \approx)$ are Morita equivalent.

Example 4.18. Again, let $E = \overline{V(v^2 - u^3 - u^2)} \subset \mathbb{P}^2$ be a plane nodal cubic. Consider the involution $E \xrightarrow{\tau} E$ given by the rule $(u, v) \mapsto (u, -v)$. Let $E \xrightarrow{\pi} E/G \cong \mathbb{P}^1$ be the projection map, where $G = \langle \tau \rangle \cong \mathbb{Z}_2$. Next, let $\mathcal{A} := \mathcal{O}_E * G$ and $\mathbb{E} = (\mathbb{P}^1, \mathcal{A})$ be the corresponding non-commutative nodal curve. Again, the categories $\mathsf{Coh}^G(E)$ and $\mathsf{Coh}(\mathbb{E})$ are equivalent. Let us choose homogeneous coordinates on $\mathbb{P}^1$ in such a way that $\pi^{-1}(\tilde{o}^+)$ is the singular point of E and $\pi^{-1}(\tilde{o}^-)$ is its infinite point. Consider the length function $\mathbb{P}^1 \xrightarrow{l} \mathbb{N}$ given by the rule:

$$l(\tilde{x}) = \left\{ \begin{array}{ll} 2 & \text{if } \tilde{x} = \tilde{o}^- \\ 1 & \text{otherwise.} \end{array} \right.$$

We define the relation $\approx$ on the set Π by setting $(\tilde{o}^+, 1) \approx (\tilde{o}^+, 1)$. Obviously, the datum $(\mathbb{P}^1, l, \approx)$ is admissible. According to Lemma 3.25 and Theorem 4.2, the datum $(\mathbb{P}^1, l, \approx)$ defines a non-commutative nodal curve, which is Morita equivalent to $\mathbb{E}$.

Example 4.19. In this example, we give a description of stacky cycles of projective lines used in the paper of Lekili and Polishchuk [35] in the language of non-commutative nodal curves. Let $r \in \mathbb{N}$, $\vec{n} = (n_1, \dots, n_r) \in \mathbb{N}^r$ and $\vec{c} = (c_1, \dots, c_r) \in \mathbb{N}^r$ be such that $\mathsf{gcd}(n_k, c_k) = 1$ for any $1 \le k \le r$. Let E_r be a cycle of r projective lines and $\widetilde{X} \xrightarrow{\pi} E_r$ be its normalization. Then $\widetilde{X} = \widetilde{X}_1 \sqcup \dots \sqcup \widetilde{X}_r$ is a disjoint union of r projective lines. Let $\{o_1, \dots, o_r\}$ be the set of singular points of E_r, where we choose their labeling in such a way that $\pi^{-1}(o_k) = \{\tilde{o}_k^-, \tilde{o}_{k+1}^+\}$, where $\tilde{o}_k^- = (1:0) \in \widetilde{X}_k$ and $\tilde{o}_{k+1}^+ = (0:1) \in \widetilde{X}_{k+1}$. The completion of the local ring of E_r at each point o_k is isomorphic to the commutative nodal ring $\Bbbk[\![u, v]\!]/(uv)$. For any $1 \le k \le n$, consider the action of the cyclic group $G_{n_k} = \langle \rho \,|\, \rho^{n_k} = e \rangle$ on $D = \Bbbk[\![u, v]\!]/(uv)$ given by the rule

$$\left\{ \begin{array}{lcl} \rho \circ u & = & \zeta_k u \\ \rho \circ v & = & \zeta_k^{c_k} v, \end{array} \right. \tag{53}$$

where ζ_k is some primitive n_k-th root of 1. The category of coherent sheaves $\mathsf{Coh}(\mathbb{E})$ on a stacky cycle of projective lines $\mathbb{E} := \mathbb{E}_r(\vec{n}, \vec{c})$ is an abelian category satisfying the following property: the category $\mathsf{Tor}(\mathbb{E})$ of finite length objects of $\mathsf{Coh}(\mathbb{E})$ splits into a direct sum of blocks:

$$\mathsf{Tor}(\mathbb{E}) \cong \biguplus_{x \in E_r} \mathsf{Tor}_x(\mathbb{E}),$$

where $\mathsf{Tor}_x(\mathbb{E})$ is equivalent to the category of finite length $\Bbbk[\![w]\!]$-modules if $x \in E_r$ smooth and to the category of finite length $D * G_{n_k}$-modules if $x = o_k$, where the action of G_{n_k} is given by the rule (53).

The stacky cycle of projective lines $\mathbb{E}_r(\vec{n}, \vec{c})$ can be understood as an appropriate cyclic gluing of weighted projective lines $\mathbb{P}^1(n_1, n_2), \dots, \mathbb{P}^1(n_{r-1}, n_r), \mathbb{P}^1(n_r, n_1)$. Now let us proceed with a formal definition of $\mathbb{E}_r(\vec{n}, \vec{c})$ viewed as a non-commutative nodal curve. As above, let $\widetilde{X}$ be a disjoint union of r projective lines. Consider the length function $\widetilde{X} \xrightarrow{l} \mathbb{N}$ given by the rule:

$$l(\tilde{x}) = \begin{cases} n_k & \text{if } \tilde{x} \in \left\{\tilde{o}_k^-, \tilde{o}_{k+1}^+\right\} \\ 1 & \text{otherwise.} \end{cases}$$

Let $\Pi_k^\pm = \Pi_{\tilde{o}_k^\pm}$. It is convenient to use the identification

$$\Pi_k^- = \{\bar{1}, \dots, \bar{n}_k\} = \Pi_{k+1}^+,$$

given by the rule: $(\tilde{o}_k^-, j) = \bar{j} = (\tilde{o}_{k+1}^+, j)$ for $1 \le j \le n_k$. We have a bijection

$$\Pi_k^- \xrightarrow{\tau_k} \Pi_{k+1}^+, \; \bar{j} \mapsto \overline{c_k \cdot j}.$$

Let $\approx$ be a relation on the set Π, given by the rule: $(\tilde{o}_k^-, j) \approx \bigl(\tilde{o}_{k+1}^+, \tau_k(j)\bigr)$. Note that the set Π does not contain reflexive elements, hence $\Pi^\ddagger = \Pi$ in this case.

Next, we claim that the datum $(\widetilde{X}, l, \approx)$ is admissible. Indeed, let $m := n_1 \dots n_r$. Then we can define a compatible weight function $\Pi \xrightarrow{\mathsf{wt}} \mathbb{N}$ by the following rules:

$$\mathsf{wt}(\tilde{x}, j) = \begin{cases} \dfrac{m}{n_{k-1}} & \text{if } \tilde{x} = \tilde{o}_k^+ \text{ and } \; \le j \le n_{k-1}, \\ \dfrac{m}{n_k} & \text{if } \tilde{x} = \tilde{o}_k^- \text{ and } 1 \le j \le n_k, \\ m & \text{otherwise.} \end{cases}$$

Then the stacky cycle of projective lines $\mathbb{E}_r(\vec{n}, \vec{c})$ can be defined as the non-commutative nodal curve corresponding to the datum $(\widetilde{X}, l, \approx, \mathsf{wt})$. Recall that, up to Morita equivalence, the non-commutative nodal curve $\mathbb{E}_r(\vec{n}, \vec{c})$ does not depend on the choice of the weight function wt; see Theorem 4.2.

4.5. **Auslander curve of a non-commutative nodal curve.**

Definition 4.20. Let $\mathbb{X} = (X, \mathcal{A})$ be a non-commutative nodal curve and $\mathcal{H}$ be the hereditary cover of $\mathcal{A}$. The *conductor ideal sheaf* $\mathcal{C}$ is defined as follows: for any open subset $U \subseteq X$ we put:

$$\Gamma(U, \mathcal{C}) := \Big\{ f \in \Gamma(U, \mathcal{A}) \,\Big|\, fg \in \Gamma(U, \mathcal{A}) \text{ for all } g \in \Gamma(U, \mathcal{H}) \Big\}. \tag{54}$$

Lemma 4.21. *The following results are true.*

- *The canonical morphism of $\mathcal{O}_X$-modules $\mathcal{C} \to \mathit{Hom}_{\mathcal{A}}(\mathcal{H}, \mathcal{A})$ given on the level of local sections by the rule* (36) *is an isomorphism.*
- *$\mathcal{C}$ is a sheaf of two-sided ideals both in $\mathcal{A}$ and $\mathcal{H}$.*
- *The $\Bbbk$-algebras $\bar{A} := \Gamma(X, \mathcal{A}/\mathcal{C})$ and $\bar{H} := \Gamma(X, \mathcal{H}/\mathcal{C})$ are finite dimensional and semisimple.*

Proof. All statements follow directly from the corresponding local statements; see Proposition 3.29 and Corollary 3.30. □

Definition 4.22. For a nodal curve $\mathbb{X} = (X, \mathcal{A})$ as above, we call the sheaf of orders

$$\mathcal{B} := \begin{pmatrix} \mathcal{A} & \mathcal{H} \\ \mathcal{C} & \mathcal{H} \end{pmatrix} \tag{55}$$

the *Auslander sheaf* of $\mathcal{A}$. The corresponding non-commutative nodal curve $\mathbb{Y} := (X, \mathcal{B})$ is called the *Auslander curve* of $\mathbb{X}$.

Consider the following idempotent sections $e := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $f := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in \Gamma(X, \mathcal{B})$. Then we get the following sheaves of locally projective left $\mathcal{B}$-modules:

$$\mathcal{P} := \mathcal{B}e \cong \begin{pmatrix} \mathcal{A} \\ \mathcal{C} \end{pmatrix} \text{ and } \mathcal{Q} := \mathcal{B}f \cong \begin{pmatrix} \mathcal{H} \\ \mathcal{H} \end{pmatrix}. \tag{56}$$

We denote $\mathcal{P}^\vee := \mathit{Hom}_{\mathcal{B}}(\mathcal{P}, \mathcal{B}) \cong e\mathcal{B}$ and $\mathcal{Q}^\vee := \mathit{Hom}_{\mathcal{B}}(\mathcal{Q}, \mathcal{B}) \cong f\mathcal{B}$.

Similarly to the local case, we introduce the following functors.

- An exact functor $\mathsf{G} := \mathsf{Hom}_{\mathcal{B}}(\mathcal{P}, -) \simeq \mathcal{P}^\vee \otimes_{\mathcal{B}} -$ from $\mathsf{QCoh}(\mathbb{Y})$ to $\mathsf{QCoh}(\mathbb{X})$.
- Similarly, we have an exact functor $\widetilde{\mathsf{G}} = \mathsf{Hom}_{\mathcal{B}}(\mathcal{Q}, -) \simeq \mathcal{Q}^\vee \otimes_{\mathcal{B}} -$ from $\mathsf{QCoh}(\mathbb{Y})$ to $\mathsf{QCoh}(\widetilde{\mathbb{X}})$.
- We have functors $\mathsf{F} := \mathcal{P} \otimes_{\mathcal{A}} -$ and $\mathsf{H} := \mathit{Hom}_{\mathcal{A}}(\mathcal{P}^\vee, -)$ from $\mathsf{QCoh}(\mathbb{X})$ to $\mathsf{QCoh}(\mathbb{Y})$.
- Similarly, we have functors $\widetilde{\mathsf{F}} := \mathcal{Q} \otimes_{\mathcal{H}} -$ and $\widetilde{\mathsf{H}} := \mathit{Hom}_{\mathcal{H}}(\mathcal{Q}^\vee, -)$ from $\mathsf{QCoh}(\widetilde{\mathbb{X}})$ to $\mathsf{QCoh}(\mathbb{Y})$.
- Let $\mathsf{DG}, \mathsf{D}\widetilde{\mathsf{G}}$, LF, $\mathsf{L}\widetilde{\mathsf{F}}$, RH and $\mathsf{R}\widetilde{\mathsf{H}}$ be the corresponding derived functors.

Remark 4.23. It is clear that all functors $\mathsf{G}, \widetilde{\mathsf{G}}$, F, $\widetilde{\mathsf{F}}$, H and $\widetilde{\mathsf{H}}$ can be restricted to the corresponding subcategories of coherent sheaves. The functor $\widetilde{\mathsf{F}}$ is exact and transforms locally projective $\mathcal{H}$-modules to locally projective $\mathcal{B}$-modules.

All statements and proofs of Theorem 2.4 and Corollary 3.38 can be generalized to the global setting in a straightforward way. In particular, we have the following results.

Theorem 4.24. *Let $\mathbb{X} = (X, \mathcal{A})$ be a non-commutative nodal curve as in Definition 4.7, $\widetilde{\mathbb{X}} = (X, \mathcal{H})$ be its hereditary cover and $\mathbb{Y} = (X, \mathcal{B})$ be the corresponding Auslander curve. Then the following results are true.*

- *We have:* $\mathsf{gl.dim}\big(\mathsf{Coh}(\mathbb{Y})\big) = 2$.
- *We have a recollement diagram*

$$D^b\big(\bar{A}\text{-}\mathsf{mod}\big) \begin{array}{c} \xleftarrow{\;\mathsf{I}^*\;} \\ \xrightarrow{\;\mathsf{I}\;} \\ \xleftarrow[\;\mathsf{I}^!\;]{} \end{array} D^b\big(\mathsf{Coh}(\mathbb{Y})\big) \begin{array}{c} \xleftarrow{\;\mathsf{L}\widetilde{\mathsf{F}}\;} \\ \xrightarrow{\;\mathsf{D}\widetilde{\mathsf{G}}\;} \\ \xleftarrow[\;\mathsf{R}\widetilde{\mathsf{H}}\;]{} \end{array} D^b\big(\mathsf{Coh}(\widetilde{\mathbb{X}})\big). \tag{57}$$

Here, the exact functor I is determined by the rule $\mathsf{I}(\bar{A}) = \mathcal{S}$, where $\mathcal{S}$ is given by the locally projective resolution

$$0 \longrightarrow \begin{pmatrix} \mathcal{C} \\ \mathcal{C} \end{pmatrix} \longrightarrow \begin{pmatrix} \mathcal{A} \\ \mathcal{C} \end{pmatrix} \longrightarrow \mathcal{S} \longrightarrow 0. \tag{58}$$

In particular, we have a semi-orthogonal decomposition

$$D^b\big(\mathsf{Coh}(\mathbb{Y})\big) = \Big\langle \mathsf{Im}(\mathsf{I}),\, \mathsf{Im}(\mathsf{L}\widetilde{\mathsf{F}}) \Big\rangle = \Big\langle D^b\big(\bar{A}\text{-}\mathsf{mod}\big),\, D^b\big(\mathsf{Coh}(\widetilde{\mathbb{X}})\big) \Big\rangle. \tag{59}$$

- *Moreover, we have the following commutative diagram of categories and functors:*

$$\begin{tikzcd} D^b\big(\mathsf{Coh}(\widetilde{\mathbb{X}})\big) \arrow[r, hook, "\mathsf{L}\widetilde{\mathsf{F}}"] \arrow[dr, "\nu_*"'] & D^b\big(\mathsf{Coh}(\mathbb{Y})\big) \arrow[d, "\mathsf{DG}"] & \mathsf{Perf}(\mathbb{X}) \arrow[l, hook', "\mathsf{LF}"'] \arrow[dl, hook', "\mathsf{E}"] \\ & D^b\big(\mathsf{Coh}(\mathbb{X})\big) & \end{tikzcd} \tag{60}$$

where $\mathsf{Perf}(\mathbb{X})$ is the perfect derived category of coherent sheaves on $\mathbb{X}$, E is the canonical inclusion functor, LF and $\mathsf{L}\widetilde{\mathsf{F}}$ are fully faithful, DG is an appropriate localization functor and ν_* is the functor induced by the "normalization map" $\widetilde{\mathbb{X}} \xrightarrow{\nu} \mathbb{X}$.

5. Tilting on rational non-commutative nodal projective curves

5.1. **Tilting on projective hereditary curves.** We begin with a brief description of the standard tilting bundle on a weighted projective line due to Geigle and Lenzing [23] reexpressed in the language of non-commutative hereditary curves. Let $\widetilde{X} = \mathbb{P}^1$ and $\widetilde{X} \xrightarrow{l} \mathbb{N}$ be any length function. Let us fix any weight function $\Pi \xrightarrow{\mathsf{wt}} \mathbb{N}$ compatible with l. As usual, we put

$$\vec{p}(\tilde{x}) := \big(\mathsf{wt}(\tilde{x}, 1), \dots, \mathsf{wt}(\tilde{x}, l(\tilde{x}))\big) \quad \text{for any} \quad \tilde{x} \in \widetilde{X}.$$

Let $m := \big|\vec{p}(\tilde{x})\big|$ for some (hence for any) point $\tilde{x} \in \widetilde{X}$ and $\widetilde{H}_{\tilde{x}} := H\big(O_{\tilde{x}}, \vec{p}(\tilde{x})\big) \subseteq \mathsf{Mat}_m(O_{\tilde{x}})$ be the standard hereditary order, defined by the vector $\vec{p}(\tilde{x})$. Next, let $\widetilde{Q}_{(\tilde{x},1)}, \dots, \widetilde{Q}_{(\tilde{x},l(\tilde{x}))}$

be the standard indecomposable projective left $\widetilde{H}_{\tilde{x}}$-modules, i.e. we have a direct sum decomposition

$$\widetilde{H}_{\tilde{x}} \cong \widetilde{Q}_{(\tilde{x},1)}^{\oplus \,\mathsf{wt}(\tilde{x},1)} \oplus \cdots \oplus \widetilde{Q}_{(\tilde{x},l(\tilde{x}))}^{\oplus \,\mathsf{wt}(\tilde{x},l(\tilde{x}))}.$$

Here, we have:

$$\widetilde{Q}_{(\tilde{x},1)} := \begin{pmatrix} O_{\tilde{x}} \\ \vdots \\ O_{\tilde{x}} \end{pmatrix} \cong \begin{pmatrix} \mathfrak{m}_{\tilde{x}} \\ \vdots \\ \mathfrak{m}_{\tilde{x}} \end{pmatrix} =: \widetilde{Q}'_{(\tilde{x},1)}.$$

Note that there is a chain of embeddings of $H_{\tilde{x}}$-modules $\widetilde{Q}'_{(\tilde{x},1)} \subset \widetilde{Q}_{(\tilde{x},l(\tilde{x}))} \subset \cdots \subset \widetilde{Q}_{(\tilde{x},1)}$.

Let $\widetilde{\mathcal{H}} := \widetilde{\mathcal{H}}(l,\mathsf{wt})$ be the sheaf of hereditary orders on $\widetilde{X}$ defined by (l,wt) and $\widetilde{\mathbb{X}} = (\widetilde{X},\widetilde{\mathcal{H}})$ the corresponding non-commutative hereditary curve. Recall that $\widetilde{\mathcal{H}}$ is a subsheaf of the sheaf of maximal orders $Mat_m(\mathcal{O}_{\widetilde{X}})$ such that $\widetilde{\mathcal{H}}_{\tilde{x}} = \widetilde{H}_{\tilde{x}}$ for any $\tilde{x} \in \widetilde{X}$. First of all, note that we have an exact fully faithful functor

$$\mathsf{Coh}(\widetilde{X}) \xrightarrow{\bar{\mathbb{F}}} \mathsf{Coh}(\widetilde{\mathbb{X}}), \quad \mathcal{E} \mapsto \begin{pmatrix} \mathcal{E} \\ \vdots \\ \mathcal{E} \end{pmatrix},$$

which transforms locally free sheaves on $\widetilde{X}$ into locally projective $\widetilde{\mathcal{H}}$-modules. We put:

$$\widetilde{\mathcal{L}} := \bar{\mathbb{F}}(\mathcal{O}_{\widetilde{X}}) = \begin{pmatrix} \mathcal{O}_{\widetilde{X}} \\ \vdots \\ \mathcal{O}_{\widetilde{X}} \end{pmatrix}.$$

Next, for any $\tilde{x} \in \widetilde{X}$ such that $l(\tilde{x}) \geq 2$ and $2 \leq i \leq l(\tilde{x})$, we have a locally projective $\mathcal{H}$-module $\widetilde{\mathcal{L}}_{(\tilde{x},i)}$ uniquely determined by the following properties:

- The sheaf $\widetilde{\mathcal{L}}_{(\tilde{x},i)}$ is a subsheaf of $\widetilde{\mathcal{L}}$.
- For any $\tilde{y} \in \widetilde{X}$ we have:

$$\left(\widetilde{\mathcal{L}}_{(\tilde{x},i)}\right)_{\tilde{y}} = \begin{cases} \widetilde{\mathcal{L}}_{\tilde{y}} & \text{if} \quad \tilde{y} \neq \tilde{x} \\ \widetilde{Q}_{(\tilde{x},i)} & \text{if} \quad \tilde{y} = \tilde{x}. \end{cases}$$

For a convenience of notation, we also define for any point $\tilde{x} \in \widetilde{X}$ the locally projective subsheaf $\widetilde{\mathcal{L}}_{(\tilde{x},1)}$ of $\widetilde{\mathcal{L}}$ defined by the following condition on the stalkes:

$$\left(\widetilde{\mathcal{L}}_{(\tilde{x},1)}\right)_{\tilde{y}} = \begin{cases} \widetilde{\mathcal{L}}_{\tilde{y}} & \text{if} \quad \tilde{y} \neq \tilde{x} \\ \widetilde{Q}'_{(\tilde{x},1)} & \text{if} \quad \tilde{y} = \tilde{x}. \end{cases}$$

It is clear that

$$\widetilde{\mathcal{L}}_{(\tilde{x},1)} := \bar{\mathbb{F}}\big(\mathcal{O}_{\widetilde{X}}(-\tilde{x})\big) = \begin{pmatrix} \mathcal{O}_{\widetilde{X}}(-\tilde{x}) \\ \vdots \\ \mathcal{O}_{\widetilde{X}}(-\tilde{x}) \end{pmatrix}.$$

Note that $\widetilde{\mathcal{L}}_{(\tilde{x}',1)} \cong \widetilde{\mathcal{L}}_{(\tilde{x}'',1)}$ for any $\tilde{x}', \tilde{x}'' \in \widetilde{X}$. Abusing the notation, we shall denote this sheaf by $\widetilde{\mathcal{L}}(-1)$. Next, let

$$\Phi := \bigl\{\tilde{x} \in \widetilde{X} \,\big|\, l(\tilde{x}) \geq 2\bigr\} =: \bigl\{\tilde{x}_1, \dots, \tilde{x}_r\bigr\} \tag{61}$$

be the set of weighted points of $\widetilde{X}$. Let us choose some homogeneous coordinates $(z : w)$ on $\widetilde{X}$. Then we have a pair of distinguished sections $z, w \in \mathsf{Hom}_{\widetilde{X}}\bigl(\mathcal{O}_{\widetilde{X}}(-1), \mathcal{O}_{\widetilde{X}}\bigr)$ vanishing, respectively at the points $\tilde{o}^+ := (0:1)$ and $\tilde{o}^- = (1:0)$. Let $\tilde{x}_i = (\lambda_i : \mu_i)$ and $l_i = l(\tilde{x}_i)$ for $1 \leq i \leq r$.

The following theorem is a restatement of the classical result of Geigle and Lenzing; see [23, Proposition 4.1].

Theorem 5.1. *The locally projective $\mathcal{H}$-module*

$$\widetilde{\mathcal{T}} := \bigl(\widetilde{\mathcal{L}} \oplus \widetilde{\mathcal{L}}(-1)\bigr) \oplus \bigoplus_{\tilde{x} \in \Phi} \bigoplus_{i=2}^{l(\tilde{x})} \widetilde{\mathcal{L}}_{(\tilde{x},i)} \tag{62}$$

is a tilting object in the derived category $D^b\bigl(\mathsf{Coh}(\widetilde{\mathbb{X}})\bigr)$ (called, in what follows, the standard tilting bundle *on $\widetilde{\mathbb{X}}$) and the corresponding algebra $\Gamma := \bigl(\mathsf{End}_{\widetilde{\mathbb{X}}}(\widetilde{\mathcal{T}})\bigr)^\circ$ is isomorphic to the Ringel canonical algebra $\Gamma = \Gamma\bigl((\tilde{x}_1, l_1), \dots, (\tilde{x}_r, l_r)\bigr)$ which is the path algebra of the following quiver*[1]

$$\begin{tikzcd}
& \circ \arrow[r, "u_{12}"] & \circ \arrow[r] & \cdots \arrow[r] & \circ \arrow[ddr, "u_{1l_1}"] & \\
& \circ \arrow[r, "u_{22}"] & \circ \arrow[r] & \cdots \arrow[r] & \circ \arrow[dr, "u_{2l_2}"'] & \\
\circ \arrow[uur, "u_{11}"] \arrow[ur, "u_{21}"'] \arrow[rrrrr, bend left=10, "z"] \arrow[rrrrr, bend right=10, "w"'] \arrow[ddr, "u_{r1}"'] & & & & & \circ \\
& \vdots & \vdots & & \vdots & \\
& \circ \arrow[r, "u_{r2}"] & \circ \arrow[r] & \cdots \arrow[r] & \circ \arrow[uur, "u_{rl_r}"'] &
\end{tikzcd} \tag{63}$$

modulo the relations

$$u_{il_i} \dots u_{i1} = \lambda_i w - \mu_i z \quad \text{for} \quad 1 \leq i \leq r. \tag{64}$$

In other words, the derived functor

$$\mathsf{RHom}_{\widetilde{\mathbb{X}}}(\widetilde{\mathcal{T}}, -) : D^b\bigl(\mathsf{Coh}(\widetilde{\mathbb{X}})\bigr) \longrightarrow D^b(\Gamma\text{-}\mathsf{mod})$$

is an equivalence of triangulated categories.

[1] In this picture the leftmost vertex corresponds to the sheaf $\mathcal{L}$, the rightmost one corresponds to the sheaf $\mathcal{L}(-1)$, while the internal vertices of the k-th branch correspond to the sheaves $\widetilde{\mathcal{L}}_{(\tilde{x}_k,i)}$ for a special point $\tilde{x}_k$ and $l_k = l(\tilde{x}_k)$.

Remark 5.2. In what follows, we shall call the arrows u_{ij} in the quiver (63) for $1 \le i \le r$, $1 \le j \le l_i$ *essential*, whereas the arrows z and w will be called *redundant*. Note that the relations (63) defining the canonical algebra $\Gamma\bigl((\tilde{x}_1, l_1), \dots, (\tilde{x}_r, l_r)\bigr)$ generate an admissible ideal if and only if the set Φ is empty (in this case, the canonical algebra is the path algebra of the Kronecker quiver).

Note also that we can formally add to the set Φ any point $\tilde{x} \in \widetilde{X}$ of *length one*, which correspond to a formal addition of another redundant arrow and does not change the corresponding canonical algebra. We will use this procedure in the study of nodal curves.

Example 5.3. Let $\lambda \in \Bbbk \setminus \{0, 1\}$, $\Phi := \bigl\{(0:1), (1:0), (1:1), (\lambda:1)\bigr\}$ and $l(\tilde{x}) = 2$ for any $\tilde{x} \in \Phi$. Then the corresponding canonical algebra (63) is the tubular algebra (1) from the introduction.

5.2. **Tilting on non-commutative rational projective nodal curves.** We begin with an admissible datum $(\widetilde{X}, l, \approx)$, where $\widetilde{X} = \widetilde{X}_1 \sqcup \dots \sqcup \widetilde{X}_r$ is a smooth rational projective curve (each $\widetilde{X}_i \simeq \mathbb{P}^1$). Let wt be any compatible weight function, $\mathbb{X} = (X, \mathcal{A})$ be the corresponding non-commutative nodal curve, $\widetilde{\mathbb{X}} = (X, \mathcal{H}) = \widetilde{\mathbb{X}}_1 \sqcup \dots \sqcup \widetilde{\mathbb{X}}_r$ be its hereditary cover and $\mathbb{Y} = (X, \mathcal{B})$ be the corresponding Auslander curve.

Theorem 5.4. *Let $\widetilde{\mathcal{T}}_i$ be the standard tilting bundle on $\widetilde{\mathbb{X}}_i$ defined by* (62)*, $\widetilde{\mathcal{T}} := \widetilde{\mathcal{T}}_1 \oplus \dots \oplus \widetilde{\mathcal{T}}_r$ and $\mathcal{T} := \widetilde{\mathsf{F}}(\widetilde{\mathcal{T}}) = \begin{pmatrix} \widetilde{\mathcal{T}} \\ \widetilde{\mathcal{T}} \end{pmatrix}$. Consider the complex $\mathcal{X}^\bullet := \mathcal{T} \oplus \mathcal{S}[-1]$ in the derived category $D^b\bigl(\mathsf{Coh}(\mathbb{Y})\bigr)$, where $\mathcal{S}$ is the torsion sheaf defined by the short exact sequence* (58)*. Then $\mathcal{X}^\bullet$ is a tilting complex in $D^b\bigl(\mathsf{Coh}(\mathbb{Y})\bigr)$. In particular, if $\Lambda := \bigl(\mathsf{End}_{D^b(\mathbb{Y})}(\mathcal{X}^\bullet)\bigr)^\circ$ then the derived categories $D^b\bigl(\mathsf{Coh}(\mathbb{Y})\bigr)$ and $D^b(\Lambda\text{-}\mathsf{mod})$ are equivalent.*

Proof. The fact that $\mathcal{X}^\bullet$ generates the derived category $D^b\bigl(\mathsf{Coh}(\mathbb{Y})\bigr)$ follows from the recollement diagram (57) and the facts that $\widetilde{\mathcal{T}}$ generates $D^b\bigl(\mathsf{Coh}(\widetilde{\mathbb{X}})\bigr)$ and $\bar{A}$ generates $D^b(\bar{A}\text{-}\mathsf{mod})$. Since the functors I and $\mathsf{L}\widetilde{\mathsf{F}}$ are fully faithful, we have:

$$\mathsf{Ext}^i_{\mathbb{Y}}(\mathcal{S}, \mathcal{S}) = 0 = \mathsf{Ext}^i_{\mathbb{Y}}(\mathcal{T}, \mathcal{T})$$

for $i \ge 2$. Since the functor $\mathsf{L}\widetilde{\mathsf{F}}$ is left adjoint to DG and $\mathsf{DG}(\mathcal{S}) = 0$, we have:

$$\mathsf{Ext}^i_{\mathbb{Y}}(\mathcal{T}, \mathcal{S}) \cong \mathsf{Hom}_{D^b(\widetilde{\mathbb{X}})}\bigl(\widetilde{\mathcal{T}}, \mathsf{DG}(\mathcal{S})[i]\bigr) = 0 \quad \text{for all } i \in \mathbb{Z}.$$

Finally, for any $i \in \mathbb{Z}$ we have: $\mathsf{Ext}^i_{\mathbb{Y}}(\mathcal{S}, \mathcal{T}) \cong \Gamma\bigl(X, \mathit{Ext}^i_{\mathbb{Y}}(\mathcal{S}, \mathcal{T})\bigr)$. Since $\mathcal{S}$ is torsion and $\mathcal{T}$ is locally projective, we have: $\mathit{Hom}_{\mathbb{Y}}(\mathcal{S}, \mathcal{T}) = 0$. It follows from the exact sequence (58) that $\mathit{Ext}^i_{\mathbb{Y}}(\mathcal{S}, \mathcal{T}) = 0$ for $i \ge 2$. Therefore, $\mathsf{Hom}_{D^b(\mathbb{Y})}\bigl(\mathcal{X}^\bullet, \mathcal{X}^\bullet[i]\bigr) = 0$ for $i \ne 0$ and

$$\Lambda := \bigl(\mathsf{End}_{D^b(\mathbb{Y})}(\mathcal{X}^\bullet)\bigr)^\circ \cong \begin{pmatrix} \Gamma & 0 \\ W & \bar{A} \end{pmatrix}, \tag{65}$$

where $\Gamma = \bigl(\mathsf{End}_{\widetilde{\mathbb{X}}}(\widetilde{\mathcal{T}})\bigr)^\circ$, $\bar{A} = \bigl(\mathsf{End}_{\mathbb{Y}}(\mathcal{S})\bigr)^\circ$ and $W := \Gamma\bigl(X, \mathit{Ext}^1_{\mathbb{Y}}(\mathcal{S}, \mathcal{T})\bigr)$ (viewed as an $\bar{A}$-Γ-bimodule). □

Corollary 5.5. *Let $(\widetilde{X}, l, \approx)$ be an admissible datum, where $\widetilde{X}$ is a disjoint union of projective lines. Then we have the following commutative diagram of categories and functors:*

$$
\begin{tikzcd}
 & D^b\big(\mathsf{Coh}(\widetilde{\mathbb{X}})\big) \arrow[dl, "\nu_*"'] \arrow[d, hook, "\mathsf{L}\widetilde{\mathsf{F}}"] & \\
D^b\big(\mathsf{Coh}(\mathbb{X})\big) & D^b\big(\mathsf{Coh}(\mathbb{Y})\big) \arrow[l, two heads, "\mathsf{DG}"'] \arrow[r, "\mathsf{T}"] & D^b(\Lambda\text{-}\mathsf{mod}) \\
 & \mathsf{Perf}(\mathbb{X}) \arrow[ul, hook, "\mathsf{E}"] \arrow[u, hook, "\mathsf{LF}"'] &
\end{tikzcd}
\tag{66}
$$

in which T is an exact equivalence of triangulated categories, LF and $\mathsf{L}\widetilde{\mathsf{F}}$ are fully faithful exact functors, E is the canonical inclusion, DG is an appropriate Verdier localization functor and ν_* is induced by the forgetful functor $\mathsf{Coh}(\widetilde{\mathbb{X}}) \longrightarrow \mathsf{Coh}(\mathbb{X})$ (normalization).

5.3. **Tame non-commutative nodal curves and tilting.** We are especially interested in studying those finite dimensional $\Bbbk$-algebras Λ arising in the diagram (66) for which the derived category $D^b(\Lambda\text{-}\mathsf{mod})$ has tame representation type. Since $D^b(\Lambda\text{-}\mathsf{mod})$ contains the category of vector bundles $\mathsf{VB}(\mathbb{X})$ as a full subcategory, the non-commutative nodal curve $\mathbb{X}$ has to be vector bundle tame, i.e. of the form $\mathbb{X}(\vec{p}, \vec{q}, \approx)$, where $(\vec{p}, \vec{q}, \approx)$ is a datum from Definition 4.12; see Theorem 4.13.

In this subsection we are going to elaborate one step further an explicit description of the corresponding algebras $\Lambda(\vec{p}, \vec{q}, \approx)$.

Definition 5.6. Let us start with a pair of tuples

$$
\vec{p} = \big((p_1^+, p_1^-), \dots, (p_r^+, p_r^-)\big) \in \big(\mathbb{N}^2\big)^r \quad \text{and} \quad \vec{q} = (q_1, \dots, q_s) \in \mathbb{N}^s,
$$

where $r, s \in \mathbb{N}_0$ (either of this tuples is allowed to be empty).

For any $1 \le i \le r$, let $\Xi_i^{\pm} := \big\{x_{i,1}^{\pm}, \dots, x_{i,p_i^{\pm}}^{\pm}\big\}$ and

$$
\Gamma\big(p_i^+, p_i^-\big) =
\begin{tikzcd}
 & \circ \arrow[r, "x_{i,2}^+"] & \circ & \cdots & \circ \arrow[r] & \circ \arrow[dr, "x_{i,p_i^+}^+"] & \\
\circ \arrow[ur, "x_{i,1}^+"] \arrow[d, "x_{i,1}^-"'] & & & & & & \circ \\
\circ \arrow[r, "x_{i,2}^-"'] & \circ \arrow[r] & \circ & \cdots & \circ \arrow[r] & \circ \arrow[r] & \circ \arrow[u, "x_{i,p_i^-}^-"']
\end{tikzcd}
\tag{67}
$$

Next, for any $1 \le j \le s$, let $\Xi_j^\circ := \{w_{j,1}, \dots, w_{j,q_j}\}$ and

$$\Gamma\bigl(2,2,q_j\bigr) = \begin{tikzcd}
 & & & \circ \arrow[rrrd, bend left, "v_j^+"] & & & \\
\circ \arrow[rrru, bend left, "u_j^+"] \arrow[rrrd, bend right, "u_j^-"'] \arrow[r, "w_{j,1}"] & \circ \arrow[r, "w_{j,2}"] & {} & \cdots & \circ \arrow[r] & \circ \arrow[r, "w_{j,q_j}"] & \circ \\
 & & & \circ \arrow[rrru, bend right, "v_j^-"'] & & &
\end{tikzcd} \tag{68}$$

modulo the relation $v_j^+u_j^+ + v_j^-u_j^- + w_{j,q_j} \dots w_{j,1} = 0$.
Let $\approx$ be a symmetric relation on the set

$$\Xi := \bigl((\Xi_1^+ \cup \Xi_1^-) \cup \dots \cup (\Xi_r^+ \cup \Xi_r^-)\bigr) \cup \bigl(\Xi_1^\circ \cup \dots \cup \Xi_s^\circ\bigr)$$

such that for any $\xi \in \Xi$ there exists at most one $\xi' \in \Xi$ such that $\xi \approx \xi'$. Then the datum $(\vec{p}, \vec{q}, \approx)$ defines a finite dimensional $\Bbbk$-algebra $\Lambda = \Lambda(\vec{p}, \vec{q}, \approx)$ which is obtained from the disjoint union of quivers with relatioins $\Gamma\bigl(p_i^+, p_i^-\bigr)$ and $\Gamma\bigl(2,2,q_j\bigr)$ by the following combinatorial procedure.

- For any pair of tied elements $\varrho' \approx \varrho''$ of Ξ, we add a new vertex and two arrows ending in it:

$$\begin{tikzcd}
\circ \arrow[r, "\varrho'"] & \circ \arrow[dr, "\vartheta'"] & \\
 & & \bullet \\
\circ \arrow[r, "\varrho''"] & \circ \arrow[ur, "\vartheta''"'] &
\end{tikzcd} \tag{69}$$

  The new arrows satisfy the following zero relations: $\vartheta'\varrho' = 0 = \vartheta''\varrho''$.
- For each reflexive element $\varrho \in \Xi$, we add two new vertices and two arrows ending in each new vertex:

$$\begin{tikzcd}
 & & \bullet \\
\circ \arrow[r, "\varrho"] & \circ \arrow[ur, "\vartheta_+"] \arrow[dr, "\vartheta_-"'] & \\
 & & \bullet
\end{tikzcd} \tag{70}$$

  The new arrows satisfy the following zero relations: $\vartheta_\pm\varrho = 0$.

Remark 5.7. In the case when $s = 0$ (i.e. when the tuple $\vec{q}$ is void) the algebra Λ is skew-gentle [24]. If additionally $\xi \not\approx \xi$ for all $\xi \in \Xi$, then the algebra Λ is gentle [2]. We also refer to [14] for a survey of results on the derived categories of gentle and skew-gentle algebras.

Theorem 5.8. *Let $\mathbb{X} = \mathbb{X}(\vec{p}, \vec{q}, \approx)$ be the non-commutative nodal curve attached to an admissible datum $(\vec{p}, \vec{q}, \approx)$ from Definition 4.12, $\mathbb{Y}$ be the Auslander curve of $\mathbb{X}$ and $\Lambda = \Lambda(\vec{p}, \vec{q}, \approx)$ be the finite dimensional algebra from Definition 5.6. Then the following results hold:*

- *The derived categories $D^b\bigl(\mathsf{Coh}(\mathbb{Y})\bigr)$ and $D^b(\Lambda-\mathsf{mod})$ are equivalent.*
- *Moreover, $D^b\bigl(\mathsf{Coh}(\mathbb{Y})\bigr)$ and $D^b\bigl(\mathsf{Coh}(\mathbb{X})\bigr)$ have tame representation type.*

Proof. According to Theorem 5.4, there exists a tilting complex $\mathcal{X}^\bullet := \mathcal{T} \oplus \mathcal{S}[-1]$ in the derived category $D^b\bigl(\mathsf{Coh}(\mathbb{Y})\bigr)$ such that

$$\widetilde{\Lambda} := \bigl(\mathsf{End}_{D^b(\mathbb{Y})}(\mathcal{X}^\bullet)\bigr)^\circ \cong \begin{pmatrix} \Gamma & 0 \\ W & \bar{A} \end{pmatrix},$$

Note, that $\Gamma \cong \bigl(\Gamma(p_1^+, p_1^-) \times \dots \times \Gamma(p_r^+, p_r^-)\bigr) \times \bigl(\Gamma(2,2,q_1) \times \dots \times \Gamma(2,2,q_s)\bigr)$. Next, $\bar{A}$ is a product of several copies of the semisimple algebras $\Bbbk$ and $\Bbbk \times \Bbbk$. Namely, each pair $\omega', \omega'' \in \Pi$ of tied elements gives a factor $\Bbbk$, whereas each reflexive element $\omega \in \Pi$ gives a factor $\Bbbk \times \Bbbk$. Taking into account the description of the space $W = \Gamma\bigl(X, \mathit{Ext}^1_{\mathbb{Y}}(\mathcal{S}, \mathcal{T})\bigr)$ viewed as right Γ-module given by Lemma 3.40, we can conclude that actually $\widetilde{\Lambda} = \Lambda$, giving the first statement.

Since the derived category $D^b(\Lambda - \mathsf{mod})$ is representation tame (it can be deduced as in [12]), the derived category $D^b\bigl(\mathsf{Coh}(\mathbb{Y})\bigr)$ is representation tame too. Since $D^b\bigl(\mathsf{Coh}(\mathbb{X})\bigr)$ can be obtained as a Verdier localization of $D^b\bigl(\mathsf{Coh}(\mathbb{Y})\bigr)$ (see Theorem 4.24), one can conclude that $D^b\bigl(\mathsf{Coh}(\mathbb{X})\bigr)$ is representation tame as well.[2] $\square$

6. Tilting exercises with some tame non-commutative nodal curves

In this section we are going to study in more details several special cases of the setting of Corollary 5.5.

6.1. **Elementary modifications.** We are going to introduce two "elementary modifications", which allow to replace the algebra $\Lambda = \Lambda(\vec{p}, \vec{q}, \approx)$ by a derived-equivalent algebra.

Lemma 6.1. *Any fragment of Λ of the form (69) can be replaces by the fragment*

$$\begin{tikzcd} \circ \arrow[dr, "\varrho_1'"'] & & \circ \\ & \bullet \arrow[ur, "\varrho_2'"'] \arrow[dr, "\varrho_2''"] & \\ \circ \arrow[ur, "\varrho_1''"] & & \circ \end{tikzcd} \qquad \varrho_2''\varrho_1' = 0, \varrho_1''\varrho_2'' = 0$$

Proof. Let j be the common target of the arrows ϑ' and ϑ'', i' be the source of ϑ' and i'' be the source of ϑ''. Consider the complex

$$T_j := (\dots \longrightarrow 0 \longrightarrow P_j \xrightarrow{(\vartheta', \vartheta'')} \underline{P_{i'} \oplus P_{i''}} \longrightarrow 0 \longrightarrow \dots),$$

[2] Another approach to establish the representation tameness of $D^b\bigl(\mathsf{Coh}(\mathbb{X})\bigr)$ is given in [22].

where the underlined term of T_* is located in the zero degree. Let Ω be the set of vertices of the quiver of the algebra Λ. Then $T := T_j \oplus \bigl(\oplus_{i \in \Omega \setminus \{j\}} P_i\bigr)$ is a tilting object of $D^b(\Lambda\text{-}\mathsf{mod})$. Let $\Gamma := \bigl(\mathsf{End}_{D^b(\Lambda)}(T)\bigr)^{\circ}$. Then on the level of quivers and relations we get precisely the transformation described in the statement of Lemma. $\square$

Example 6.2. Let Λ be the path algebra of the following quiver

$$
\begin{tikzcd}
& \circ \arrow[rr, "x_2"] \arrow[dr, "u_1"] & & \circ \arrow[dr, "x_3"] & & \\
\circ \arrow[ur, "x_1"] \arrow[dr, "y_1"'] & & \bullet & & \circ \arrow[r, bend left, "u_3"] \arrow[r, bend right, "v_3"'] & \bullet \\
& \circ \arrow[rr, "y_2"'] & & \circ \arrow[ul, "v_2"'] \arrow[ur, "y_3"'] & &
\end{tikzcd}
\tag{71}
$$

modulo the relations: $u_i x_i = 0$ for $i \in \{1, 3\}$ and $v_j y_j = 0$ for $j \in \{2, 3\}$.
Making an elementary transformation at both bullets, we get a derived equivalent algebra Γ, which is the path algebra of the following quiver

$$
\begin{tikzcd}
& \circ \arrow[rr, "x_2"] & & \circ \arrow[dr, "x_3^{(1)}"] & & \\
\circ \arrow[rr, "x_1^{(1)}"] \arrow[dr, "y_1"'] & & \bullet \arrow[ul, "x_1^{(2)}"'] \arrow[dr, "y_2^{(2)}"'] & & \circ \arrow[r, bend left, "x_3^{(2)}"] \arrow[r, bend right, "y_3^{(2)}"'] & \bullet \\
& \circ \arrow[ur, "y_2^{(1)}"'] & & \circ \arrow[ur, "y_3^{(1)}"'] & &
\end{tikzcd}
\tag{72}
$$

modulo the relations: $x_1^{(2)} y_2^{(1)} = y_2^{(2)} x_1^{(1)} = x_3^{(2)} y_3^{(1)} = y_3^{(2)} x_3^{(1)} = 0$.

Lemma 6.3. *Any fragment of Λ of the form (70) can be replaced by the fragment*

$$
\begin{tikzcd}
& \bullet \arrow[dr, "\varrho_+^{(2)}"] & \\
\circ \arrow[ur, "\varrho_+^{(1)}"] \arrow[dr, "\varrho_-^{(1)}"'] & & \circ \\
& \bullet \arrow[ur, "\varrho_-^{(2)}"'] &
\end{tikzcd}
\qquad \varrho_+^{(2)} \varrho_+^{(1)} = \varrho_-^{(2)} \varrho_-^{(1)}
$$

Proof. Let $j_\pm$ be the target of $\vartheta_\pm$ and i be their common source. Consider the complexes

$$T_{j_\pm} := \bigl(\ldots \longrightarrow 0 \longrightarrow P_{j_\pm} \xrightarrow{\vartheta_\pm} P_i \longrightarrow 0 \longrightarrow \ldots\bigr).$$

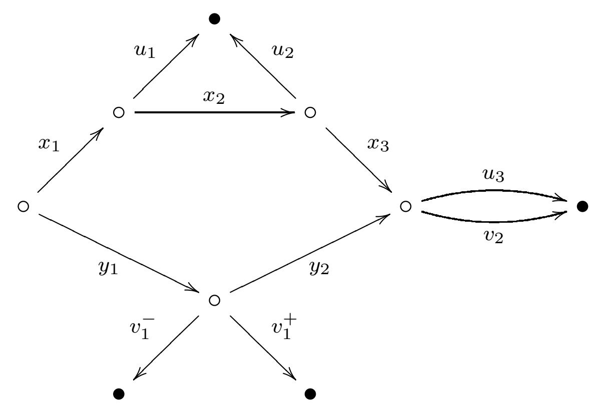
Again, let Ω be the set of vertices of the quiver of the algebra Λ. Then

$$T := \bigl(T_{j_+} \oplus T_{j_-}\bigr) \oplus \bigl(\oplus_{i \in \Omega \setminus \{j_+, j_-\}} P_i\bigr)$$

is a tilting object in $D^b(\Lambda\text{-}\mathsf{mod})$. If $\Gamma := \bigl(\mathsf{End}_{D^b(\Lambda)}(T)\bigr)^{\circ}$, then on the level of quivers and relations the passage from Λ to Γ gives the desired elementary transformation. $\square$

Example 6.4. Let Λ be the path algebra of the following quiver

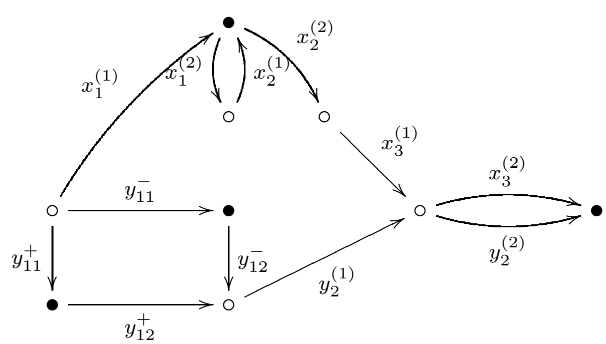
(73)



subject to the relations: $u_i x_i = 0$ for all $1 \le i \le 3$, $v_2 y_2 = 0$ and $v_1^{\pm} y_1 = 0$.

Performing the elementary transformations at all bullets, we get a derived equivalent algebra Γ, given as the path algebra of the following quiver

(74)



subject to the relations:

$$x_2^{(2)} x_1^{(1)} = x_1^{(2)} x_2^{(1)} = x_3^{(2)} y_2^{(1)} = y_2^{(2)} x_3^{(1)} = 0 \quad \text{and} \quad y_{12}^+ y_{11}^+ = y_{12}^- y_{11}^-.$$

6.2. **Degenerate tubular algebra.** Let $E = V\big(zy^2 - x^2(x-z)\big) \subset \mathbb{P}^2$ be a plane nodal cubic and $G = \langle \tau \rangle \cong \mathbb{Z}_2$, where $E \xrightarrow{\tau} E$ is the involution given by the rule $(x : y : z) \mapsto (x : -y : z)$. Then the category $\mathsf{Coh}^G(E)$ of G-equivariant coherent sheaves on E is equivalent to the category of coherent sheaves on the non-commutative nodal curve $\mathbb{E} = \mathbb{X}(\vec{p}, \vec{q} \approx)$ described in Example 4.18. Recall that the vector $\vec{p}$ is void, $\vec{q} = (1)$ and $(\tilde{o}, 1) \approx (\tilde{o}, 1)$. Then the corresponding algebra $\Lambda = \Lambda(\vec{p}, \vec{q}, \approx)$ is the path algebra of the following quiver

$$\begin{tikzcd}
& \circ \arrow[dr, "x_2"] & & \bullet \\
\circ \arrow[ur, "x_1"] \arrow[rr, "w"] \arrow[dr, "y_1"'] & & \circ \arrow[ur, "u_+"] \arrow[dr, "u_-"'] & \\
& \circ \arrow[ur, "y_2"'] & & \bullet
\end{tikzcd}$$

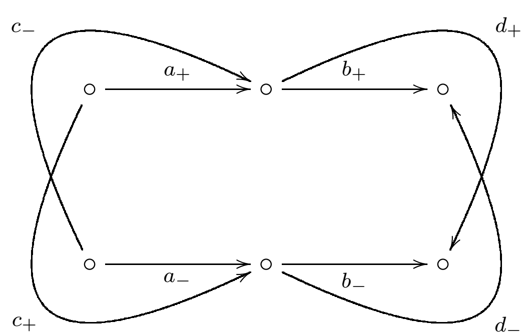
modulo the relations $x_2x_1 + y_2y_1 + w = 0$ and $u_\pm w = 0$. Note that the corresponding ideal in the path algebra is not admissible and the arrow w is redundant. Applying the elementary transformation from Lemma 6.3 to the arrow w, we end up with the the path algebra T of the following quiver

$$\begin{tikzcd}
& & & \circ \arrow[dlll, "a_1"'] \arrow[dl, "a_2"] \arrow[dr, "a_3"'] \arrow[drrr, "a_4"] & & & \\
\circ \arrow[drrr, "b_1"'] & & \circ \arrow[dr, "b_2"] & & \circ \arrow[dl, "b_3"'] & & \circ \arrow[dlll, "b_4"] \\
& & & \circ & & &
\end{tikzcd} \tag{75}$$

modulo the relations $b_1a_1 + b_2a_2 + b_3a_3 = 0$ and $b_1a_1 = b_4a_4$, i.e. the degenerate tubular algebra from Introduction. Since the derived categories $D^b(\Lambda\text{-}\mathsf{mod})$ and $D^b(T\text{-}\mathsf{mod})$ are equivalent, the commutative diagram of categories and functors (3) is a special case of the setting from Corollary 5.5.

Let S be the path algebra of the following quiver

(76)



modulo the following set of relations:

$$d_\pm a_\pm = b_\mp c_\pm \quad \text{and} \quad b_\pm a_\pm = d_m p a_\mp,$$

i.e. any two paths with the same source and target are equal.

Proposition 6.5. *The derived categories $D^b(T\text{-}\mathsf{mod})$ and $D^b(S\text{-}\mathsf{mod})$ are equivalent.*

Proof. Let A be the path algebra of the affine Dynkin quiver

$$
\begin{tikzcd}
\circ \arrow[drr, "b_1"'] & \circ \arrow[dr, "b_2"] & & \circ \arrow[dl, "b_3"'] & \circ \arrow[dll, "b_4"] \\
& & \circ & &
\end{tikzcd}
\tag{77}
$$

and M be the left A-module corresponding to the representation

$$
\begin{tikzcd}
\Bbbk \arrow[drr, "\binom{1}{0}"'] & \Bbbk \arrow[dr, "\binom{1}{1}"] & & \Bbbk \arrow[dl, "\binom{1}{1}"'] & \Bbbk \arrow[dll, "\binom{0}{1}"] \\
& & \Bbbk^2 & &
\end{tikzcd}
\tag{78}
$$

Then we have: $T \cong \begin{pmatrix} \Bbbk & 0 \\ M & A \end{pmatrix}$, i.e. T is a so-called one-point extension of the algebra A by the left A-module M. Next, consider the following left A-modules:

$$
V_1 = \begin{tikzcd}
0 \arrow[drr] & 0 \arrow[dr] & & \Bbbk \arrow[dl, "1"'] & 0 \arrow[dll] \\
& & \Bbbk & &
\end{tikzcd}
\qquad
V_2 = \begin{tikzcd}
0 \arrow[drr] & \Bbbk \arrow[dr, "1"] & & 0 \arrow[dl] & 0 \arrow[dll] \\
& & \Bbbk & &
\end{tikzcd}
$$

$$
V_3 = \begin{tikzcd}
\Bbbk \arrow[drr, "\binom{1}{1}"'] & \Bbbk \arrow[dr, "\binom{1}{0}"] & & \Bbbk \arrow[dl, "\binom{0}{1}"'] & 0 \arrow[dll] \\
& & \Bbbk^2 & &
\end{tikzcd}
\qquad
V_4 = \begin{tikzcd}
0 \arrow[drr] & \Bbbk \arrow[dr, "\binom{1}{0}"] & & \Bbbk \arrow[dl, "\binom{0}{1}"'] & \Bbbk \arrow[dll, "\binom{1}{1}"] \\
& & \Bbbk^2 & &
\end{tikzcd}
$$

and

$$
V_5 = \begin{tikzcd}
0 \arrow[drr] & \Bbbk \arrow[dr, "1"] & & \Bbbk \arrow[dl, "1"'] & 0 \arrow[dll] \\
& & \Bbbk & &
\end{tikzcd}
$$

Then $V := V_1 \oplus V_2 \oplus V_3 \oplus V_4 \oplus V_5$ is a tilting module in $D^b(A\text{-}\mathsf{mod})$ and

$$\mathsf{T} := \mathsf{RHom}_A(T, -) : D^b(A\text{-}\mathsf{mod}) \longrightarrow D^b(B\text{-}\mathsf{mod})$$

is an equivalence of triangulated categories, where $B := \big(\mathsf{End}_A(V)\big)^\circ$. Note that B is isomophic to the path algebra of the following quiver

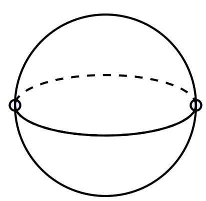
$$
\begin{tikzcd}
& \circ \arrow[r, "b_+"'] \arrow[ddr, bend left=60, "d_+"] & \circ \\
\circ \arrow[ur, "u_+"] \arrow[dr, "u_-"'] & & \\
& \circ \arrow[r, "b_-"] \arrow[uur, bend right=60, "d_-"'] & \circ
\end{tikzcd}
\tag{79}
$$

modulo the relations: $b_\pm u_\pm = d_\mp u_\mp$. It is not difficult to see that $\mathsf{T}(M) \cong N$, where N is the following representation of the quiver (79):

$$\begin{tikzcd}
 & \Bbbk \arrow[r, "1"'] \arrow[dr, bend left=60, "1"] & \Bbbk \\
0 \arrow[ur] \arrow[dr] & & \\
 & \Bbbk \arrow[r, "1"] \arrow[uur, bend right=60, "1"'] & \Bbbk
\end{tikzcd}$$

Observe that $S \cong \begin{pmatrix} \Bbbk & 0 \\ N & B \end{pmatrix}$. The statement follows now a result of Barot and Lenzing [5, Theorem 1] on derived equivalences of one-point extensions. $\square$

Remark 6.6. According to [32, Theorem 8.1.10 and Exercise 8.1], the derived category $D^b(S\text{-}\mathsf{mod})$ is equivalent to the derived category of constructible sheaves $D^b_\Sigma\big(\mathsf{ConSh}(\mathbb{S}^2)\big)$ on the two-dimensional real sphere $\mathbb{S}^2$ with respect to the stratification Σ described in the following picture:



Putting together all results obtained in this subsection, we get the following commutative diagram of triangulated categories and exact functors:

$$\begin{tikzcd}
D^b(T\text{-}\mathsf{mod}) \arrow[r, "\mathsf{T}"] & D^b_\Sigma\big(\mathsf{ConSh}(\mathbb{S}^2)\big) \arrow[r, two heads, "\mathsf{P}"] & D^b\big(\mathsf{Coh}^G(E)\big) \\
 & \mathsf{Perf}^G(E) \arrow[ul, hook, "\mathsf{E}"] \arrow[ur, hook', "\mathsf{I}"'] &
\end{tikzcd} \tag{80}$$

where I is the canonical inclusion functor, E is a fully faithful functor, T is an equivalence of categories and P is an appropriate localization functor. It would be quite interesting to give an interpretation of this result in terms of the homological mirror symmetry in the spirit of the approach of [38].

6.3. **A purely commutative application of non-commutative nodal curves.** Again, let $E = V\big(zy^2 - x^2(x-z)\big) \subset \mathbb{P}^2$ be a plane nodal cubic. Consider the action of the cyclic group $G \cong \mathbb{Z}_2$ from Example 4.15. As an application of the technique of non-commutative nodal curves, we give a direct proof of the following known result; see [48, Example 1.4].

Proposition 6.7. *Let $\widetilde{E}$ be the cycle of two projective lines. Then the derived categories $D^b\big(\mathsf{Coh}(\widetilde{E})\big)$ and $D^b\big(\mathsf{Coh}^G(E)\big)$ are equivalent.*

Proof. Let $\mathbb{E}$ be the non-commutative nodal curve from Example 4.15 (i.e. the categories $\mathsf{Coh}(\mathbb{E})$ and $\mathsf{Coh}^G(E)$ are equivalent). Let $\mathbb{Y}$ (respectively, $\widetilde{\mathbb{Y}}$) be the Auslander curve of $\mathbb{E}$ (respectively, of $\widetilde{E}$). Let $\mathsf{K} := \mathsf{Ker}\Big(D^b\big(\mathsf{Coh}(\mathbb{X})\big) \xrightarrow{\mathsf{P}} D^b\big(\mathsf{Coh}(\mathbb{E})\big)\Big)$ and $\widetilde{\mathsf{K}} := \mathsf{Ker}\Big(D^b\big(\mathsf{Coh}(\widetilde{\mathbb{X}})\big) \xrightarrow{\tilde{\mathsf{P}}} D^b\big(\mathsf{Coh}(\widetilde{E})\big)\Big)$ be the kernels of the corresponding localization functors from Corollary 5.5. We are going to construct an equivalence of triangulated categories $D^b\big(\mathsf{Coh}(\mathbb{X})\big) \xrightarrow{\mathsf{E}} D^b\big(\mathsf{Coh}(\widetilde{\mathbb{X}})\big)$, which induces a commutative diagram of categories and functors

$$
\begin{tikzcd}
\mathsf{K} \arrow[r, "\mathsf{E}"] \arrow[d, hook] & \widetilde{\mathsf{K}} \arrow[d, hook] \\
D^b\big(\mathsf{Coh}(\mathbb{Y})\big) \arrow[r, "\mathsf{E}"] \arrow[d, "\mathsf{P}"'] & D^b\big(\mathsf{Coh}(\widetilde{\mathbb{Y}})\big) \arrow[d, "\tilde{\mathsf{P}}"] \\
D^b\big(\mathsf{Coh}(\mathbb{E})\big) \arrow[r, "\bar{\mathsf{E}}"] & D^b\big(\mathsf{Coh}(\widetilde{E})\big),
\end{tikzcd}
$$

where all horizontal arrows are equivalences of triangulated categories.

Let $D^b\big(\mathsf{Coh}(\mathbb{Y})\big) \xrightarrow{\mathsf{T}} D^b(\Lambda\text{-}\mathsf{mod})$ and $D^b\big(\mathsf{Coh}(\widetilde{\mathbb{Y}})\big) \xrightarrow{\widetilde{\mathsf{T}}} D^b(\widetilde{\Lambda}\text{-}\mathsf{mod}))$ be the equivalences of triangulated categories, where the algebras Λ and $\widetilde{\Lambda}$ are the algebras corresponding, respectively, to $\mathbb{Y}$ and to $\widetilde{\mathbb{Y}}$ as in Corollary 5.5. Recall that

$$
\Lambda = \begin{tikzcd}
 & 2_- \arrow[d, "b_-"] \arrow[dr, "c_+"] & & \\
1 \arrow[ur, "a_+"] \arrow[dr, "a_-"'] & 3 & 4 \arrow[r, bend left, "d_+"] \arrow[r, bend right, "d_-"'] & 5 \\
 & 2_- \arrow[u, "b_+"'] \arrow[ur, "c_-"'] & &
\end{tikzcd}
\qquad b_\pm a_\mp = 0 \ \text{ and } \ d_\pm c_\mp = 0 \tag{81}
$$

whereas

$$
\widetilde{\Lambda} = \begin{tikzcd}
 & & 3_+ & & \\
1 \arrow[r, bend left, "u_+"] \arrow[r, bend right, "u_-"'] & 2 \arrow[ur, "v_+"] \arrow[dr, "v_-"'] & & 4 \arrow[ul, "w_+"'] \arrow[dl, "w_-"] & 5 \arrow[l, bend right, "z_+"'] \arrow[l, bend left, "z_-"] \\
 & & 3_- & &
\end{tikzcd}
\qquad v_\pm u_\mp = 0 \ \text{ and } \ w_\pm z_\mp = 0. \tag{82}
$$

Consider the third gentle algebra

$$\Gamma = \begin{tikzcd} & & 3_+ \arrow[dr, "c_+"] & & \\ 1 \arrow[r, bend left, "u_+"] \arrow[r, bend right, "u_-"'] & 2 \arrow[ur, "v_+"] \arrow[dr, "v_-"'] & & 4 \arrow[r, bend left, "d_+"] \arrow[r, bend right, "d_-"'] & 5 \\ & & 3_- \arrow[ur, "c_-"'] & & \end{tikzcd} \qquad v_\pm u_\mp = 0 \text{ and } d_\pm c_\mp = 0.$$

We construct now a pair of equivalences of triangulated categories:

$$D^b(\Lambda\text{-}\mathsf{mod}) \xrightarrow{\mathsf{T}_1} D^b(\Gamma\text{-}\mathsf{mod}) \xleftarrow{\mathsf{T}_2} D^b(\widetilde{\Lambda}\text{-}\mathsf{mod}).$$

- The first equivalence T_1 is just the elementary modification from Lemma 6.1, applied to the third vertex.
- The second equivalence T_2 is given by the tilting complex

$$X^\bullet := S_1[-2] \oplus S_2[-1] \oplus P_{3_+} \oplus P_{3_-} \oplus P_4 \oplus P_5.$$

The image of the localizing subcategory $\mathsf{K} \subset D^b\bigl(\mathsf{Coh}(\mathbb{Y})\bigr)$ in $D^b(\Lambda\text{-}\mathsf{mod})$ under the tilting equivalence T is the triangulated envelope $\bigl\langle X_+, X_-, Y_+, Y_-\bigr\rangle$, where

$$X_\pm = \bigl(\dots \longrightarrow 0 \longrightarrow P_5 \xrightarrow{d_\pm} P_4 \xrightarrow{c_\mp} \underline{P_{2_\pm}} \longrightarrow 0 \longrightarrow \dots\bigr)$$

and $Y_\pm = \bigl(\dots \longrightarrow 0 \longrightarrow P_3 \xrightarrow{b_\mp} P_{2_\mp} \xrightarrow{a_\pm} \underline{P_{1_\pm}} \longrightarrow 0 \longrightarrow \dots\bigr)$. One can check that $\mathsf{T}_1(X_\pm) \cong S_{3_\pm}$, whereas $\mathsf{T}_1(Y_\pm) \cong Z_\pm$, where

$$Z_\pm = \bigl(\dots \longrightarrow 0 \longrightarrow P_3 \xrightarrow{v_\mp} P_{2_\mp} \xrightarrow{u_\pm} \underline{P_{1_\pm}} \longrightarrow 0 \longrightarrow \dots\bigr).$$

In an analogous way one can check that the image of the localizing subcategory $\widetilde{\mathsf{K}}$ under the chain of equivalences of derived categories

$$D^b\bigl(\mathsf{Coh}(\widetilde{\mathbb{Y}})\bigr) \xrightarrow{\widetilde{\mathsf{T}}} D^b(\widetilde{\Lambda}\text{-}\mathsf{mod}) \xrightarrow{\mathsf{T}_2} D^b(\Gamma\text{-}\mathsf{mod})$$

is again the triangulated category $\bigl\langle S_{3_+}, S_{3_-}, Z_+, Z_-\bigr\rangle$. It proves the proposition. □

6.4. **Tilting on Zhelobenko's non-commutative cycles of projective lines.** For $n \in \mathbb{N}$, let $E = E_{2n}$ be a cycle of $2n$ projective lines. It is convenient to label the irreducible components of E by the natural numbers $\bigl\{1, 2, \dots, 2n\bigr\}$. Let $\widetilde{E} \xrightarrow{\nu} E$ be the normalization of E, $\widetilde{\mathcal{O}} := \nu_*\bigl(\mathcal{O}_{\widetilde{E}}\bigr)$ and $\mathcal{C} := Ann_E\bigl(\widetilde{\mathcal{O}}/\mathcal{O}\bigr) \cong Hom_E\bigl(\widetilde{\mathcal{O}}, \mathcal{O}\bigr)$ be the corresponding conductor ideal sheaf. In this particular case, $\mathcal{C}$ is just the ideal sheaf of the set $\bigl\{o_1, o_2, \dots, o_{2n}\bigr\}$ of the singular points of E. Let $\mathbb{E} = (E, \mathcal{B})$ be the Auslander curve of E, where $\mathcal{B} = \begin{pmatrix} \mathcal{O} & \widetilde{\mathcal{O}} \\ \mathcal{C} & \widetilde{\mathcal{O}} \end{pmatrix}$. Recall that for any $1 \le k \le 2n$ we have: $\widehat{\mathcal{B}}_{o_k} \cong B$, where

B is the Auslander order (23). Let $\mathcal{S}_k^{\pm}$ be the simple torsion sheaf on $\mathbb{E}$ supported at the singular point o_k which corresponds to the vertex $\pm$ of the quiver (24). Note that

$$\mathsf{Ext}^p_{\mathbb{E}}\big(\mathcal{S}_k^{\pm}, \mathcal{S}_k^{\pm}\big) \cong \left\{ \begin{array}{ll} \Bbbk & \text{if } p = 0 \\ 0 & \text{otherwise.} \end{array} \right. \tag{83}$$

Let $e := \begin{pmatrix} 1 & 0 \\ 0 & 1' \end{pmatrix} \in \Gamma(E, \mathcal{B})$, where $1 = 1' + 1'' \in \Gamma(E, \widetilde{\mathcal{O}})$ is the decomposition of the identity section corresponding to the decomposition $\widetilde{\mathcal{O}} = \widetilde{\mathcal{O}}' \oplus \widetilde{\mathcal{O}}''$ of $\widetilde{\mathcal{O}}$ in the direct sum of "even" and "odd" components. Let

$$\mathcal{A} := \mathit{Hom}_{\mathcal{B}}(\mathcal{B}e) \cong \begin{pmatrix} \mathcal{O} & \widetilde{\mathcal{O}}' \\ \mathcal{C} & \widetilde{\mathcal{O}}' \end{pmatrix}$$

and $\mathbb{A} := (E, \mathcal{A})$ be the corresponding non-commutative nodal curve. Note that for any $1 \le k \le 2n$ we have: $\widehat{\mathcal{A}}_{o_k} \cong A$, where A is the Zhelobenko order (21), so we call $\mathbb{A}$ *Zhelobenko's non-commutative cycle of projective lines*. Next, we have a splitting $\mathcal{C} = \mathcal{C}' \oplus \mathcal{C}''$, where $\mathcal{C}' \subset \widetilde{\mathcal{O}}'$ (respectively, $\mathcal{C}'' \subset \widetilde{\mathcal{O}}''$). Under these notations, the following sequences of sheaves are exact: $0 \longrightarrow \mathcal{C}' \longrightarrow \mathcal{O} \longrightarrow \widetilde{\mathcal{O}}'' \longrightarrow 0$ and $0 \longrightarrow \mathcal{C}'' \longrightarrow \mathcal{O} \longrightarrow \widetilde{\mathcal{O}}' \longrightarrow 0$. Next, let

$$\mathcal{I} := \mathsf{Im}\Big(\mathcal{B}e \otimes_{e\mathcal{B}e} e\mathcal{B} \xrightarrow{\mathsf{mult}} \mathcal{B}\Big) = \begin{pmatrix} \mathcal{O} & \widetilde{\mathcal{O}}' \oplus \widetilde{\mathcal{O}}'' \\ \mathcal{C} & \widetilde{\mathcal{O}}' \oplus \widetilde{\mathcal{C}}'' \end{pmatrix}$$

and $\bar{\mathcal{B}} := \mathcal{B}/\mathcal{I}$. Then we have: $\bar{O}'' := \Gamma\big(E, \widetilde{\mathcal{O}}''/\mathcal{C}''\big) \cong \Gamma(E, \bar{\mathcal{B}}) \cong \underbrace{\Bbbk \times \cdots \times \Bbbk}_{2n \text{ times}}$.

Proposition 6.8. *We have a recollement diagram*

$$D^b\big(\bar{O}''\text{-}\mathsf{mod}\big)\big) \rightleftarrows D^b\big(\mathsf{Coh}(\mathbb{E})\big) \rightleftarrows D^b\big(\mathsf{Coh}(\mathbb{A})\big). \tag{84}$$

In particular, there exists an equivalence of triangulated categories:

$$D^b\big(\mathsf{Coh}(\mathbb{A})\big) \longrightarrow \big\langle \mathcal{S}_1^+, \mathcal{S}_2^-, \dots, \mathcal{S}_{2n-1}^+, \mathcal{S}_{2n}^- \big\rangle^{\perp} \subset \big(\mathsf{Coh}(\mathbb{E})\big). \tag{85}$$

Proof. It is a consequence of the corresponding local statement (see Theorem 2.4) combined with the fact (following from (83)) that the functor

$$D^b\big(\bar{O}''\text{-}\mathsf{mod}\big)\big) \longrightarrow D^b_{\bar{\mathcal{B}}}\big(\mathsf{Coh}(\mathbb{E})\big) = \big\langle \mathcal{S}_1^+, \mathcal{S}_2^-, \dots, \mathcal{S}_{2n-1}^+, \mathcal{S}_{2n}^- \big\rangle$$

is an equivalence of triangulated categories. □

Theorem 6.9. *Let $\Upsilon = \Upsilon_n$ be the gentle algebra given by* (9)*. Then the derived categories $D^b\big(\mathsf{Coh}(\mathbb{A})\big)$ and $D^b(\Upsilon\text{-}\mathsf{mod})$ are equivalent.*

Proof. Let $D^b\big(\mathsf{Coh}(\mathbb{E})\big) \xrightarrow{\mathsf{T}} D^b(\Lambda\text{-}\mathsf{mod})$ be the tilting equivalence from Corollary 5.5. Recall from [13, Section 5.2] that $\Lambda = \Lambda_{2n}$ is the path algebra of the following quiver

$$
\begin{tikzcd}
\alpha_1 \arrow[r, bend left, "a_1^+"] \arrow[r, bend right, "a_1^-"'] & \beta_1 \arrow[r, "b_1^+"] \arrow[dddddddr, bend right=10, "b_1^-"'] & \phi_1^+ & & \\
& & \phi_1^- & \gamma_1 \arrow[l, "c_1^+"'] \arrow[ul, "c_1^-"'] & \delta_1 \arrow[l, bend right, "d_1^+"'] \arrow[l, bend left, "d_1^-"] \\
\alpha_2 \arrow[r, bend left, "a_2^+"] \arrow[r, bend right, "a_2^-"'] & \beta_2 \arrow[r, "b_2^+"] \arrow[ur, "b_2^-"] & \phi_2^+ & & \\
& & \phi_2^- & \gamma_2 \arrow[l, "c_2^+"'] \arrow[ul, "c_2^-"'] & \delta_2 \arrow[l, bend right, "d_2^+"'] \arrow[l, bend left, "d_2^-"] \\
& & \vdots & & \\
& & \phi_{n-1}^- & & \\
\alpha_n \arrow[r, bend left, "a_n^+"] \arrow[r, bend right, "a_n^-"'] & \beta_n \arrow[r, "b_n^+"] \arrow[ur, "b_n^-"] & \phi_n^+ & & \\
& & \phi_n^- & \gamma_n \arrow[l, "c_n^+"'] \arrow[ul, "c_n^-"'] & \delta_n \arrow[l, bend right, "d_n^+"'] \arrow[l, bend left, "d_n^-"]
\end{tikzcd}
$$

modulo the relations $b_k^{\pm} a_k^{\mp} = 0 = c_k^{\pm} d_k^{\mp}$ for all $1 \le k \le n$. For any $1 \le k \le n$, consider the complexes

$$A_k^{\pm} := \big(\ldots \longrightarrow 0 \longrightarrow P_{\phi_k^{\pm}} \xrightarrow{c_k^{\mp}} P_{\gamma_k} \xrightarrow{d_k^{\pm}} \underline{P_{\delta_k}} \longrightarrow 0 \longrightarrow \ldots\big).$$

Then we have:

$$\left\{ \begin{array}{rcl} \mathsf{T}\big(\mathcal{S}_{2k-1}^{+}\big) & \cong & A_k^{+} \\ \mathsf{T}\big(\mathcal{S}_{2k}^{-}\big) & \cong & A_k^{-} \end{array} \right. \quad 1 \le k \le n.$$

For any $1 \le k \le n$, consider the following objects of $D^b(\Lambda\text{-}\mathsf{mod})$:

$$B_k = \big(\ldots \longrightarrow 0 \longrightarrow P_{\phi_{k-1}^+} \oplus P_{\phi_k^-} \xrightarrow{\left(b_k^- \; b_k^+\right)} P_{\beta_k} \longrightarrow 0 \longrightarrow \ldots\big),$$

$$C_k = \big(\ldots \longrightarrow 0 \longrightarrow P_{\phi_{k-1}^+} \oplus P_{\phi_k^-} \xrightarrow{\left(b_k^- a_k^- \; b_k^+ a_k^+\right)} P_{\alpha_k} \longrightarrow 0 \longrightarrow \ldots\big)$$

and

$$H := \bigoplus_{k=1}^{n} \bigl(P_{\gamma_k} \oplus P_{\delta_k} \oplus B_k \oplus C_k\bigr).$$

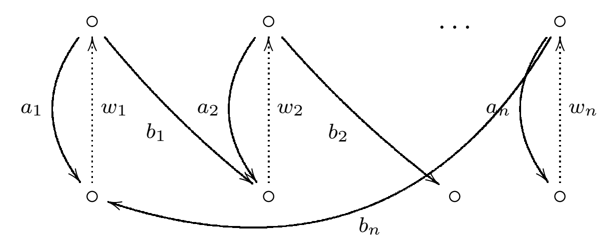
It is not difficult to check that $\mathsf{Hom}_{D^b(\Lambda)}\bigl(H, H[p]\bigr) = 0$ and H generates the triangulated category $\bigl\langle A_1^+, A_1^-, \dots, A_n^+, A_n^-\bigr\rangle^{\perp}$. Hence H is a tilting object of the latter category. Moreover, one can show that $\bigl(\mathsf{End}_{D^b(\Lambda)}(H)\bigr)^{\circ} \cong \Upsilon$. Summing up, we have a chain of equivalences of triangulated categories

$$D^b\bigl(\mathsf{Coh}(\mathbb{A})\bigr) \longrightarrow \bigl\langle A_1^+, A_1^-, \dots, A_n^+, A_n^-\bigr\rangle^{\perp} \longrightarrow D^b(\Upsilon\text{-}\mathsf{mod}),$$

which yields the desired statement. $\square$

For any $n \in \mathbb{N}$, consider the *graded* gentle algebra $\Theta = \Theta_n$, given as the path algebra of the following quiver

(86)



modulo the relations $b_k w_k = 0 = w_k a_k$ for all $1 \le k \le n$, where the grading is given by the rule $\mathsf{deg}(a_k) = \mathsf{deg}(b_k) = 0$, whereas $\mathsf{deg}(w_k) = 1$.

Proposition 6.10. *The triangulated categories $D^b(\Upsilon\text{-}\mathsf{mod})$ and $D^b(\Theta)$ are equivalent, where $D^b(\Theta)$ denotes the derived category of Θ viewed a differential graded category with trivial differential. As a consequence, the triangulated categories $D^b\bigl(\mathsf{Coh}(\mathbb{A})\bigr)$ and $D^b(\Theta)$ are equivalent, too.*

Proof. Let S be a Serre functor of the derived category $D^b\bigl(\mathsf{Coh}(\mathbb{E})\bigr)$. In [16, Lemma 5.2] it was observed that $\mathsf{S}(\mathcal{S}_k^{\pm}) \cong \mathcal{S}_k^{\mp}[2]$ for any $1 \le k \le 2n$. Moreover, for $p \in \mathbb{N}_0$ and $\varepsilon, \delta \in \{+, -\}$ we have:

$$\mathsf{Ext}^p_{\mathbb{E}}\bigl(\mathcal{S}_k^{\varepsilon}, \mathcal{S}_k^{\delta}\bigr) = \left\{ \begin{array}{ll} \Bbbk & \text{if} \quad p = 0 \text{ and } \varepsilon = \delta \\ 0 & \text{if} \quad p = 2 \text{ and } \varepsilon \ne \delta. \end{array} \right.$$

In other words, for any $1 \le k \le 2n$, the pair of objects S_k^+, S_k^- forms a generalized 2-spherical collection. Let $\mathsf{T}_k : D^b\bigl(\mathsf{Coh}(\mathbb{E})\bigr) \longrightarrow D^b\bigl(\mathsf{Coh}(\mathbb{E})\bigr)$ be the corresponding Seidel-Thomas twist functor. According to [46, Proposition 2.10] (see also [9, Theorem 2], [44, Remark 2.5] and [1]), the functor T_k is an auto-equivalence of $D^b\bigl(\mathsf{Coh}(\mathbb{E})\bigr)$. For any $1 \le k, l \le 2n$ we have:

$$\mathsf{T}_k\bigl(\mathcal{S}_l^{\pm}\bigr) \cong \left\{ \begin{array}{ll} \mathcal{S}_l^{\pm} & \text{if} \quad l \ne k \\ \mathcal{S}_l^{\mp}[2] & \text{if} \quad l = k. \end{array} \right.$$

It follows that the composition $\widetilde{\mathsf{T}} := \mathsf{T}_1 \circ \mathsf{T}_3 \circ \dots \circ \mathsf{T}_{2n-1}$ induces an equivalence of triangulated categories

$$D^b(\Upsilon\text{-}\mathsf{mod}) \longrightarrow \big\langle \mathcal{S}_1^+, \mathcal{S}_2^-, \dots, \mathcal{S}_{2n-1}^+, \mathcal{S}_{2n}^- \big\rangle^\perp \xrightarrow{\widetilde{\mathsf{T}}} \big\langle \mathcal{S}_1^-, \mathcal{S}_2^-, \dots, \mathcal{S}_{2n-1}^-, \mathcal{S}_{2n}^- \big\rangle^\perp.$$

Let $\widetilde{A}_k^- := \big(\dots \longrightarrow 0 \longrightarrow P_{\phi_k^+} \xrightarrow{b_k^+} P_{\beta_k} \xrightarrow{a_k^-} \underline{P_{\alpha_k}} \longrightarrow 0 \longrightarrow \dots\big)$. Then we have: $\mathsf{T}\big(\mathcal{S}_{2k-1}^-\big) \cong \widetilde{A}_k^-$ for all $1 \le k \le n$. As a consequence, the categories $D^b(\Upsilon\text{-}\mathsf{mod})$ and $\big\langle A_1^-, \widetilde{A}_1^-, \dots, A_n^-, \widetilde{A}_n^- \big\rangle^\perp$ are equivalent.

For any $1 \le k \le n$, consider the following object in $D^b(\Lambda\text{-}\mathsf{mod})$:

$$\left\{\begin{array}{lcl} X_k & = & \big(\dots \longrightarrow 0 \longrightarrow P_{\beta_k} \xrightarrow{a_k^-} P_{\alpha_k} \longrightarrow 0 \longrightarrow \dots\big) \\ Y_k & = & \big(\dots \longrightarrow 0 \longrightarrow P_{\gamma_k} \xrightarrow{d_k^-} P_{\delta_k} \longrightarrow 0 \longrightarrow \dots\big) \\ U_k & = & \big(\dots \longrightarrow 0 \longrightarrow P_{\phi_k^-} \xrightarrow{b_k^-} P_{\beta_{k+1}} \longrightarrow 0 \longrightarrow \dots\big) \\ V_k & = & \big(\dots \longrightarrow 0 \longrightarrow P_{\phi_k^+} \xrightarrow{c_k^-} P_{\gamma_k} \longrightarrow 0 \longrightarrow \dots\big). \end{array}\right.$$

One can show that

$$G := \bigoplus_{k=1}^{n} \big(X_k \oplus Y_k \oplus U_k \oplus V_k\big)$$

generates the orthogonal category $\big\langle A_1^-, \widetilde{A}_1^-, \dots, A_n^-, \widetilde{A}_n^- \big\rangle^\perp$. Moreover, we have an isomorphism of graded algebras $\Theta \cong \big(\mathsf{Ext}^*_{D^b(\Lambda)}(G)\big)^\circ$.

A result of Bardzell (see [4, Theorem 4.1]) allows to write down a minimal resolution of Θ viewed as a module over its enveloping algebra $\Theta^e := \Theta \otimes_{\mathbb{k}} \Theta^\circ$. From the explicit form of this resolution one can conclude that $\mathsf{gl.dim}\,\Theta = 3$. Moreover, one can show that in the category of *graded* left Θ^e-modules the following vanishing is true:

$$\mathsf{Ext}^3_{\mathsf{gr}(\Theta^e)}\big(\Theta, \Theta(-1)\big) = 0.$$

A result of Kadeishvili [31] implies that the algebra Θ is *intrinsically formal*, i.e. that any minimal A_∞-structure on Θ is equivalent to the trivial one. According to Keller's work [33], the categories $\big\langle A_1^-, \widetilde{A}_1^-, \dots, A_n^-, \widetilde{A}_n^- \big\rangle^\perp$ and $D^b(\Theta)$ are equivalent, which implies the statement. $\square$

Remark 6.11. It follows from Bardzell's resolution [4, Theorem 4.1] that the third Hochschild cohomology $\mathsf{HH}^3(\Upsilon)$ is non-vanishing. Assume that Γ is a finite dimensional algebra, which is derived equivalent to Υ. According to a result of Schröer and Zimmermann [45], Γ is a gentle algebra. Rickard's derived Morita theorem [42] implies that $\mathsf{HH}^3(\Gamma) \cong \mathsf{HH}^3(\Upsilon) \ne 0$ (see for instance [34, Section 2.4] for a detailed argument). It follows from [29, Section 1.5] that $\mathsf{gl.dim}(\Gamma) \ge 3$. In particular, the gentle algebra Υ can not be derived equivalent to a gentle algebra $\Lambda(\vec{p}, \approx)$ from Definition 5.6. As a consequence, a Zhelobenko's non-commutative cycle of projective lines $\mathbb{A}$ is not derived equivalent to the Auslander curve of a non-commutative projective nodal curve.

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