NON-COMMUTATIVE NODAL CURVES AND DERIVED TAME ALGEBRAS

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Abstract. In this paper, we develop a geometric approach to study derived tame finite dimensional associative algebras, based on the theory of non-commutative nodal curves.

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1. Introduction

Let $\mathbb{k}$ be an algebraically closed field such that $\text{char}(\mathbb{k}) \neq 2$. For $\lambda \in \mathbb{k} \setminus \{0, 1\}$, let

$$Y_\lambda = V(zy^2 - x(x - z)(x - \lambda z)) \subset \mathbb{P}^2$$

be the corresponding elliptic curve and $Y_\lambda \to Y_\lambda$ be the involution, given by the rule $(x : y : z) \mapsto (x : -y : z)$. Let $T_\lambda$ be the tubular canonical algebra of type $((2, 2, 2); \lambda)$, i.e. the path algebra of the following quiver

![Quiver Diagram](image)

modulo the relations $b_1a_1 - b_2a_2 = b_3a_3$ and $b_1a_1 - \lambda b_2a_2 = b_4a_4$. According to Geigle and Lenzing [23, Example 5.8], there exists an exact equivalence of derived categories

$$D^b(\text{Coh}^G(Y_\lambda)) \to D^b(T_\lambda\text{-mod}),$$

where $G = \langle i \rangle \cong \mathbb{Z}_2$ and $\text{Coh}^G(Y_\lambda)$ is the category of $G$-equivariant coherent sheaves on $Y_\lambda$. It is well-known that $D^b(\text{Coh}^G(Y_\lambda))$ and $D^b(T_\lambda\text{-mod})$ have tame representation type; see [3, 28, 37]. At this place one can ask the following natural

**Question.** Is there any link between $D^b(\text{Coh}^G(Y_\lambda))$ and $D^b(T_\lambda\text{-mod})$ when the parameter $\lambda \in \mathbb{k}$ takes the “forbidden” value $0$?

Let $E := Y_0 = V(zy^2 - x^2(x - z)) \subset \mathbb{P}^2$ be the plane nodal cubic and $T := T_0$ be the corresponding degenerate tubular algebra. Both derived categories $D^b(\text{Coh}^G(E))$ and $D^b(T\text{-mod})$ are known to be representation tame. Moreover, it follows from our previous papers [11, 12] that the indecomposable objects in both categories can be described by very similar combinatorial patterns. However, since $\text{gl.dim}(\text{Coh}^G(E)) = \infty$ and $\text{gl.dim}(T) = 2$, the derived categories $D^b(\text{Coh}^G(E))$ and $D^b(T\text{-mod})$ can not be equivalent. Nevertheless, it turns out that the following result is true:

**Proposition** (see Remark 6.6). There exists a commutative diagram of triangulated categories and functors

$$\begin{array}{ccc}
D^b(T\text{-mod}) & \xrightarrow{P} & D^b(\text{Coh}^G(E)) \\
\downarrow{E} & & \downarrow{I} \\
\text{Perf}^G(E) & & \\
\end{array}$$

where $\text{Perf}^G(E)$ is the perfect derived category of $\text{Coh}^G(E)$, $I$ is the canonical inclusion functor, $E$ is a fully faithful functor and $P$ is an appropriate Verdier localization functor.
The main goal of this work is to extend the above result to a broader class of derived tame algebras. Let us start with a pair of tuples \( \vec{p} = ((p_1^+, p_1^-), \ldots, (p_r^+, p_r^-)) \in (\mathbb{N}^2)^r \) and \( \vec{q} = (q_1, \ldots, q_s) \in \mathbb{N}^s \), where \( r, s \in \mathbb{N}_0 \) (either of this tuples is allowed to be empty). For any \( 1 \leq i \leq r \) and \( 1 \leq j \leq s \), consider the following sets: \( \Xi^x_i := \{ x_{i,1}^+, \ldots, x_{i,p_i^+}^+ \} \) and \( \Xi^x_j := \{ w_{j,1}, \ldots, w_{j,q_j} \} \). Let \( \approx \) be a symmetric relation (not necessarily an equivalence) on the set \( \Xi := ((\Xi^x_1 \cup \Xi^x_1) \cup \cdots \cup (\Xi^x_1 \cup \Xi^x_1)) \cup (\Xi^x_1 \cup \cdots \cup \Xi^x_1) \) such that for any \( \xi \in \Xi \), there exists at most one \( \xi' \in \Xi \) such that \( \xi \approx \xi' \). Then the datum \( (\vec{p}, \vec{q}, \approx) \) defines a derived tame finite dimensional \( \mathbb{k} \)-algebra \( \Lambda = \Lambda(\vec{p}, \vec{q}, \approx) \), obtained from the canonical algebras \( \Gamma(p_1^+, p_1^-), \ldots, \Gamma(p_r^+, p_r^-), \Gamma(2, 2, q_1), \ldots, \Gamma(2, 2, q_s) \) by a certain “gluing/blowing-up process” determined by the relation \( \approx \). In the case when \( s = 0 \) (i.e., when the tuple \( \vec{q} \) is void), the algebra \( \Lambda \) is skew-gentle [24]. If additionally \( \xi \not\approx \xi \) for all \( \xi \in \Xi \), the algebra \( \Lambda \) is gentle [2]. Instead of defining \( \Lambda(\vec{p}, \vec{q}, \approx) \) now, we refer to Definition 5.6 below and give here two examples explaining characteristic features of these class of algebras.

**Example.** Let \( \vec{p} = (3, 2), \vec{q} \) be void (i.e. \( r = 1 \) and \( s = 0 \)) and \( \approx \) be given by the rule \( x_{1,1}^+ \approx x_{1,1}^- \) and \( x_{1,3}^+ \approx x_{1,2}^- \). Then the corresponding gentle algebra \( \Lambda(\vec{p}, \approx) \) is the path algebra of the following quiver

![Quiver 4](image)

subject to the relations: \( u_1 x_{1,1}^+ = 0 = v_1 x_{1,1}^- \) and \( u_2 x_{1,3}^+ = 0 = v_2 x_{1,2}^- \).

**Example.** Let \( \vec{p} = ((1, 1), (1, 1)), \vec{q} = (2) \) and \( \approx \) be given by the rule: \( x_{1,1}^+ \approx w_{1,1}, x_{1,1}^- \approx x_{2,1}^+ \) and \( w_{1,2} \approx w_{1,1} \). Then the corresponding algebra \( \Lambda(\vec{p}, \vec{q}, \approx) \) is the path algebra of the following quiver

![Quiver 5](image)
modulo the relations:

\[
\begin{align*}
    z_{1,2}z_{1,1} + w_{1,2}w_{1,1} + t_{1,2}t_{1,1} &= 0 \\
    v_3^\pm w_{1,2} &= 0 \\
    u_1x_{1,1}^\pm &= 0 = v_1x_{2,1}^\pm \\
    u_2x_{1,1}^\pm &= 0 = v_2w_{1,1}
\end{align*}
\]

Let \((\vec{p}, \vec{q}, \approx)\) be a datum as in the definition of \(\Lambda(\vec{p}, \vec{q}, \approx)\), additionally satisfying a certain admissibility condition. It turns out that it defines (uniquely up to Morita equivalence) a tame non-commutative projective nodal curve \(\mathbb{X} = \mathbb{X}(\vec{p}, \vec{q}, \approx)\). Conversely, any tame non-commutative projective nodal curve is Morita equivalent to \(\mathbb{X}(\vec{p}, \vec{q}, \approx)\) for an appropriate admissible datum \((\vec{p}, \vec{q}, \approx)\), see [21]. The main result of this paper is the following.

**Theorem** (see Corollary 5.5). Let \((\vec{p}, \vec{q}, \approx)\) be an admissible datum, \(\mathbb{X}\) be the corresponding tame non-commutative nodal curve and \(\Lambda\) be the corresponding \(k\)-algebra. Next, let \(\mathbb{Y}\) be the Auslander curve of \(\mathbb{X}\) (which is another tame non-commutative projective nodal curve) and \(\mathbb{X} \xrightarrow{\nu} \mathbb{X}\) be the hereditary cover of \(\mathbb{X}\). Then there exists the following commutative diagram of triangulated categories and exact functors:

\[
\begin{array}{cccc}
  & D^b(\text{Coh}(\mathbb{X})) & \xrightarrow{\nu_*} & D^b(\text{Coh}(\mathbb{X})) \\
  \downarrow{\nu} & \downarrow{E} & & \downarrow{E} \\
  D^b(\text{Coh}(\mathbb{Y})) & \leftarrow & D^b(\text{Coh}(\mathbb{Y})) & \xrightarrow{T} D^b(\Lambda\text{-mod}) \\
  \downarrow{I} & & \downarrow{E} & \leftarrow \\
  \text{Perf}(\mathbb{X}) & & \text{Perf}(\mathbb{X}) &
\end{array}
\]

where \(T\) is an equivalence of triangulated categories, \(E\) and \(\bar{E}\) are fully faithful functors, \(I\) is the canonical inclusion, \(P\) is an appropriate Verdier localization functor and \(\nu_*\) is induced by the forgetful functor \(\text{Coh}(\mathbb{X}) \rightarrow \text{Coh}(\mathbb{X})\).

This theorem generalizes an earlier results of the authors [13], where \(\mathbb{X}\) was a commutative tame nodal curve (i.e. a chain or a cycle of projective lines [20, 11]).

In [35], Lekili and Polishchuk proved a version of the homological mirror symmetry for punctured Riemann surfaces. According to their work, for any (in appropriate sense graded) compact Riemann surface \(\Sigma\) with finitely many punctures \(x_1, \ldots, x_n \in \Sigma\), there exists either a stacky chain or a stacky cycle of projective lines \(\mathbb{V}\) (actually, not uniquely determined) and equivalences of triangulated categories

\[
\begin{align*}
    D^b(\text{Coh}(\mathbb{V})) & \xrightarrow{\cong} \text{WFuk}(\Sigma) \\
    \text{Perf}(\mathbb{V}) & \xrightarrow{\cong} \text{Fuk}(\Sigma)
\end{align*}
\]
where $\text{Fuk}(\Sigma)$ (respectively, $\text{WFuk}(\Sigma)$) is the Fukaya category (respectively, the wrapped Fukaya category) of the punctured Riemann surface $\Sigma = \Sigma \setminus \{x_1, \ldots, x_n\}$ (which is also viewed as a Riemann surface with boundary). The proof of [25, Theorem B] was based on properties of a partially wrapped Fukaya category $\text{PWFuk}(\Sigma, \vec{m})$ introduced by Haiden, Katzarkov and Kontsevich [27], where $\vec{m} \in \mathbb{N}^n$ is the vector describing the numbers of marked points on the boundary components of $\Sigma$. The partially wrapped Fukaya category $\text{PWFuk}(\Sigma, \vec{m})$ has the following two key features (see [27, Sections 3.4 and 3.5]):

- It is related with the wrapped Fukaya category via an appropriate localization functor $\text{PWFuk}(\Sigma, \vec{m}) \longrightarrow \text{WFuk}(\Sigma)$.
- There exists a (graded) gentle algebra $\Lambda = \Lambda(\Sigma, \vec{m})$ of finite global dimension such that we have an equivalence of triangulated categories $D^b(\Lambda) \longrightarrow \text{PWFuk}(\Sigma, \vec{m})$, where $D^b(\Lambda)$ stands for the derived category of $\Lambda$ viewed as a graded dg-algebra with trivial differential. In the when the grading of $\Lambda$ is trivial, $D^b(\Lambda)$ is equivalent to the conventional derived category $D^b(\Lambda\text{-mod})$.

Given a punctured Riemann surface $\Sigma$, Lekili and Polishchuk describe a stacky cycle/chain of projective lines $V$ as well as a marking of the boundary of $\Sigma$ and grading of $\Sigma$, for which there exists the following commutative diagram of categories and functors

\[
\begin{array}{ccc}
D^b(\text{Coh}(\mathcal{W})) & \longrightarrow & \text{PWFuk}(\Sigma, \vec{m}) \\
\downarrow & & \downarrow \\
D^b(\text{Coh}(V)) & \longrightarrow & \text{WFuk}(\Sigma) \\
\downarrow & & \downarrow \\
\text{Perf}(V) & \longrightarrow & \text{Fuk}(\Sigma),
\end{array}
\]

where all horizontal arrows are equivalences of triangulated categories. The first equivalence $D^b(\text{Coh}(\mathcal{W})) \longrightarrow \text{PWFuk}(\Sigma, \vec{m})$ is defined as the composition of two equivalences of triangulated categories

\[
D^b(\text{Coh}(\mathcal{W})) \longrightarrow D^b(\Lambda\text{-mod}) \longrightarrow \text{PWFuk}(\Sigma, \vec{m}),
\]

both given by appropriate tilting complexes in $D^b(\text{Coh}(\mathcal{W}))$ and $\text{PWFuk}(\Sigma, \vec{m})$, respectively. We are going to explain how a stacky cycle/chain of projective lines $V$ can be naturally viewed as a tame non-commutative nodal curve. In these terms, $\mathcal{W}$ is the Auslander curve of $V$ and the left-hand side of (7) is a subpart of the diagram (6).

Let $\mathcal{Y}$ be the Auslander curve of an arbitrary tame non-commutative nodal curve $X(\vec{p}, \approx)$ (in this notation, the vector $\vec{q}$ is void for the admissible datum $(\vec{p}, \vec{q}, \approx)$), which need not
be a stacky cycle or chain of projective lines. Let \( \Lambda = \Lambda(\vec{p}, \approx) \) be the corresponding gentle algebra. Then there exists a graded punctured marked Riemann surface \((\Sigma, \vec{m})\) as well as equivalences of triangulated categories as in diagram (8); see [27, 36] for further details. From this perspective, our work provides further examples for the homological mirror symmetry for partially wrapped Fukaya categories of marked punctured and appropriately graded Riemann surfaces.

At this place we want to stress that the introduced class of algebras \( \Lambda(\vec{p}, \vec{q}, \approx) \) does not exhaust (even up to derived equivalence) all derived tame algebras which are derived equivalent to an appropriate non-commutative tame projective nodal curve. For example, in the paper [8, Theorem 2.1] it was observed that on a chain of projective lines there exists a tilting bundle whose endomorphism algebra is a gentle algebra of infinite global dimension. In this paper, we have found another class of gentle algebras which are derived equivalent to appropriate non-commutative tame projective nodal curves. For any \( n \in \mathbb{N} \), let \( \Upsilon_n \) be the path algebra of the following quiver

\[
\begin{array}{ccccccc}
\bullet & \rightarrow & \bullet & \rightarrow & \bullet & \rightarrow & \bullet \\
| & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
| & \equiv & \equiv & \equiv & \equiv & \equiv & \equiv \\
| & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
| & \equiv & \equiv & \equiv & \equiv & \equiv & \equiv \\
| & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
| & \equiv & \equiv & \equiv & \equiv & \equiv & \equiv \\
\end{array}
\]

modulo the relations

\[
b_i^\pm a_i^- = 0, c_i^- b_i^+ = 0 \quad \text{and} \quad c_i^+ b_i^- = 0 \quad \text{for} \quad 1 \leq i \leq n.
\]

Since \( \text{HH}^3(\Upsilon_n) \neq 0 \), the algebra \( \Upsilon_n \) can not be derived equivalent to any gentle algebra of the form \( \Lambda(\vec{p}, \approx) \). On the other hand, we prove that \( D^b(\Upsilon_n \text{-mod}) \) is equivalent to the derived category of coherent sheaves on the so-called Zhelobenko non-commutative cycle of projective lines; see Theorem 6.9.

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2. SOME ALGEBRAIC PREREQUISITIES

2.1. **Brief review of the theory of minors.** Throughout this subsection, let $R$ be a commutative noetherian ring. For any $R$-algebra $C$, we denote by $C^\circ$ the opposite $R$-algebra, by $C\text{-mod}$ (respectively, by $\text{mod-}C$) the category of finitely generated left (respectively, right) $C$-modules and by $C\text{-Mod}$ (respectively, $\text{Mod-}C$) the category of all left (respectively, right) $C$-modules. For any $C$-module $X$ we denote by $\text{add}(X)$ the additive closure of $X$, i.e. the full subcategory of $C\text{-Mod}$ consisting of all direct summands of all finite direct sums of $X$.

In what follows, $B$ is an $R$-algebra, which is finitely generated as $R$-module.

**Definition 2.1.** Let $P$ be a finitely generated projective left $B$-module. Then the $R$-algebra $A := (\text{End}_B(P))^\circ$ is called a *minor* of $B$; see [18, 15].

It is clear that $P$ is a ($B-A$)-bimodule and we have an exact functor

$$G = \text{Hom}_B(P, -) : B\text{-Mod} \to A\text{-Mod}.$$  

In the case $P$ is a *projective generator* of the category $B\text{-Mod}$ (meaning that for any object $X$ of $B\text{-Mod}$ there exists an epimorphism $P^n \to X$ for some $n \in \mathbb{N}$), Morita theorem asserts that the functor $G$ is an equivalence of categories.

It is not difficult to prove the following result.

**Lemma 2.2.** Consider the dual (right) $B$-module $P^\vee := \text{Hom}_B(P, B)$. The following statements hold.

- The canonical morphism $P \to P^{\vee\vee} = \text{Hom}_B(P^{\vee}, B)$ is an isomorphism of left $B$-modules. Moreover, the canonical morphism of $R$-algebras

$$\text{End}_B(P) \to (\text{End}_B(P^{\vee}))^\circ$$

is an isomorphism too.

- For any object $X$ of $B\text{-Mod}$, the canonical morphism of left $A$-modules

$$P^{\vee} \otimes_B X \to \text{Hom}_B(P, X), \quad l \otimes x \mapsto (y \mapsto l(y) \cdot x)$$

is an isomorphism, i.e. we have an isomorphism of functors $G \cong - \otimes_B P^\vee$. As a consequence, $P^\vee$ is a flat (actually, even projective) right $B$-module.

- In particular, the canonical morphism

$$P^\vee \otimes_B P \to \text{Hom}_B(P, P), \quad l \otimes y \mapsto (x \mapsto l(y) \cdot x)$$

is an isomorphism of $(A-A)$-bimodules.

Using Lemma 2.2, one can deduce the following results.

**Theorem 2.3.** Consider the functors $F := P \otimes_A -$ and $H := \text{Hom}_A(P^\vee, -)$ from $A\text{-Mod}$ to $B\text{-Mod}$. Then the following statements hold.

- The functors $(F, G, H)$ form an adjoint triple, i.e. $(F, G)$ and $(G, H)$ form adjoint pairs.
The functors $F$ and $H$ are fully faithful, whereas $G$ is essentially surjective.

Let $I_P := \text{Im}(P \otimes_A P^\vee \xrightarrow{ev} B)$. Then $I_P$ is a $(B-B)$-bimodule and

$$\ker(G) := \{ X \in B\text{-Mod} \mid I_P X = 0 \}.$$ 

In other words, the kernel of the exact functor $G$ can be identified with the essential image of the (fully faithful) restriction functor $B\text{-Mod} \to B\text{-Mod}$, where $B = B/I_P$. Moreover, the functor $G$ induces an equivalence of categories $B\text{-Mod}/\ker(G) \to A\text{-Mod}$.

The same results remain true, when we consider $G$, $F$ and $H$ as functors between the categories $B\text{-mod}$ and $A\text{-mod}$ of finitely generated modules.

The essential image of the functor $F : A\text{-mod} \to B\text{-mod}$ is the category $P\text{-mod} := \{ X \in B\text{-mod} \mid \text{there exists } P_1 \to P_0 \to X \to 0 \text{ with } P_0, P_1 \in \text{add}(P) \}.$

It turns out that the relation between $B\text{-Mod}$, $A\text{-Mod}$ and $\check{B}\text{-Mod}$ becomes even more transparent, when we pass to the setting of derived categories.

**Theorem 2.4.** Let $DG : D(B\text{-Mod}) \to D(A\text{-Mod})$ be the derived functor of (an exact) functor $G$, $LF : D(A\text{-Mod}) \to D(B\text{-Mod})$ be the left derived functor of (a right exact functor) $F$ and $RH : D(A\text{-Mod}) \to D(B\text{-Mod})$ be the right derived functor of (a left exact functor) $H$. Then the following results hold.

- $(LF, DG, RH)$ is an adjoint triple of functors.
- The functors $LF$ and $RH$ are fully faithful, whereas $DG$ is essentially surjective.
- The essential image of $LF$ is equal to the left orthogonal category $\perp \check{B} := \{ X^* \in \text{Ob}(D(B\text{-Mod})) \mid \text{Hom}(X^*, \check{B}[i]) = 0 \text{ for all } i \in \mathbb{Z} \}$ of $\check{B}$ (viewed as a left $B$-module). Similarly, the essential image of $RH$ is equal to the right orthogonal category $\check{B}^\perp$.
- We have a recollement diagram

\[
\begin{array}{ccc}
D_B(B\text{-Mod}) & \xrightarrow{I^*} & D(B\text{-Mod}) \\
\text{Id} & \xrightleftharpoons{\text{LF}} & \text{DG} & \xleftleftharpoons{\text{RH}} & D(A\text{-Mod}) \\
\end{array}
\]

where $D_B(B\text{-Mod})$ is the full subcategory of $D(B\text{-Mod})$ consisting of those complexes whose cohomologies belong to $B\text{-Mod}$.

Assume that the $(B-B)$-bimodule $I_P$ is flat viewed as a right $B$-module.

- Then the functor $D(\check{B}\text{-Mod}) \to D_B(B\text{-Mod})$ is an equivalence of triangulated categories.
- We have: $\text{gl.dim}B \leq \max\{\text{gl.dim}\check{B} + 2, \text{gl.dim}A\}$. 
Finally, suppose that $\text{gl.dim} \bar{B} < \infty$ and $\text{gl.dim} A < \infty$. Then we have a recollement diagram

$$
D^b(B\text{-Mod}) \xrightarrow{I^*} D^b(B\text{-Mod}) \xleftarrow{I} D^b(A\text{-Mod}) \xrightarrow{\text{DG}} D^b(A\text{-Mod}).
$$

Remark 2.5. In the case $P = \text{Be}$ for an idempotent $e \in B$, most of the results from this subsection are due to Cline, Parshall and Scott [17]. The “abelian” theory of minors attached to an arbitrary finitely generated projective $B$-module $P$ was for the first time suggested in [18]. Detailed proofs of Theorems 2.3 and 2.4 can also be found in [15].

2.2. Generalities on orders. From now on, let $R$ be an excellent reduced ring of Krull dimension one and $K := \text{Quot}(R) \cong K_1 \times \cdots \times K_r$ be the corresponding total ring of fractions, where $K_1, \ldots, K_r$ are fields.

Definition 2.6. An $R$-algebra $A$ is an $R$-order if the following conditions hold.

- $A$ is a finitely generated torsion free $R$-module.
- $A_K := K \otimes_R A$ is a semisimple $K$-algebra having finite length as a $K$-module.

Lemma 2.7. Let $R$ be as above, $R' \subseteq R$ be a finite ring extension and $A$ be an $R$-algebra. Then $A$ is an $R$-order if and only if $A$ is an $R'$-order. Moreover, if $K := \text{Quot}(R)$ then we have: $A_K \cong A_{K'}$.

Proof. It is clear that $A$ is finitely generated and torsion free over $R$ if and only if it is finitely generated and torsion free over $R'$. Next, note that the ring extension $R' \subseteq R$ induces a finite ring extension $K' \subseteq K$ of the corresponding total rings of fractions. Moreover, Chinese remainder theorem implies that the multiplication map $K' \otimes_{R'} R \rightarrow K$ is an isomorphism. Therefore, for any finitely generated $R$-module $M$, the natural map $K' \otimes_{R'} M \rightarrow K \otimes_R M$ is an isomorphism of $K'$-modules. In particular, we get an isomorphism of $K'$-algebras $A_{K'} \rightarrow A_K$. □

Definition 2.8. Let $A$ be a ring.

- $A$ is a one-dimensional order (or just an order) provided its center $R = Z(A)$ is a reduced excellent ring of Krull dimension one, and $A$ is an $R$-order.
- Let $K := \text{Quot}(R)$. Then $A_K := K \otimes_R A$ is called the rational envelope of $A$.
- A ring $\tilde{A}$ is called an overorder of $A$ if $A \subseteq \tilde{A} \subseteq A_K$ and $\tilde{A}$ is finitely generated as (a left) $A$-module.

Note that for any overorder $\tilde{A}$ of $A$, the map $K \otimes_R \tilde{A} \rightarrow A_K$ is automatically an isomorphism. Hence, $A_K = \tilde{A}_K$ and $\tilde{A}$ is an order over $R$.

Theorem 2.9. Let $H$ be a hereditary order (i.e. $\text{gl.dim} H = 1$) and $R = Z(H)$ be the center of $H$. Then the following results are true.

- We have: $R \cong R_1 \times \cdots \times R_r$, where $R_i$ is a Dedekind domain for all $1 \leq i \leq r$. 
• Let $K_i$ be the quotient field of $R_i$. Then we have: $H_K \cong \mathcal{Y}_1 \times \cdots \times \mathcal{Y}_r$, where $\mathcal{Y}_i$ is a finite dimensional central simple $K_i$-algebra. Moreover, we have a decomposition $H = H_1 \times \cdots \times H_r$, where each factor $H_i$ is a hereditary order whose rational envelope is $\mathcal{Y}_i$.

• If $H$ is an order over $H$ then $H$ is hereditary too.

• If $H'$ is a minor of $H$ then $H'$ is a hereditary order too.

• Assume that $R$ is semilocal. Let $J$ be the radical of $H$ and $\hat{H} = \lim_{k \to \infty} (H/J^k)$ be the radical completions of $H$. Then $\hat{H}$ is a hereditary order.

Proofs of all these results can be for instance found in [39, 40].

Theorem 2.10. Let $k$ be an algebraically closed field and $K$ be either $k((w))$ or the function field of an algebraic curve over $k$. Let $\mathcal{Y}$ be a finite dimensional central simple algebra over $K$. Then $\mathcal{Y} \cong \text{Mat}_t(K)$ for some $t \in \mathbb{N}$.

Proof. This is a restatement of the fact that the Brauer group of the field $K$ is trivial; see for instance [26, Proposition 6.2.3, Theorem 6.2.8 and Theorem 6.2.11].

The following result is well-known; see [40].

Lemma 2.11. Let $R$ be a discrete valuation ring, $m$ be its maximal ideal, $k := R/m$ the corresponding residue field and $K$ the field of fractions of $R$. For any sequence of natural numbers $\vec{p} = (p_1, \ldots, p_r)$, consider the $R$-algebra

$$H(R, \vec{p}) := \begin{pmatrix}
R & \cdots & R & m & \cdots & m & \cdots & m & \cdots & m \\
\vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
R & \cdots & R & m & \cdots & m & \cdots & m & \cdots & m \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
R & \cdots & R & \cdots & R & m & \cdots & m \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
R & \cdots & R & \cdots & R & \cdots & R & \cdots & R & \cdots & R
\end{pmatrix} \subseteq \text{Mat}_p(K),$$

(13)

where the size of the $i$-th diagonal block is $(p_i \times p_i)$ for each $1 \leq i \leq r$ and $p := |\vec{p}| = p_1 + \cdots + p_r$. Then $H(R, \vec{p})$ is a hereditary $R$-order.

In what follows, $H(R, \vec{p})$ will be called standard hereditary $R$-order of type $\vec{p}$. For any $M \in H(R, \vec{p})$ and any pair $1 \leq i, j \leq r$, we denote by $M^{(i,j)}$ the corresponding block of $M$, which is a matrix of size $p_i \times p_j$. In particular, $M^{(i,i)}(0) \in \text{Mat}_{p_i}(k)$. In the simplest case when $\vec{p} = \vec{p}_i := (1, \ldots, 1)$, we shall use the notation $H_r(R) := H(R, \vec{p}_r)$.
**Theorem 2.12.** Let $\mathbb{k}$ be an algebraically closed field, $R = \mathbb{k}[w]$ and $K = \mathbb{k}((w))$. Then the following results are true.

- Assume that $H$ is a hereditary $R$-order in the central simple $K$-algebra $\text{Mat}_p(K)$. Then there exists $S \in \text{Mat}_p(K)$ such that $H = \text{Ad}_S(H(R, \vec{p})) := S \cdot H(R, \vec{p}) \cdot S^{-1}$ for some tuple $\vec{p} = (p_1, \ldots, p_r)$ such that $p = |\vec{p}|$. Moreover, such a tuple $\vec{p}$ is uniquely determined up to a permutation.
- Let $H$ be a hereditary $R$-order. Then we have: $H \cong H_1 \times \cdots \times H_s$, where each $H_i$ is some standard $R$-order for any $1 \leq i \leq s$.
- For any vector $\vec{p}$, the orders $H(R, \vec{p})$ and $H_r(R)$ are Morita equivalent.

Proofs of these results can be found in [40].

### 3. Nodal orders

Nodal orders are appropriate non-commutative generalizations of the commutative nodal ring $D := \mathbb{k}[u, v]/(uv)$.

#### 3.1. Definition and basic properties of nodal orders.

**Definition 3.1.** An order $A$ is called *nodal* if its center is a semilocal excellent ring and there exists a hereditary overorder $H \supseteq A$ such that the following conditions are satisfied.

- $J := \text{rad}(A) = \text{rad}(H)$.
- For any finitely generated simple left $A$-module $U$ we have: $l_A(H \otimes_A U) \leq 2$.

**Remark 3.2.** In what follows, hereditary orders will be considered as special cases of nodal orders. Next, it is clear that an order $A$ is nodal if and only if its radical completion $\hat{A}$ is nodal. Moreover, it is not difficult to show that for a nodal order $A$, the hereditary overorder $H$ from Definition 3.1 is in fact *uniquely determined* and admits the following intrinsic description:

$$H = \{ x \in A_K \mid xJ \subseteq J \} \cong \text{End}_A(J),$$

where $J$ is viewed as a right $A$-module and $A_K$ is the rational envelope of $A$. For a nodal order $A$, the order $H$ will be called the *hereditary cover* of $A$.

**Remark 3.3.** The classical commutative nodal ring $D = \mathbb{k}[u, v]/(uv)$ is a nodal order in the sense of Definition 3.1. Indeed, we have an embedding $D \subseteq \hat{D} := \mathbb{k}[u] \times \mathbb{k}[v]$ and $\text{rad}(D) = \text{rad}(\hat{D}) = (u, v)$. Moreover, $\dim_k(\hat{D} \otimes_D \mathbb{k}) = 2$. Thus $\hat{D}$ is the hereditary cover of $D$.

Nodal orders were introduced by the second-named author in [19]. In that work it was shown that the category of finite length modules over a (non-hereditary) nodal order is representation tame and conversely an order of nonwild representation type is automatically nodal. In our previous joint work [10] we proved that even the derived category of a nodal order has tame representation type.

**Theorem 3.4.** Let $A$ be a nodal order. Then the following statements are true.

- Any overorder of $A$ is nodal.
• Any minor of \( A \) is nodal. In particular, if \( A' \) is Morita equivalent to \( A \) then \( A' \) is a nodal order too.

• Let \( G \) be a finite group acting on \( A \). If \( |G| \) is invertible in \( A \) then the skew group product \( A \ast G \) is a nodal order.

Proofs of the above statements can be found or easily deduced from the results of \([19]\).

3.2. Combinatorics of nodal orders. Let \( \mathbb{k} \) be an algebraically closed field. It turns out that nodal orders over the discrete valuation ring \( \mathbb{k}[x] \) can be completely classified.

**Definition 3.5.** Let \( \Omega \) be a finite set and \( \approx \) be a symmetric but not necessarily reflexive relation on \( \Omega \) such that for any \( \omega \in \Omega \) there exists at most one \( \omega' \in \Omega \) (possibly, \( \omega' = \omega \)) such that \( \omega \approx \omega' \) (note that \( \approx \) is automatically transitive). We say that \( \omega \in \Omega \) is

• **simple** if \( \omega \not\approx \omega' \) for all \( \omega' \in \Omega \);

• **reflexive** if \( \omega \approx \omega \);

• **tied** if there exists \( \omega' \neq \omega \in \Omega \) such that \( \omega \approx \omega' \).

It is clear that any element of \( \Omega \) is either simple, or reflexive, or tied with respect to the relation \( \approx \).

Given \((\Omega, \approx)\) as above, we define two new sets \( \Omega^\dagger \) and \( \tilde{\Omega}^\dagger \) by the following procedures.

• We get \( \Omega^\dagger \) from \( \Omega \) by replacing each reflexive element \( \omega \in \Omega \) by a pair of new simple elements \( \omega_+ \) and \( \omega_- \). The tied elements of \( \Omega^\dagger \) are the same as for \( \Omega \).

• The set \( \tilde{\Omega}^\dagger \) is obtained from \( \Omega^\dagger \) by replacing each pair of tied elements \( \{\omega', \omega''\} \) by a single element \( \{\omega', \omega''\} \).

A map \( \Omega^\dagger \xrightarrow{\text{wt}^\dagger} \mathbb{N} \) is called a **weight function** provided \( \text{wt}^\dagger(\omega') = \text{wt}^\dagger(\omega'') \) for all \( \omega' \approx \omega'' \) in \( \Omega^\dagger \). It is clear that \( \text{wt}^\dagger \) descends to a map \( \tilde{\Omega}^\dagger \xrightarrow{\text{wt}^\dagger} \mathbb{N} \). A weight function \( \Omega^\dagger \xrightarrow{\text{wt}^\dagger} \mathbb{N} \) determines a map (also called weight function) \( \Omega \xrightarrow{\text{wt}} \mathbb{N} \) given by the rule \( \text{wt}(\omega) := \text{wt}^\dagger(\omega_+) + \text{wt}^\dagger(\omega_-) \) for any reflexive point \( \omega \in \Omega \). Abusing the notation, we shall drop the symbol \( \dagger \) in the notation of \( \text{wt}^\dagger \) and write \( \text{wt} \) for all weight functions introduced above.

**Remark 3.6.** The simplest weight function \( \Omega^\dagger \xrightarrow{\text{wt}_o} \mathbb{N} \) is given by the rule \( \text{wt}_o(\omega) = 1 \) for all \( \omega \in \Omega^\dagger \).

Let \((\Omega, \approx)\) be as in Definition 3.5 and \( \Omega \xrightarrow{\sigma} \Omega \) be a permutation. Then we have a decomposition

\[
\Omega = \Omega_1 \sqcup \cdots \sqcup \Omega_t,
\]

where \( \sigma(\Omega_i) = \Omega_i \) and the restricted permutation \( \sigma|_{\Omega_i} \) is cyclic for any \( 1 \leq i \leq t \). In a similar way, we get a decomposition \( \Omega^\dagger = \Omega^\dagger_1 \sqcup \cdots \sqcup \Omega^\dagger_t \).

**Definition 3.7.** Let \( \Omega \) be a finite set and \( \Omega \xrightarrow{\sigma} \Omega \) a permutation. A **marking** \( m \) of \((\Omega, \sigma)\) is a choice of an element \( \omega_i \in \Omega_i \) for any \( 1 \leq i \leq t \), where \( \Omega_i \) are given by (14).
Note that a choice of marking $m$ makes each set $\Omega_i$ totally ordered:

$$\omega_i < \sigma(\omega_i) < \cdots < \sigma^{l_i-1}(\omega_i),$$

where $l_i := |\Omega_i|$. Let $\approx$ be a relation on $\Omega$ as in Definition \[\text{3.5}\] and $\Omega^+ \xrightarrow{\text{wt}} \mathbb{N}$ be a weight function. Then for any $1 \leq i \leq t$, we get a vector

$$\vec{p}_i := (\text{wt}(\omega_i), \text{wt}(\sigma(\omega_i)), \ldots, \text{wt}(\sigma^{l_i-1}(\omega_i))) \in \mathbb{N}^{l_i}.$$  

**Definition 3.8.** Let $(\Omega, \approx, \sigma)$ be a datum as in Definition \[\text{3.7}\] $m$ be a marking of $(\Omega, \sigma)$ and $\Omega^+ \xrightarrow{\text{wt}} \mathbb{N}$ be a weight function. For any $1 \leq i \leq t$, let $H_i := H(R, \vec{p}_i)$ be the corresponding standard hereditary order (\[\text{13}\]). Then we put:

$$H = H(R, (\Omega, \sigma, \approx, m, \text{wt})) := H_1 \times \cdots \times H_t.$$  

It is clear that $H$ is a hereditary order, whose rational envelope is the semisimple algebra

$$\Lambda := \text{Mat}_{s_1}(K) \times \cdots \times \text{Mat}_{s_t}(K),$$

where $K$ is the fraction field of $R$ and $s_i = |\vec{p}_i|$ for $1 \leq i \leq t$.

**Remark 3.9.** According to the definition (\[\text{13}\]) of a standard hereditary order, any matrix belonging to $H_i$ is endowed with a division into vertical and horizontal stripes labelled by the elements of the set $\Omega_i$. Moreover, for any reflexive element $\omega \in \Omega_i$, the corresponding vertical and horizontal stripes have further subdivisions labelled by the elements $\omega_{\pm} \in \Omega_i^\pm$.

**Definition 3.10.** Let $R$ be a discrete valuation ring and $(\Omega, \sigma, \approx, m, \text{wt})$ be a datum as above. Then we have a ring $A = A(R, (\Omega, \sigma, \approx, m, \text{wt})) \subseteq H$ defined as follows:

$$A := \left\{(X_1, \ldots, X_t) \in H \mid \begin{array}{l}
X_{i'}^{(\omega', \omega'')} (0) = X_{i''}^{(\omega'', \omega'')} (0) \text{ for all } 1 \leq i' \leq t, \omega' \in \Omega_{i'}, \omega'' \in \Omega_{i''} \\
\omega' \approx \omega'', \omega' \neq \omega'' \\
x_1^{(\omega, \omega_1)} (0) = 0 \text{ and } x_i^{(\omega, \omega_1)} (0) = 0 \text{ for all } 1 \leq i \leq t, \omega \in \Omega_i, \omega \approx \omega
\end{array} \right\}.$$  

The proof of the following results is straightforward.

**Theorem 3.11.** Let $R$ be a discrete valuation ring, $(\Omega, \sigma, \approx, m, \text{wt})$ a datum as in Definition \[\text{3.7}\] and $A = A(R, (\Omega, \sigma, \approx, m, \text{wt}))$ the corresponding ring from Definition \[\text{3.10}\]. Then the following statements hold.

- The ring $A$ is connected if and only if there exists a surjection

  $$\{1, \ldots, m\} \xrightarrow{\tau} \{1, \ldots, t\}$$

  for some $m$ as well as elements $v_i \in \Omega_{\tau(i)}$ for $1 \leq i \leq m - 1$ such that each $v_i \approx v'_i$ for some $v'_i \in \Omega_{\tau(i+1)}$. If $A$ is connected and $R = \mathbb{k}[w]$ then the center of $A$ is isomorphic to $\mathbb{k}[w_1, \ldots, w_t]/(w_iw_j, 1 \leq i \neq j \leq t)$.

- The ring $A$ is an order whose rational envelope is the semisimple algebra $\Lambda$ given by (\[\text{13}\]). The order $H$ defined by (\[\text{17}\]) is an overorder of $A$. 

• Let $\tilde{m}$ be any other marking of $(\Omega, \sigma)$ and $\tilde{A} = A(R, (\Omega, \sigma, \approx, \tilde{m}, \text{wt}))$ be the corresponding order. Then there exists $S \in \Lambda$ such that $\tilde{A} = \text{Ad}_S(A)$, i.e. the orders $\tilde{A}$ and $A$ are conjugate in $\Lambda$. It means that the order $A(R, (\Omega, \sigma, \approx, m, \text{wt}))$ does not depend (up to a conjugation) on the choice of marking of $(\Omega, \sigma)$, so in what follows it will be denoted by $A(R, (\Omega, \sigma, \approx, \text{wt}))$.

• The orders $A(R, (\Omega, \sigma, \approx, \text{wt}))$ and $A(R, (\Omega', \sigma, \approx, \text{wt})$) are Morita equivalent.

• Let

$$J := \left\{ (X_1, \ldots, X_t) \in A \mid X_i^{(\omega, \omega)}(0) = 0 \text{ for all } 1 \leq i \leq t, \omega \in \Omega_i \right\}.$$  \hfill (19)

Then we have: $J = \text{rad}(A) = \text{rad}(H)$. Moreover, the natural map

$$H \rightarrow \text{End}_A(J), \quad X \mapsto (Y \mapsto XY)$$

is an isomorphism, where $J$ is viewed as a right $A$-module.

• For any $\omega \in \Omega$, let $\bar{H}_\omega := \text{Mat}_{m_\omega}(k)$, where $m_\omega = \text{wt}(\omega)$. Similarly, for any $\gamma \in \tilde{\Omega}^\dagger$, let $\bar{A}_\gamma := \text{Mat}_{m_\gamma}(k)$, where $m_\gamma = \text{wt}(\gamma)$. Then we have a commutative diagram:

$$A/J \xrightarrow{\cong} \prod_{\gamma \in \tilde{\Omega}^\dagger} \bar{A}_\gamma \quad \text{and} \quad H/J \xrightarrow{\cong} \prod_{\omega \in \Omega} \bar{H}_\omega,$$  \hfill (20)

where the components of the embedding $i$ are defined as follows.

– Let $\omega \in \Omega$ be a simple element and $\gamma$ be the corresponding element of $\tilde{\Omega}^\dagger$. Then the corresponding component $\bar{A}_\gamma \rightarrow \bar{H}_\omega$ of $i$ is the identity map.

– Let $\omega \in \Omega$ be reflexive. Then the corresponding component of the map $i$ is given by the rule:

$$\bar{A}_{\omega^+} \times \bar{A}_{\omega^-} \rightarrow \bar{H}_\omega, \quad (X, Y) \mapsto \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}.$$  

– Finally, let $\omega', \omega'' \in \Omega$ be a pair of tied elements and $\gamma := \{\omega', \omega''\}$ be the corresponding element of $\tilde{\Omega}^\dagger$. Then the corresponding component of $i$ is the diagonal embedding

$$\bar{A}_\gamma \rightarrow \bar{H}_{\omega'} \times \bar{H}_{\omega''}, \quad X \mapsto (X, X).$$

• The order $A(R, (\Omega, \sigma, \approx, \text{wt}))$ is nodal and $H(R, (\Omega, \sigma, \approx, \text{wt}))$ is its hereditary cover.

• Let $(\Omega, \sigma, \approx, \text{wt}), (\Omega', \sigma', \approx', \text{wt'})$ be two data as in Definition 3.7. If $R$ is complete then we have:

$$A(R, (\Omega, \sigma, \approx, \text{wt})) \cong A(R, (\Omega', \sigma', \approx', \text{wt'})).$$
if and only if there exists a bijection \( \Omega \xrightarrow{\varphi} \Omega' \) such that the diagram

\[
\begin{array}{ccc}
\Omega & \xrightarrow{\sigma} & \Omega \\
\varphi \downarrow & & \varphi \downarrow \\
\Omega' & \xrightarrow{\sigma'} & \Omega' \\
\end{array}
\]

is commutative and \( \omega_1 \approx \omega_2 \) in \( \Omega \) if and only if \( \varphi(\omega_1) \approx \varphi(\omega_2) \) in \( \Omega' \).

**Remark 3.12.** For the weight function \( \Omega \xrightarrow{\text{wt}} \mathbb{N} \), given by the rule \( \text{wt}_\sigma(\omega) = 1 \) for all \( \omega \in \Omega \), we shall write \( A(R, (\Omega, \sigma, \approx)) := A(R, \Omega, \sigma, \approx, \text{wt}_\sigma) \) or simply \( A(\Omega, \sigma, \approx) \) in the case when it is clear which ring \( R \) is meant.

In the examples below we take \( R = \mathbb{k}[w] \).

**Example 3.13.** Let \( \Omega = \{1, 2\} \), \( \sigma \) is identity and \( 1 \approx 2 \). Then we have:

\[
A(\Omega, \sigma, \approx) \cong D := \mathbb{k}[x, y]/(xy).
\]

As already mentioned, the hereditary cover of \( D \) is \( \tilde{D} := \mathbb{k}[x] \times \mathbb{k}[y] \).

**Example 3.14.** Let \( \Omega = \{1, 2, 3\} \), \( \sigma = (\begin{smallmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{smallmatrix}) \) and \( 1 \approx 2 \). Then we have:

\[
H = H(\Omega, \sigma, \approx) = \mathbb{k}[x] \times \left( \frac{\mathbb{k}[y]}{\mathbb{k}[y]} \right.
\]

and

\[
A = A(\Omega, \sigma, \approx) = \left\{ (X, Y) \in H \mid X(0) = Y(1) \right\}.
\]

Note that \( A \cong \text{End}_D(\mathbb{k}[x] \oplus D) \), where \( D = \mathbb{k}[x, y]/(xy) \). Alternatively, one can identify \( A \) with the arrow completion of the path algebra of the following quiver with relations:

\[
\begin{array}{ccc}
\bullet & \xrightarrow{a} & \bullet \\
\bullet & \xleftarrow{b} & \bullet \\
\bullet & \xrightarrow{c} & \bullet \\
\end{array} \quad ba = 0, \quad ac = 0,
\]

The order given by (21) will be called Zhelobenko order, since it appeared for the first time in a work of Zhelobenko [50] dedicated to the study of admissible finite length representations of the Lie group \( \text{SL}_2(\mathbb{C}) \).

**Example 3.15.** Let \( \Omega = \{1, 2, 3, 4\} \), \( \sigma = (\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{smallmatrix}) \) and \( 2 \approx 3 \). Then we have:

\[
A = A(\Omega, \sigma, \approx) \cong \left( \begin{array}{ccc}
D & \tilde{D} \\
I & \tilde{D} \end{array} \right) \cong \text{End}_D(D \oplus I),
\]
where $D = k[[x, y]]/(xy)$, $I = (x, y)$ and $\bar{D} = k[[x]] \times k[[y]]$. The order $A$ is isomorphic to the arrow completion of the following quiver with relations:

$$\begin{align*}
(24) \quad & - \begin{array}{c} u_- \\ v_- \end{array} \to \begin{array}{c} u_+ \\ v_+ \end{array} + v_- u_+ = 0.
\end{align*}$$

Following the terminology of our previous work \[\text{[13]}\], $A$ will be called the Auslander order of $D$, or just Auslander order.

**Example 3.16.** Let $\Omega = \{1\}$ with $1 \approx 1$ (of course, $\sigma = \text{id}$ in this case). Then we have:

$$\begin{align*}
(25) \quad A = A(\Omega, \sigma, \approx) & \cong \left( \begin{array}{cc} k[[w]] & (w) \\ (w) & k[[w]] \end{array} \right),
\end{align*}$$

The hereditary cover $H$ of $A$ is just the matrix algebra $\text{Mat}_2(k[[w]])$.

**Example 3.17.** Let $\Omega = \{1, 2\}$, $\sigma = (1 \ 2)$ and $2 \approx 2$. Then we have:

$$\begin{align*}
(26) \quad A = A(\Omega, \sigma, \approx) & \cong \left( \begin{array}{ccc} k[[w]] & (w) & (w) \\ (w) & k[[w]] & (w) \\ (w) & (w) & k[[w]] \end{array} \right).
\end{align*}$$

Note that $A$ is isomorphic to the arrow completion of the following quiver with relations:

$$\begin{align*}
(27) \quad & - \begin{array}{c} a_- \\ b_- \end{array} \to \begin{array}{c} a_+ \\ b_+ \end{array} + a_+ b_+ = a_- b_-.
\end{align*}$$

The order \[\text{[26]}\] appeared for the first time in the 1970 ICM talk of I. Gelfand \[\text{[25]}\] in the context of the study of admissible finite length representations of the Lie group $\text{SL}_2(\mathbb{R})$. In what follows, it will be called Gelfand order. The hereditary cover of $A$ is

$$H = H(\Omega, \sigma, \approx) \cong \left( \begin{array}{ccc} k[[w]] & (w) & (w) \\ (w) & k[[w]] & (w) \\ (w) & (w) & k[[w]] \end{array} \right).$$

**Theorem 3.18.** Let $k$ be an algebraically closed field, $R = k[[w]], K = k((w))$ and

$$\Lambda = \text{Mat}_{s_1}(K) \times \cdots \times \text{Mat}_{s_t}(K)$$

for some $s_1, \ldots, s_t \in \mathbb{N}$. If $A$ is a nodal order whose rational envelope is $\Lambda$ then there exists a datum $(\Omega, \sigma, \approx, \text{wt})$ and $S \in \Lambda$ such that $A = A_{\text{Ad}}(A(\Omega, \sigma, \approx, \text{wt}))$, where $A(\Omega, \sigma, \approx, \text{wt})$ is the nodal order from Definition \[\text{[3.10]}\].

**Proof.** Let $A \subseteq \Lambda$ be a nodal order, whose rational envelope is $\Lambda$ and $A^\circ$ be a basic order Morita equivalent to $A$. Then $A^\circ$ is also nodal, see Theorem \[\text{3.4}\]. Let $H^\circ$ be the hereditary cover of $A^\circ$. Then $H^\circ \cong H(R, \tilde{p}_1) \times \cdots \times H(R, \tilde{p}_t)$, where each component $H(R, \tilde{p}_i)$ is a standard hereditary order given by \[\text{[13]}\]. Let $J^\circ = \text{rad}(A^\circ) = \text{rad}(H^\circ)$ be the common radical of $A^\circ$ and $H^\circ$, $\bar{A}^\circ = A^\circ/J^\circ$ and $\bar{H}^\circ = H^\circ/J^\circ$. Since $A^\circ$ is basic and $A^\circ$ is isomorphic to a product of several copies of the field $k$. By
the definition of nodal orders, the embedding of semisimple algebras $\tilde{A}^\circ \hookrightarrow \tilde{H}^\circ$ has the following property: for any simple $\tilde{A}^\circ$-module $U$ we have: $l_{\tilde{A}^\circ}(\tilde{H}^\circ \otimes \tilde{A}^\circ U) \leq 2$. From this property one can easily deduce that

- Each simple component of $\tilde{H}^\circ$ is either $\mathbb{k}$ or $\text{Mat}_2(\mathbb{k})$. In other words, for any $1 \leq i \leq r$, each entry of the vector $\vec{p}_i$ is either 1 or 2.
- The embedding $\tilde{A}^\circ \hookrightarrow \tilde{H}^\circ$ splits into the product of the following components:

$$\mathbb{k} \xrightarrow{id} \mathbb{k}, \quad \mathbb{k} \xrightarrow{\text{diag}} \mathbb{k} \times \mathbb{k} \quad \text{or} \quad (\mathbb{k} \times \mathbb{k}) \xrightarrow{\text{diag}} \text{Mat}_2(\mathbb{k}).$$

Let $1 \leq i \leq r$ and $\vec{p}_i = (p_{i,1}, \ldots, p_{i,k_i})$. Then we put:

$$\Omega_i := \{(i,1), \ldots, (i,k_i)\}, \quad \Omega := \Omega_1 \sqcup \cdots \sqcup \Omega_r \quad \text{and} \quad \sigma(i,j) := (i, j + 1 \mod k_i).$$

Note that the set $\Omega$ parameterizes simple components of the algebra $\tilde{H}^\circ$. If $p_{i,j} = 2$ for $(i,j) \in \Omega_i$, we put: $(i,j) \approx (i,j)$. If $(i,j), (i',j') \in \Omega$ are such that the corresponding component of the embedding $r$ is $\mathbb{k} \xrightarrow{\text{diag}} \mathbb{k} \times \mathbb{k}$, we put: $(i,j) \approx (i',j')$.

Note that the following diagram of $\mathbb{k}$-algebras

$$\begin{array}{ccc}
A^\circ & \xrightarrow{i} & \tilde{A}^\circ \\
\downarrow & & \downarrow i \\
H^\circ & \xrightarrow{\pi} & \tilde{H}^\circ
\end{array}$$

is a pull-back diagram, i.e. $A^\circ \cong \pi^{-1}(\text{Im}(i))$. From this it is not difficult to deduce that

$$A^\circ = A(\Omega, \sigma, \approx, \text{wt}_\circ),$$

where $\text{wt}_\circ$ is the trivial weight function.

Since the order $A$ is Morita equivalent to $A^\circ$, there exists a projective left $A^\circ$-module $P$ such that $A \cong (\text{End}_{A^\circ}(P))^\circ$. Recall that the isomorphism classes of indecomposable $A^\circ$-modules are parameterized by the elements of the set $\tilde{\Omega}^\dagger$. Let $P \cong \bigoplus_{\gamma \in \tilde{\Omega}^\dagger} P_{\gamma}^{\oplus m_{\gamma}}$ be a decomposition of $P$ into a direct sum of indecomposable modules. Then we get a weight function $\tilde{\Omega}^\dagger \xrightarrow{\text{wt}} \mathbb{N}, \gamma \mapsto m_{\gamma}$. It is easy to see that $A^\circ \cong A(\Omega, \sigma, \approx, \text{wt})$. □

### 3.3. Skew group products of $\mathbb{k}[u,v]/(uv)$ with a finite group.

Let $\mathbb{k}$ be an algebraically closed field of characteristic zero, $\zeta \in \mathbb{k}$ be a primitive $n$-th root of 1, $G := \langle \rho \mid \rho^n = e \rangle$ be a cyclic group of order $n \in \mathbb{N}_{\geq 2}$ and $\mathbb{k}[G]$ be the corresponding group algebra. The following results is well-known.

**Lemma 3.19.** For $1 \leq k \leq n$, let $\xi_k := \zeta^k$ and $\varepsilon_k := \frac{1}{n} \sum_{j=0}^{n-1} \xi_k^j \rho^j \in \mathbb{k}[G]$. Then we have:

$$\begin{cases}
1 & = \varepsilon_1 + \cdots + \varepsilon_n \\
\varepsilon_k \cdot \varepsilon_l & = \delta_{kl} \varepsilon_k
\end{cases}$$

In other words, $\{\varepsilon_1, \ldots, \varepsilon_n\}$ is a complete set of primitive idempotents of $\mathbb{k}[G]$. 
Proposition 3.20. Consider the action of the cyclic group $G$ on the polynomial algebra $\mathbb{k}[u]$ given by the rule: $\rho \circ u := \zeta u$. Then the skew product $\mathbb{k}[u] \ast G$ is isomorphic to the path algebra of the cyclic quiver $\vec{C}_n$

\[
\begin{array}{c}
1 \quad \rho
\end{array}
\]
(28)

Proof. An isomorphism $\mathbb{k}[u] \ast G \longrightarrow \mathbb{k}[\vec{C}_n]$ is given by the rule:

\[
\begin{cases}
\varepsilon_k & \mapsto e_k \\
\varepsilon_k u \varepsilon_{k+1} & \mapsto a_k,
\end{cases}
\]
where $e_k \in \mathbb{k}[\vec{C}_n]$ is the trivial path corresponding to the vertex $k$.

Corollary 3.21. Let $R := \mathbb{k}[u]$. Then the skew group product $R \ast G$ is isomorphic to the arrow completion of the path algebra of (28). Note that the latter algebra can in its turn be identified with the algebra of matrices

\[
T_n(R) := \left( \begin{array}{cccc}
R & R & \ldots & R \\
m & R & \ldots & R \\
\vdots & \vdots & \ddots & \vdots \\
m & m & \ldots & R
\end{array} \right),
\]

where the primitive idempotent $e_k$ corresponding to the vertex $k$ of $\vec{C}_n$ is sent to the $k$-th diagonal matrix unit of $T_n(R)$.

Remark 3.22. Let us notice that, strictly speaking, $R \ast G$ depends on the choice of an $n$-th primitive root of unity $\zeta$. On the other hand, the primitive idempotent

\[
\varepsilon = \varepsilon_n := \frac{1}{n}(e + \rho + \ldots + \rho^{n-1}) \in \mathbb{k}[G] \cong R \ast G / \text{rad}(R \ast G)
\]
does not depend on the choice of $\zeta$. Therefore, identifying the skew group product $R \ast G$ with the completed path algebra of a cyclic quiver, we shall always choose a labeling of vertices of $\vec{C}_n$ such that the idempotent $\varepsilon$ is identified with the trivial path corresponding to the vertex labeled by $n$.

Note also that the orders $T_n(R)$ and $H_n(R)$ are isomorphic.

Let $0 < c < n$ be such that $\text{gcd}(n, c) = 1$. Then we have a permutation

\[
\{\tilde{1}, \ldots, \tilde{n}\} \xrightarrow{\tau c} \{\tilde{1}, \ldots, \tilde{n}\} \quad \tilde{k} \mapsto \tilde{c} \cdot \tilde{k},
\]
where $\tilde{k}$ denotes the remainder of $k$ modulo $n$. 
Proposition 3.23. For any $0 < c < n$ such that $\gcd(n, c) = 1$, consider the action of the cyclic group $G = \langle \rho \mid \rho^n = e \rangle$ on the nodal algebra $D = \mathbb{k}[u, v]/(uv)$, given by the rule

$$
\begin{cases}
\rho \circ u &= \zeta u \\
\rho \circ v &= \zeta^c v.
\end{cases}
$$

Then the nodal order $A := D * G$ has the following description:

$$
A \cong \left\{ (U, V) \in T_n(R) \times T_n(R) \mid U^{\tau_c(k)}V^{\tau_c(k)}(0) = V^{kk}(0) \text{ for } 1 \leq k \leq n \right\}.
$$

Proof. Let $\tilde{D} = \mathbb{k}[u] \times \mathbb{k}[v]$ be the hereditary cover of $D$. Then

$$
H := \tilde{D} * G \cong (\mathbb{k}[u] * G) \times (\mathbb{k}[v] * G) \cong T_n(R) \times T_n(R)
$$

is the hereditary cover of $A$. For any $1 \leq k \leq n$, let

$$
\varepsilon_k := \varepsilon_{\tau_c,k} = \frac{1}{n}(1 + \zeta^{ck} + \zeta^{2ck} + \cdots + \zeta^{(n-1)ck} + \zeta^{n-1}) \in \mathbb{k}[G].
$$

We consider the elements $\varepsilon_k$ and $\tilde{\varepsilon}_k$ as elements, respectively, of $\mathbb{k}[[u]] * G$ and $\mathbb{k}[[v]] * G$. It is convenient, taking into account that the actions of $G$ on $\mathbb{k}[u]$ and on $\mathbb{k}[v]$ are actually different.

We have the following commutative diagram of algebras and algebra homomorphisms:

$$
\begin{array}{ccc}
A & \xrightarrow{\cong} & \mathbb{k}[G] \\
\downarrow & & \downarrow \text{diag} \\
H & \xrightarrow{\cong} & \mathbb{k}[G] \times \mathbb{k}[G]
\end{array}
$$

Viewing $\varepsilon_k$ and $\tilde{\varepsilon}_k$ as elements of $(\mathbb{k}[u] * G)/(u * G) \cong \mathbb{k}[G] \cong (\mathbb{k}[v] * G)/(v * G)$, we get: $\text{diag}(\tilde{\varepsilon}_k) = (\varepsilon_{\tau_c,k}, \varepsilon_k) = (\varepsilon_{\sigma_c(k)}, \varepsilon_k)$. Taking into account the rules for the isomorphisms

$$
\mathbb{k}[u] * G \xrightarrow{\cong} T_n(R) \xleftarrow{\cong} \mathbb{k}[v] * G,
$$

we get the description (31) of the nodal order $A$. \hfill \Box

Remark 3.24. In the terms of Theorem 3.11, the order $A$ given by (31) has the following description.

- Let $\Omega = \{1, \ldots, n, \tilde{1}, \ldots, \tilde{n}\}$.
- The relation $\approx$ is given by the rule: $\tilde{k} \approx \tau_c(k)$ for $1 \leq k \leq n$.
- The permutation $\Omega \xrightarrow{\sigma} \Omega$ is given by the formula

$$
\sigma = \begin{pmatrix}
1 & 2 & \cdots & n \\
\tilde{1} & \tilde{2} & \cdots & \tilde{n} \\
\nu & \nu & \cdots & \nu
\end{pmatrix}.
$$

Then we have: $A \cong A(\Omega, \sigma, \approx, \text{wt}_c)$. It is not difficult to derive the quiver description of the order $A$. Of the major interest is the case $c = n - 1$. Then $A$ is isomorphic to the
arrow completion of the path algebra of the following quiver

\begin{center}
\begin{tikzpicture}
\node (n1) at (0,0) [shape=circle,fill=white,draw] {}; \node (n2) at (1,1) [shape=circle,fill=white,draw] {}; \node (n3) at (2,0) [shape=circle,fill=white,draw] {}; \node (n4) at (1,-1) [shape=circle,fill=white,draw] {}; \node (n5) at (0,0) [shape=circle,fill=white,draw] {};
\draw [thick] (n1) to [bend left=45] (n2);
\draw [thick] (n2) to [bend right=45] (n3);
\draw [thick] (n3) to [bend left=45] (n4);
\draw [thick] (n4) to [bend right=45] (n1);
\node (a1) at (0.5,0.5) {$a_1$}; \node (a2) at (1.5,0.5) {$a_2$}; \node (b1) at (0.5,-0.5) {$b_1$}; \node (b2) at (1.5,-0.5) {$b_2$};
\end{tikzpicture}
\end{center}

modulo the relations $a_k b_k = 0 = b_k a_k$ for all $1 \leq k \leq n$.

**Lemma 3.25.** Consider the action of the cyclic group $G = \langle \tau \mid \tau^2 = e \rangle$ on the nodal algebra $D = k[u, v]/(uv)$, given by the rule $\tau(u) = v$. Then the nodal order $A := D * G$ has the following description:

\[
A \cong \left( \frac{k[w]}{(w)} \right).
\]

**Proof.** Consider the elements $e_\pm := \frac{1 \pm \tau}{2} \in A$. Then the following statements are true.

- $e_\pm^2 = e_\pm, e_+ e_- = 0$ and $1 = e_+ + e_-$. Moreover, $\tau \cdot e_\pm = \pm e_\pm$.
- For any $s, t \in \{+, -\}$ we have: $e_s A e_t = e_s De_t$.

Therefore we have the Peirce decomposition:

\[
A \cong \left( \begin{array}{cc}
e_+ A e_+ & e_+ A e_- \\
e_- A e_+ & e_- A e_- \end{array} \right).
\]

For any $m \in \mathbb{N}_0$, set $w_\pm^{(m)} := \frac{1}{2^m} (u^m \pm v^m)$ (within this notation, $w_\pm^{(0)} = 0$). It is easy to see that $w_\pm^{(m_1)} \cdot w_\pm^{(m_2)} = w_\pm^{(m_1 + m_2)}$ for any $m_1, m_2 \in \mathbb{N}_0$ and $s_1, s_2 \in \{+, -\}$. Moreover, one can check that

$e_\pm De_\pm = \left\langle w_\pm^{(m)} e_\pm \mid m \in \mathbb{N}_0 \right\rangle_k$ and $e_\pm De_\mp = \left\langle w_\mp^{(m)} e_\mp \mid m \in \mathbb{N} \right\rangle_k$.

One can check that the linear map

\[
A \cong \left( \begin{array}{cc}
e_+ D e_+ & e_+ D e_- \\
e_- D e_+ & e_- D e_- \end{array} \right) \rightarrow \left( \begin{array}{cc}
k[w] & (w) \\
(w) & k[w] \end{array} \right)
\]

given by the following rules:

\[
\begin{array}{c}
w_+^{(m)} e_+ \mapsto w^m e_{11} \\
w_-^{(m)} e_- \mapsto w^m e_{22}
\end{array} \quad m \in \mathbb{N}_0
\]

and

\[
\begin{array}{c}
w_-^{(m)} e_- \mapsto w^m e_{12} \\
w_+^{(m)} e_+ \mapsto w^m e_{21}
\end{array} \quad m \in \mathbb{N}
\]
is an algebra isomorphism we are looking for.

\[ \text{Proposition 3.26.} \quad \text{For any } n \in \mathbb{N}, \text{ let } G := \langle \rho, \tau \mid \rho^n = e = \tau^2, \tau \rho \tau = \rho^{-1} \rangle \text{ be the dihedral group and } \zeta \in \mathbb{k} \text{ be a primitive } n\text{-th root of 1. Consider the action of } G \text{ on the nodal ring } D = \mathbb{k}[u, v]/(uv) \text{ given by the rules:} \\
\begin{align*}
\rho \circ u &= \zeta u \\
\rho \circ v &= \zeta^{-1}v \\
\tau \circ u &= v \\
\tau \circ v &= u.
\end{align*}

Then the nodal order } A := D * G \text{ has the following description.} \\
\text{• If } n = 2l + 1 \text{ for } l \in \mathbb{N}_0 \text{ then } A \cong A(\Omega, \sigma, \approx, \text{wt}_o), \text{ where} \\
\Omega := \{1, 2, \ldots, n\}, \quad \sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\
1 & 2 & \cdots & n-1 \\
\end{pmatrix} \\
\text{and } (2k-1) \approx 2k \text{ for } 1 \leq k \leq l, \text{ whereas } (2l+1) \approx (2l+1). \\
\text{• If } n = 2l + 2 \text{ for } l \in \mathbb{N}_0 \text{ then } A \cong A(\Omega, \sigma, \approx, \text{wt}_o), \text{ where} \\
\Omega := \{0, 1, 2, \ldots, 2l+1\}, \quad \sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\
1 & 2 & \cdots & n-1 \\
\end{pmatrix} \\
\text{and } (2k-1) \approx 2k \text{ for } 1 \leq k \leq l, \text{ whereas } 0 \approx 0 \text{ and } (2l+1) \approx (2l+1).}

\text{Sketch of the proof.} \text{ The cyclic group } K := \langle \rho \rangle \text{ is a normal subgroup of } G \text{ of index two. Let } L = \langle \tau \rangle. \text{ Then we have a commutative diagram of } \mathbb{k}\text{-algebras and algebra homomorphisms} \\
\begin{array}{ccc}
A * G & \cong & (A * K) * L \\
\downarrow & & \downarrow \\
H * G & \cong & (H * K) * L
\end{array}

\text{where the action of } L \text{ on } H * K = (\mathbb{k}[u] * K) \times (\mathbb{k}[v] * K) \text{ is given by the rule} \\
\tau \circ (u^{k_1} \rho^{l_1}, v^{k_2} \rho^{l_2}) = (u^{k_2} \rho^{-l_2}, v^{k_1} \rho^{-l_1})

\text{for any } k_1, k_2, l_1, l_2 \in \mathbb{N}_0. \text{ The nodal ring } A * K \text{ is described in Remark 3.24. In the terms of the quiver presentation } (32) \text{ we have:} \\
\begin{align*}
\tau \circ a_k &= b_{k-1} \\
\tau \circ b_k &= a_{k-1} \\
\tau \circ e_k &= e_k
\end{align*}

\text{where } k = (n-k) \text{ for } 1 \leq k \leq n \text{ and all indices are taken modulo } n. \text{ The remaining part} \\
\text{is a lengthy computation analogous to the one made in the course of the proof of Lemma 3.25} \text{ which we leave for the interested reader.}
3.4. Auslander order of a nodal order.

Definition 3.27. Let $A$ be a nodal order and $H$ be its hereditary cover. Then

\[(35) \quad C := \{ a \in A \mid ah \in A \text{ for all } h \in H \}\]

is called \textit{conductor ideal} of $A$.

Remark 3.28. It follows from the definition that $C = ACH$ and the canonical morphism

\[(36) \quad C \longrightarrow \text{Hom}_A(H, A), \quad c \mapsto (h \mapsto ch)\]

is a bijection (here, we view both $H$ and $A$ as left $A$-modules).

Proposition 3.29. Let $R$ be a discrete valuation ring and $A = A(R, (\Omega, \sigma, \approx, \text{wt}))$ be the nodal order from the Definition 3.10. Then we have:

\[C = \left\{ (X_1, \ldots, X_t) \in A \mid X_1^{(\omega', \omega')}(0) = 0 = X_t^{(\omega''', \omega''')}(0) \text{ for all } \omega' \in \Omega_{i'}, \omega'' \in \Omega_{i''} \right\}\]

and $C = \{ a \in A \mid ha \in A \text{ for all } h \in H \}$. In particular, $C$ is a two-sided ideal both in $H$ and $A$ containing the common radical $J = \text{rad}(A) = \text{rad}(H)$ of $A$ and $H$.

Corollary 3.30. Let $k$ be an algebraically closed field, $R$ be the local ring of an affine curve over $k$ at a smooth point, $H$ be a hereditary $R$-order, $A$ a nodal order whose hereditary cover is $H$ and $C$ be the corresponding conductor ideal. Then $C$ is a two-sided ideal in $H$.

Proof. We have to show that the canonical map of $R$-modules $C \longrightarrow HCH$ is surjective. For this, it is sufficient to prove the corresponding statement for the radical completions of $A$ and $H$. However, the structure of nodal orders over $\hat{R} \cong k[[w]]$ is known; see Theorem 3.18. Hence, the statement follows from Proposition 3.29. \hfill $\Box$

Lemma 3.31. Let $A = A(R, (\Omega, \sigma, \approx, \text{wt}))$ be a nodal order, $H$ be its hereditary cover, $C$ be the conductor ideal, $\hat{A} := A/C$ and $\hat{H} := H/C$. Let $\Omega_0$ be the subset of $\Omega$ whose elements are reflexive or tied elements of $\Omega$ and $\tilde{\Omega}_0$ be the subset of $\tilde{\Omega}$ defined in a similar way. Then the following diagram

\[(37) \quad \begin{array}{ccc}
\hat{A} & \xrightarrow{\cong} & \prod_{\gamma \in \tilde{\Omega}_0} \hat{A}_\gamma \\
\downarrow & & \downarrow \\
\hat{H} & \xrightarrow{\cong} & \prod_{\omega \in \Omega_0} \hat{H}_\omega
\end{array}\]

is commutative, where the components of the embedding $\iota$ are described in the same way as in diagram (20).
**Definition 3.32.** Let $A$ be a nodal order, $H$ be its hereditary cover and $C$ be the corresponding conductor ideal. The order

$$B := \begin{pmatrix} A & H \\ C & H \end{pmatrix}$$

is called *Auslander order* of $A$.

**Example 3.33.** The Auslander order of the commutative nodal ring $k[[u, v]]/(uv)$ is

$$\begin{pmatrix} k[[u, v]]/(uv) & k[[u]] \times k[[v]] \\ (u, v) & k[[u]] \times k[[v]] \end{pmatrix},$$

i.e. the order from Example 3.15.

**Example 3.34.** Let $R = k[[u]]$, $m = (w)$ and $A = \begin{pmatrix} R & m \\ m & R \end{pmatrix}$ be the nodal order from Example 3.16. The hereditary cover $H$ of $A$ is just the matrix algebra $\text{Mat}_2(R)$, whereas the corresponding conductor ideal $C = \text{Mat}_2(m)$. Therefore, the Auslander order of $A$ is

$$B = \begin{pmatrix} R & m & R & R \\ m & R & R & R \\ m & m & R & R \\ m & m & R & R \end{pmatrix}.$$ 

It is easy to see that $B$ is Morita equivalent to the Gelfand order (26).

Let $A$ be an arbitrary nodal order and $B$ be the Auslander order of $A$. For the idempotents $e := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $f := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in B$, consider the corresponding projective left $B$-modules

$$P := B e = \begin{pmatrix} A \\ C \end{pmatrix} \text{ and } Q := B f = \begin{pmatrix} H \\ H \end{pmatrix}.$$ 

Note that

$$A = e B e \cong (\text{End}_B(P))^\circ \text{ and } H = f B f \cong (\text{End}_B(Q))^\circ.$$ 

The action of $B$ on the projective left $B$-modules $\begin{pmatrix} A \\ C \end{pmatrix}$ and $\begin{pmatrix} H \\ H \end{pmatrix}$ is given by the matrix multiplication, whereas the isomorphisms $A \cong (\text{End}_B(P))^\circ$, respectively $H \cong (\text{End}_B(Q))^\circ$, are compatible with the canonical right actions on $P$, respectively $Q$. The nodal order $A$ as well as its hereditary cover $H$ are *minors* of the Auslander order $B$ in the sense of Definition 2.1. Let

$$P^\vee := \text{Hom}_B(P, B) \cong e B \text{ and } Q^\vee := \text{Hom}_B(Q, B) \cong f B.$$ 

In the terms of Subsection 2.1 we have the following functors.

- Since $P$ is a projective left $A$-module, we get an exact functor

$$G := \text{Hom}_B(P, -) \simeq P^\vee \otimes_B -$$. 

from $B\text{-Mod}$ to $A\text{-Mod}$. Of course, it restricts to an exact functor

$$B\text{-mod} \xrightarrow{G} A\text{-mod}$$
between the corresponding categories of finitely generated modules.

- Similarly, we have an exact functor \( \tilde{G} = \text{Hom}_B(Q, -) \cong Q^\vee \otimes_B - \) from \( B\text{-Mod} \) to \( H\text{-Mod} \), as well as its restriction on the full subcategories of the corresponding finitely generated modules.
- We have functors \( F := P \otimes_A - \) and \( H := \text{Hom}_A(P^\vee, -) \) from \( A\text{-Mod} \) to \( B\text{-Mod} \).
- Similarly, we have functors \( \tilde{F} := Q \otimes_H - \) and \( \tilde{H} := \text{Hom}_H(Q^\vee, -) \) from \( H\text{-Mod} \) to \( B\text{-Mod} \).

Additionally to Theorem 2.3, the following result is true.

**Proposition 3.35.** The functor \( \tilde{F} \) is exact, maps projective modules to projective modules and has the following explicit description: if \( \tilde{N} \) is a left \( H \)-module, then

\[
\tilde{F}(\tilde{N}) = \left( \begin{array}{c} \tilde{N} \\ \bar{N} \end{array} \right) \cong \tilde{N} \oplus \bar{N},
\]

where for \( b = \left( \begin{array}{cc} b_1 & b_2 \\ b_3 & b_4 \end{array} \right) \in B \) and \( z = \left( \begin{array}{c} z_1 \\ z_2 \end{array} \right) \in \left( \begin{array}{c} \tilde{N} \\ \bar{N} \end{array} \right) \), the element \( bz \) is given by the matrix multiplication.

**Remark 3.36.** Since the order \( H \) is hereditary, the conductor ideal \( C \) is projective viewed as a left \( H \)-module. The functor \( \tilde{F} \) transforms projective modules into projective modules, hence the left \( B \)-module \( \left( \begin{array}{c} C \\ C \end{array} \right) \) is projective, too. Consider the left \( B \)-modules \( S \) and \( T \), given by the projective resolutions

\[
0 \to \left( \begin{array}{c} C \\ C \end{array} \right) \to \left( \begin{array}{c} A \\ C \end{array} \right) \to S \to 0 \quad \text{and} \quad 0 \to \left( \begin{array}{c} C \\ C \end{array} \right) \to \left( \begin{array}{c} H \\ H \end{array} \right) \to T \to 0.
\]

Obviously, both \( S \) and \( T \) have finite length viewed as \( B \)-modules. Moreover, \( S \) is isomorphic to \( \bar{A} \) viewed as an \( A \)-module and \( T \) is isomorphic to \( \bar{H} \) viewed as an \( H \)-module.

**Theorem 3.37.** The following results are true.

- \( (LF, DG, RH) \) and \( (\tilde{L}F, \tilde{D}G, \tilde{R}H) \) are triples pairs of functors.
- The functors \( LF, \tilde{L}F, RH \) and \( \tilde{R}H \) are fully faithful, whereas the functors \( DG \) and \( \tilde{D}G \) are essentially surjective.
- The essential image of \( \tilde{L}F \) is equal to the left orthogonal category \( \perp S := \{ X^* \in \text{Ob}(D(B\text{-mod})) \mid \text{Hom}(X^*, S[i]) = 0 \text{ for all } i \in \mathbb{Z} \} \) of \( S \), whereas the essential image of \( \tilde{R}H \) is equal to the right orthogonal category \( S^\perp \). Similarly, the essential image of \( LF \) is equal to \( \perp T \) and the essential image of \( RH \) is equal to \( T^\perp \).
- We have a recollement diagram

\[
\begin{array}{c}
D(\bar{A}\text{-mod}) \xrightarrow{I^*} D(B\text{-mod}) \xrightarrow{L\tilde{F}} D(H\text{-mod}) \\
\xrightarrow{I^!} \xrightarrow{D\tilde{G}} \xrightarrow{R\tilde{H}} \end{array}
\]
where \( I(\overline{A}) := S \), the functor \( I^* \) is left adjoint to \( I \) and \( I^! \) is right adjoint to \( I \).

- Similarly, we have another recollement diagram

\[
\begin{array}{c}
D_T(B\text{-mod}) \quad \xrightarrow{J^*} \quad D(B\text{-mod}) \quad \xrightarrow{\text{LF}} \quad D(A\text{-mod}),
\end{array}
\]

where \( D_T(B\text{-mod}) \) is the full subcategory of the derived category \( D(B\text{-mod}) \) consisting of those complexes whose cohomologies belong to \( \text{Add}(T) \) and \( J \) is the canonical inclusion functor.

- We have: \( \text{gl.dim}B = 2 \).

**Proof.** These results are specializations of Theorem 2.4. The statements about both recollement diagrams (39) and (40) follow from the description of the kernels of the functors \( \widetilde{D} \) and \( DG \). Namely, consider the two-sided ideal

\[
I_Q := \text{Im}(Q \otimes_A Q^\vee \xrightarrow{ev} B) = \left( \begin{array}{cc} C & H \\ C & H \end{array} \right)
\]

in the algebra \( B \). As one can easily see, \( I_Q \) is projective viewed as a right \( B \)-module. Moreover, \( B/I_Q \cong A/C =: \overline{A} \) is semisimple. Hence, Theorem 2.4 gives the first recollement diagram (39). Analogously, for

\[
I_P := \text{Im}(P \otimes_H P^\vee \xrightarrow{ev} B) = \left( \begin{array}{cc} A & H \\ C & C \end{array} \right)
\]

the algebra \( B/I_P \cong H/C =: \widetilde{H} \) is again semisimple. However, this time \( I_P \) is not projective viewed as a right \( B \)-module.

To show the last statement, note that according to Theorem 2.4 we have: \( \text{gl.dim}B \leq 2 \). Since \( A \) is a non-hereditary minor of \( B \), the order \( B \) itself can not be hereditary; see Theorem 2.9. Hence \( \text{gl.dim}B = 2 \), as claimed. \( \square \)

**Corollary 3.38.** We have a recollement diagram

\[
\begin{array}{c}
D^b(\overline{A}\text{-mod}) \quad \xrightarrow{\text{I}^*} \quad D^b(B\text{-mod}) \quad \xrightarrow{\text{LF}} \quad D^b(H\text{-mod}),
\end{array}
\]

As a consequence, we have a semiorthogonal decomposition

\[
D^b(B\text{-mod}) = \langle \text{Im}(I), \text{Im}(\text{LF}) \rangle = \langle D^b(\overline{A}\text{-mod}), D^b(H\text{-mod}) \rangle.
\]

Moreover, we have the following commutative diagram of categories and functors:

\[
\begin{array}{c}
D^b(H\text{-mod}) \xrightarrow{\text{LF}} D^b(B\text{-mod}) \xrightarrow{\text{LF}} \text{Perf}(A) \\
\xrightarrow{P} \quad \text{DG} \quad \text{E} \\
D^b(A\text{-mod})
\end{array}
\]

\[
\begin{array}{c}
\xrightarrow{\text{LF}} \quad \text{DG} \quad \text{P} \\
\text{Perf}(A) \quad \text{E} \\
D^b(A\text{-mod})
\end{array}
\]
where $\text{Perf}(A)$ is the perfect derived category of $A$, $E$ is the canonical inclusion functor and $P$ is the derived functor of the restriction functor $H\text{-mod} \to A\text{-mod}$.

Proof. The recollement diagram (41) is just the restriction of the recollement diagram (39) on the corresponding full subcategories of compact objects. The isomorphism $E \simeq DG \circ LF$ follows from the fact that the adjunction unit $\text{Id}_{D(A\text{-mod})} \to DG \circ LF$ is an isomorphism of functors (already on the level of unbounded derived categories). Next, since the functor $\tilde{F}$ is exact, we have: $DG \circ L \tilde{F} \simeq D(G \circ \tilde{F})$. For any $H$-module $\tilde{N}$ we have:

$$(G \circ \tilde{F})(\tilde{N}) = \text{Hom}_B(P, \tilde{F}(\tilde{N})) = \text{Hom}_B\left(Be, \left(\frac{\tilde{N}}{\tilde{N}}\right)\right) \cong e \cdot \left(\frac{\tilde{N}}{\tilde{N}}\right) \cong \tilde{N}.$$ 

Hence, $G \circ \tilde{F}$ is isomorphic to the restriction functor $H\text{-mod} \to A\text{-mod}$, what finishes a proof of the second statement. □

Proposition 3.39. Let $A$ be a nodal order. Then the corresponding Auslander order $B$ is nodal too.

Proof. As usual, let $H$ be the hereditary cover of $A$ and $J = \text{rad}(A) = \text{rad}(H)$ be the common radical of $A$ and $H$. Consider the following orders:

$$\tilde{B} := \begin{pmatrix} A & H \\ J & H \end{pmatrix} \quad \text{and} \quad \tilde{H} := \begin{pmatrix} H & H \\ J & H \end{pmatrix}.$$ 

It is not difficult to show that

$$\tilde{J} := \text{rad}(\tilde{H}) = \begin{pmatrix} J & H \\ J & J \end{pmatrix} = \text{rad}(\tilde{B}).$$

Since $\tilde{J}$ is projective as $H$-module, $H$ is hereditary. Then the commutative diagram

$$\begin{array}{ccc}
\tilde{B}/\tilde{J} & \cong & A/J \times H/J \\
\downarrow & & \downarrow \\
\tilde{H}/\tilde{J} & \cong & H/J \times H/J
\end{array}$$

implies that $\tilde{B}$ is a nodal order and $\tilde{H}$ is its hereditary cover. Since the conductor ideal $C$ contains the radical $J$, the Auslander order $B$ is an overorder of $\tilde{B}$. It follows from Theorem 3.4 that the order $B$ is nodal, too. □

In what follows, we shall need the following result about the finite length $B$-module $S$. Assume that $R = k[w]$ and $A = A(R, (\Omega, \sigma, \approx, \text{wt}))$. Recall that

$$\tilde{A} := A/C \cong \prod_{\gamma \in \tilde{\Omega}^1_b} \tilde{A}_\gamma \cong \prod_{\gamma \in \tilde{\Omega}^1_b} \text{Mat}_{\text{wt}(\gamma)}(k).$$

It is clear that the set $\tilde{\Omega}^1_b$ also parameterizes the isomorphism classes of the simple $\tilde{A}$-modules. For any $\gamma \in \tilde{\Omega}^1_b$, let $S_\gamma$ be the simple left $B$-module which corresponds to the (unique, up to an isomorphism) simple $\tilde{A}_\gamma$-module and $P_\gamma$ be its projective cover. Then we
have: \( S \cong \bigoplus_{\gamma \in \tilde{\Omega}^+} S_{\gamma} \oplus \text{wt}(\tilde{\Omega}^+) \). Our next goal is to describe a minimal projective resolution of \( S_{\gamma} \). For any \( \omega \in \Omega \), let \( \tilde{Q}_\omega \) be the corresponding indecomposable projective left \( H \)-module and

\[
Q_\omega := \tilde{F}(\tilde{Q}_\omega) = \begin{pmatrix}
\tilde{Q}_\omega \\
Q_\omega
\end{pmatrix}
\]

be the corresponding indecomposable projective left \( B \)-module.

**Lemma 3.40.** The following statements hold.

- Let \( \omega \in \Omega \) be a reflexive element and \( \gamma = \omega_\pm \) be one of the corresponding elements of \( \tilde{\Omega}^+ \). Then

\[
0 \rightarrow Q_{\sigma(\omega)} \rightarrow P_\gamma \rightarrow S_\gamma \rightarrow 0
\]

is a minimal projective resolution of the simple \( B \)-module \( S_\gamma \). In particular, for any \( \delta \in \Omega \) we have:

\[
\text{Ext}^1_B(S_\gamma, Q_\delta) \cong \begin{cases} 
k & \text{if } \delta = \sigma(\omega) \\
0 & \text{otherwise.}
\end{cases}
\]

- Let \( \omega', \omega'' \in \Omega \) be a pair of tied elements and \( \gamma = \{\omega', \omega''\} \) be the corresponding element of \( \tilde{\Omega}^+ \). Then a minimal projective resolution of the simple \( B \)-module \( S_\gamma \) has the following form:

\[
0 \rightarrow Q_{\sigma(\omega')} \oplus Q_{\sigma(\omega'')} \rightarrow P_\gamma \rightarrow S_\gamma \rightarrow 0
\]

In particular, for any \( \delta \in \Omega \) we have:

\[
\text{Ext}^1_B(S_\gamma, Q_\delta) \cong \begin{cases} 
k & \text{if } \delta = \sigma(\omega) \text{ for } \omega \in \{\omega', \omega''\} \\
0 & \text{otherwise.}
\end{cases}
\]

**Proof.** It is not difficult to show that \( \text{rad}(P_\gamma) = Q_{\sigma(\omega)} \) for \( \gamma = \omega_\pm \) if \( \omega \in \Omega \) is reflexive and

\( \text{rad}(P_\gamma) = Q_{\sigma(\omega')} \oplus Q_{\sigma(\omega'')} \) if \( \gamma = \{\omega', \omega''\} \) for \( \omega', \omega'' \in \Omega \) tied. The formulae (45) and (47) follow from the fact that \( \text{rad}(Q_\omega) = Q_{\sigma(\omega)} \) for any \( \omega \in \Omega \).

\( \square \)

4. **Non-commutative nodal curves: global theory**

In this section, we are going to explain the construction as well as main properties of non-commutative nodal curves of tame representation type.

4.1. **The idea of a non-commutative nodal curve.** Before going to technicalities and details, let us consider the following example. Let \( k \) be a field, \( S = k[x] \), \( J = (x^2 - 1) \) and \( K = k(x) \). Consider the hereditary order

\[
H = \begin{pmatrix}
S & J & J \\
S & S & J \\
S & S & S
\end{pmatrix} \subset \mathcal{Y} := \text{Mat}_3(K).
\]
Next, consider the order

\[ A := \left\{ X \in H \mid \begin{array}{l}
X_{11}(1) = X_{22}(1) \\
X_{33}(1) = X_{33}(-1) \\
X_{21}(-1) = 0
\end{array} \right\} \subset H. \]

Let \( Z = Z(A) \) be the center of \( A \). Then we have:

\[ Z = \{ p \in S \mid p(1) = p(-1) \} = \mathbb{k}[x^2 - 1, x(x^2 - 1)] \cong \mathbb{k}[u, v]/(v^2 - u^3 - u^2). \]

The multiplication maps \( K \otimes_{\mathbb{Z}} A \rightarrow \Upsilon \) and \( K \otimes_{\mathbb{Z}} H \rightarrow \Upsilon \) are isomorphisms. In other words, \( A \) and \( H \) are \( Z \)-orders in the central simple \( \mathbb{K} \)-algebra \( \Upsilon \).

Let \( E_o = V(v^2 - u^3 - u^2) \subset \mathbb{A}^2 \) be an affine plane nodal cubic and \( s = (0,0) \in E_o \) be its unique singular point. For any \( x \in E_o \setminus \{ s \} \) we have: \( A_x = \text{Mat}_3(O_x) \), where \( O_x \) is the local ring of \( E_o \) at the point \( x \). On the other hand,

\[ \hat{A}_x = A(R, (\Omega, \sigma, \approx, \text{wt}_o)) \]

is a nodal order, where \( R = \mathbb{k}[t] \),

\[ \Omega := \{1, 2, 3, 4, 5\}, \quad \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 5 & 4 \end{pmatrix} \]

and \( 1 \approx 2, 3 \approx 5 \) and \( 4 \approx 4 \).

Let \( E = V(v^2 - u^3 - u^2) \subset \mathbb{P}^2 \) be the projective closure of \( E_o \). Then the \( Z \)-order \( A \) can be extended to a sheaf of orders \( \mathcal{A} \) on the projective curve \( E \) in such a way that the stalk of \( \mathcal{A} \) at the “infinite point” \( (0 : 1 : 0) \) of \( E \) is the maximal order (see for instance [1]).

The ringed space \( \mathfrak{E} = (E, \mathcal{A}) \) is a typical example of a projective non-commutative nodal curve of tame representation type.

Let \( H' = \begin{pmatrix} S & I & J \\ S & S & J \\ S & S & S \end{pmatrix} \), where \( J \subset I := (x - 1) \subset S \). Then \( H' \) is a hereditary order, too. Moreover, \( H' \) is the hereditary cover of the order \( A \) (the notion of the hereditary cover can be defined locally). As above, one can actually construct a sheaf of hereditary orders \( \check{\mathcal{H}} \) on \( \mathbb{P}^1 \) such that \( \mathcal{H} = \nu_*(\check{\mathcal{H}}) \), where \( \mathbb{P}^1 \overset{\nu}{\rightarrow} E \) is the normalization map. The functor \( \nu_* \) provides an equivalence between the categories of coherent sheaves on \( \check{\mathfrak{E}} \) and \( (\mathbb{P}^1, \check{\mathcal{H}}) \). In what follows, we shall consider \( \mathfrak{E} \) as the hereditary cover of the non-commutative nodal curve \( \mathfrak{E} \) what can be viewed as an appropriate non-commutative generalization of the normalization of a singular commutative nodal curve.

**Definition 4.1.** Let \( X \) be a reduced quasi-projective curve over a field \( \mathbb{k} \) and \( \mathcal{A} \) be a sheaf of orders on \( X \). Then the ringed space \( \mathfrak{X} = (X, \mathcal{A}) \) is called a *non-commutative curve*. We say that \( \mathfrak{X} \) is projective if the commutative curve \( X \) is projective. If for any point \( x \in X \) the corresponding stalk \( \mathcal{A}_x \) is a nodal order then \( \mathfrak{X} \) is a *non-commutative nodal curve*. 
Theorem 4.2. Let \( \mathbb{k} \) be an algebraically closed field, \( X \) be a quasi-projective curve over \( \mathbb{k} \), \( \mathcal{A} \) and \( \mathcal{B} \) be two sheaves of orders on \( X \) such that

\[
Z(\widehat{\mathcal{A}}_x) = \widehat{O}_x = Z(\widehat{\mathcal{B}}_x)
\]

for any \( x \in X \), where \( \widehat{O}_x \) is the completion of the local ring of \( X \) at the point \( x \) and \( Z(\widehat{\mathcal{A}}_x) \) (respectively, \( Z(\widehat{\mathcal{B}}_x) \)) is the center of \( \widehat{\mathcal{A}}_x \) (respectively, of \( \widehat{\mathcal{B}}_x \)). Assume that for any \( x \in X \) there exists a Morita equivalence

\[
\widehat{\mathcal{A}}_x \text{-mod} \xrightarrow{\Phi_x} \widehat{\mathcal{B}}_x \text{-mod}
\]

such that the following diagram

\[
\begin{array}{ccc}
\widehat{O}_x & \xrightarrow{\text{id}} & \widehat{O}_x \\
\downarrow & & \downarrow \\
Z(\widehat{\mathcal{A}}_x \text{-mod}) & \xrightarrow{\Phi_x} & Z(\widehat{\mathcal{B}}_x \text{-mod})
\end{array}
\]

is commutative, where \( \Phi_x \) denotes the induced map of centers. Then the non-commutative curves \( X = (X, \mathcal{A}) \) and \( Y = (X, \mathcal{B}) \) are Morita equivalent (i.e. the categories of quasi-coherent sheaves \( \text{QCoh}(X) \) and \( \text{QCoh}(Y) \) are equivalent).

For a proof of this result, see [15, Theorem 7.5].

4.2. Construction of non-commutative nodal curves. Let \( \mathbb{k} \) be an algebraically closed field and \( (\tilde{X}, \mathcal{O}_{\tilde{X}}) \) be a smooth quasi-projective curve over \( \mathbb{k} \).

- Let \( \tilde{X} \xrightarrow{l} \mathbb{N} \) be a function such that \( l(\tilde{x}) = 1 \) for all but finitely many points \( \tilde{x} \in \tilde{X} \) (such function will be called a length function).
- For any \( \tilde{x} \in \tilde{X} \) we put:

\[
\Pi_{\tilde{x}} := \{ (\tilde{x}, 1), \ldots, (\tilde{x}, l(\tilde{x})) \} \quad \text{and} \quad \Pi := \bigcup_{\tilde{x} \in \tilde{X}} \Pi_{\tilde{x}}.
\]

- For any function \( \Pi \xrightarrow{\text{wt}} \mathbb{N} \) and any point \( \tilde{x} \in \tilde{X} \) we denote:

\[
\tilde{p}(\tilde{x}) := (\text{wt}(\tilde{x}, 1), \ldots, \text{wt}(\tilde{x}, l(\tilde{x}))) \quad \text{and} \quad m(\tilde{x}) := |\tilde{p}(\tilde{x})| := \sum_{i=1}^{l(\tilde{x})} \text{wt}(\tilde{x}, i).
\]

- We say that \( \text{wt} \) is a weight function compatible with the given length function \( l \) if \( m(\tilde{x}') = m(\tilde{x}'') \) for any pair of points \( \tilde{x}', \tilde{x}'' \in \tilde{X} \) belonging to the same irreducible component of \( \tilde{X} \).
- For \( \tilde{x} \in \tilde{X} \), let \( O_{\tilde{x}} \) be the stalk of the structure sheaf \( \mathcal{O}_{\tilde{X}} \) at the point \( \tilde{x} \). Let

\[
\tilde{H}_{\tilde{x}} := H(O_{\tilde{x}}, \tilde{p}(\tilde{x})) \subseteq \text{Mat}_{m(\tilde{x})}(O_{\tilde{x}})
\]

be the standard hereditary order defined by [13].

Definition 4.3. Assume (for simplicity of notation) that \( \tilde{X} \) is connected, \( \tilde{X} \xrightarrow{l} \mathbb{N} \) be a length function and \( \text{wt} \) be a weight function. Let \( m = m(\tilde{x}) \) for some (hence for any) point \( \tilde{x} \in \tilde{X} \). Then we define the sheaf of hereditary orders \( \tilde{\mathcal{H}} = \tilde{\mathcal{H}}(l, \text{wt}) \) on the curve \( \tilde{X} \) as...
the subsheaf of the sheaf of maximal orders $\text{Mat}_m(\mathcal{O}_{\widetilde{X}})$ such that $\widetilde{H}_{\widetilde{X}} = \tilde{H}_{\tilde{X}}$ for all $\tilde{x} \in \tilde{X}$. The corresponding ringed space $\widetilde{X} := (\tilde{X}, \tilde{H})$ is a non-commutative hereditary curve over the field $\mathbb{k}$ defined by the datum $(\tilde{X}, l, \text{wt})$.

**Theorem 4.4.** Let $\tilde{X}$ be a smooth quasi-projective curve over $\mathbb{k}$ and $\tilde{X} \overset{l}{\to} \mathbb{N}$ be a length function. Let $\text{wt}, \text{wt}' : \Pi \to \mathbb{N}$ be two weight functions compatible with $l$ and $\tilde{X}$ and $\tilde{X}'$ be the corresponding non-commutative hereditary curves. Then the categories $\text{QCoh}(\tilde{X})$ and $\text{QCoh}(\tilde{X}')$ (respectively, $\text{Coh}(\tilde{X})$ and $\text{Coh}(\tilde{X}')$) are equivalent. Let $\tilde{Y}$ be a smooth quasi-projective curve over $\mathbb{k}$, $\tilde{Y} \overset{l}{\to} \mathbb{N}$ be a length function and $\tilde{Y}$ be the corresponding non-commutative hereditary curve. Then $\tilde{X}$ and $\tilde{Y}$ are Morita equivalent if and only if there exists an isomorphism (of commutative curves) $\tilde{X} \overset{f}{\to} \tilde{Y}$ such that the following diagram

$$
\begin{array}{ccc}
\tilde{X} & \xrightarrow{f} & \tilde{Y} \\
\downarrow l & & \downarrow l \\
\mathbb{N} & & \mathbb{N}
\end{array}
$$

is commutative. In other words, the Morita type of a non-commutative hereditary curve does not depend on the choice of a weight function $\text{wt}$ and is determined by the underlying commutative curve $\tilde{X}$ and length function $l$.

**Comment to the proof.** This result is a special case of Theorem 4.2.

**Remark 4.5.** Let $\tilde{X} = \mathbb{P}^1$, $\tilde{X} \overset{l}{\to} \mathbb{N}$ be a length function, $\Pi \overset{\text{wt}}{\to} \mathbb{N}$ be a weight functions compatible with $l$ and $\tilde{X}$ be the corresponding hereditary curve. Then $\tilde{X}$ can be identified with an appropriate weighted projective line of Geigle and Lenzing [23] in the sense that the categories of (quasi-)coherent sheaves on both objects are equivalent (see, for instance, the paper [41]).

Let us choose homogeneous coordinates on $\mathbb{P}^1$ and put: $\tilde{o}^+ := (0 : 1), \tilde{o}^- := (1 : 0)$ and $\tilde{o} := (1 : 1)$. In what follows, we shall use the following notation.

- $\mathbb{P}^1(n_+, n_-)$ is the weighted projective line corresponding to the length function $l(\tilde{x}) = \begin{cases} n_+ & \text{if } \tilde{x} = \tilde{o}^+ \\ 1 & \text{otherwise.} \end{cases}$

- $\mathbb{P}^1(n_+, n_-, n)$ is the weighted projective line corresponding to the length function $l(\tilde{x}) = \begin{cases} n_+ & \text{if } \tilde{x} = \tilde{o}^+ \\ n & \text{if } \tilde{x} = \tilde{o}^- \\ 1 & \text{otherwise,} \end{cases}$

where we additionally assume that $n_+ \geq 2$.

**Definition 4.6.** Let $\tilde{X}$ be a smooth quasi-projective curve over $\mathbb{k}$ and $\tilde{X} \overset{l}{\to} \mathbb{N}$ be a length function. Let $\approx$ be a relation on the set $\Pi$ defined by (48) such that
• For any \( \omega \in \Pi \) there exists at most one \( \omega' \in \Pi \) such that \( \omega \approx \omega' \) (such elements \( \omega, \omega' \) will be called special).

• There are only finitely many special elements in \( \Pi \).

Non-special elements of \( \Pi \) will be called simple. The set of special elements of \( \Pi \) will be denoted by \( \Pi_0 \). An element \( \omega \in \Pi_0 \) is called reflexive if \( \omega \approx \omega \) and tied if \( \omega \approx \omega' \) for some \( \omega \neq \omega' \).

Similarly to Definition 3.5 we define the set \( \Pi_\dagger \) by replacing each reflexive element \( \omega \in \Pi \) by two new simple elements \( \omega_+ \) and \( \omega_- \). The pairs of tied elements of \( \Pi_\dagger \) are the same as for \( \Pi \).

Let \( \Pi_\dagger \xrightarrow{\text{wt}_\dagger} \mathbb{N} \) be a function such that \( \text{wt}_\dagger(\omega') = \text{wt}_\dagger(\omega'') \) for all \( \omega' \approx \omega'' \in \Pi_\dagger \). Then we define the map \( \Pi \xrightarrow{\text{wt}} \mathbb{N} \) by the following rule:

\[
\text{wt}(\omega) := \begin{cases} 
\text{wt}_\dagger(\omega_+) + \text{wt}_\dagger(\omega_-) & \text{if } \omega \in \Pi \text{ is reflexive} \\
\text{wt}_\dagger(\omega) & \text{otherwise} 
\end{cases}
\]

We call such a relation \( \approx \) on the set \( \Pi \) admissible if there exists a function \( \Pi_\dagger \xrightarrow{\text{wt}_\dagger} \mathbb{N} \) for which the corresponding function \( \Pi \xrightarrow{\text{wt}} \mathbb{N} \) is a weight function compatible with the length function \( l \). Abusing the notation, we shall drop the symbol \( \dagger \) in the notation of \( \text{wt}_\dagger \) and write \( \text{wt} \) for all weight functions introduced above.

We say that two points \( \tilde{x}' \neq \tilde{x}'' \in \tilde{X} \) are tied if there are \( \omega' \in \Omega_{\tilde{x}'} \) and \( \omega'' \in \Omega_{\tilde{x}''} \) such that \( \omega' \approx \omega'' \). Let

\[
(49) \quad \tilde{Z} := \left\{ \tilde{x} \in \tilde{X} \mid \text{there exists } \tilde{y} \in \tilde{X} \setminus \{ \tilde{x} \} \text{ such that } \tilde{x} \text{ and } \tilde{y} \text{ are tied} \right\}
\]

be the set of tied points of \( \tilde{X} \). Taking the transitive closure, we get an equivalence relation \( \sim \) on \( \tilde{Z} \). We put: \( Z := \tilde{Z} / \sim \). In what follows, we shall also consider \( Z \) as a reduced subscheme of \( \tilde{X} \), \( Z \) as a reduced scheme over \( k \) and the projection map \( \tilde{Z} \xrightarrow{\tilde{\nu}} Z \) as a morphism of schemes.

Given an admissible datum \((\tilde{X}, l, \approx)\), we define a quasi-projective curve \( X \) requiring the following diagram of algebraic schemes

\[
\begin{array}{ccc}
\tilde{Z} & \xrightarrow{\tilde{\eta}} & \tilde{X} \\
\downarrow{\tilde{\nu}} & & \downarrow{\nu} \\
Z & \xrightarrow{\eta} & X.
\end{array}
\]

to be cartesian. In other words, the curve \( X \) is obtained from \( \tilde{X} \) by gluing transversally the equivalent points. It is clear that \( X \) is singular provided \( \tilde{Z} \) is non-empty and that \( \tilde{X} \xrightarrow{\nu} X \) is the normalization map. It always exists, as follows from [47].

We put: \( \mathcal{H} := \nu_*(\tilde{\mathcal{H}}) \). For any \( x \in X \) (respectively, \( \tilde{x} \in \tilde{X} \)) let \( \tilde{H}_x \) (respectively, \( \tilde{H}_{\tilde{x}} \)) be the radical completion of \( \mathcal{H}_x \) (respectively, \( \tilde{\mathcal{H}}_{\tilde{x}} \)). Note that in the notation of [13] we
have:
\[
\hat{H}_x := H(\hat{O}_x, \bar{p}(\tilde{x})),
\]
where \(\hat{O}_x\) is the completion of the local ring \(O_x\). It is clear that \(\hat{H}_x\) is also an order over the local ring \(\hat{O}_x\), which is the completion of the local ring of the structure sheaf of \(X\) at the point \(x\).

Assume now that \(x \in Z\) and \(\nu^{-1}(x) = \{\tilde{x}_1, \ldots, \tilde{x}_r\}\). Then we have a canonical isomorphism
\[
\hat{H}_x \xrightarrow{\sim} \hat{H}_{\tilde{x}_1} \times \cdots \times \hat{H}_{\tilde{x}_r}.
\]

Next, we put:
\[
\Omega_x := \Omega_{\tilde{x}_1} \cup \cdots \cup \Omega_{\tilde{x}_r}.
\]
Then we have a permutation \(\sigma_x : \tilde{x} \mapsto (\tilde{x}, i + 1 \mod l(\tilde{x}))\) for any \(\tilde{x} \in \{\tilde{x}_1, \ldots, \tilde{x}_r\}\). In the terms of Definition 3.10 we put:
\[
\hat{A}_x := A(\Omega_x, \sigma_x, \approx, wt) \subset \hat{H}_x.
\]
Then \(\hat{A}_x\) is a nodal order and \(\hat{H}_x\) is its hereditary cover. Moreover, the center of \(\hat{A}_x\) contains the local ring \(\hat{O}_x\).

**Definition 4.7.** We define the sheaf of orders \(\mathcal{A}\) on the curve \(X\) to be the subsheaf of \(\mathcal{H}\) satisfying the following conditions on the stalks:
\[
\hat{\mathcal{A}}_x := \begin{cases} 
\hat{H}_x & \text{if } x \notin Z \\
\hat{A}_x & \text{if } x \in Z.
\end{cases}
\]
We call the ringed space \(\mathbb{X} = (X, \mathcal{A})\) the non-commutative nodal curve attached to the datum \((\tilde{X}, l, \approx, wt)\). The ringed space \(\tilde{\mathbb{X}} = (X, \mathcal{H})\) will be called the hereditary cover of \(\mathbb{X}\).

Note that for \(\tilde{\mathbb{X}} : = (\tilde{X}, \tilde{\mathcal{H}})\) we have a natural morphism of ringed spaces \(\tilde{\mathbb{X}} \xrightarrow{\nu} \mathbb{X}\), which induces an equivalence of categories \(\text{Coh}(\tilde{\mathbb{X}}) \xrightarrow{\sim} \text{Coh}(\mathbb{X})\).

**Theorem 4.8.** Let \((\tilde{X}, l, \approx)\) be an admissible datum, \(\Pi^\approx \xrightarrow{wt} \mathbb{N}\) be any compatible weight function and \(\mathbb{X}\) be the corresponding non-commutative nodal curve. Then the following results hold.

- Let \(\Pi^\approx \xrightarrow{wt'} \mathbb{N}\) be any other compatible weight function and \(\mathbb{X}'\) be the corresponding non-commutative nodal curve. Then the categories of quasi-coherent sheaves \(\text{QCo}(\mathbb{X})\) and \(\text{QCo}(\mathbb{X}')\) are equivalent. That is why we often do not mention the weight \(wt\) and say that \(\mathbb{X}\) is attached to the admissible datum \((\tilde{X}, l, \approx)\).

- Let \(\approx'\) be another equivalence relation on \(\Pi\) and \(\Pi^\approx' \xrightarrow{wt'} \mathbb{N}\) be a weight function compatible with \(\approx'\). Suppose that for any \(\tilde{x} \in \tilde{X}\) there exists a cyclic permutation \(\Pi_{\tilde{x}} \xrightarrow{f_x} \Pi_{\tilde{x}}\) such that the diagram

\[
\begin{array}{ccc}
\Pi_{\tilde{x}} & \xrightarrow{f_x} & \Pi_{\tilde{x}} \\
\downarrow{wt} & & \downarrow{wt'} \\
\mathbb{N} & & \mathbb{N}
\end{array}
\]
is commutative. Then the categories $\text{QCoh}(\mathcal{X})$ and $\text{QCoh}(\mathcal{X}')$ are equivalent.

**Comment to the proof.** This result is a consequence of Theorem 4.2 and Theorem 3.11.

**Example 4.9.** Let $(\tilde{X}, l, \approx)$ be such that for any $\tilde{x} \in \tilde{X}$ with $l(\tilde{x}) \geq 2$, the set $\Pi_{\tilde{x}}$ contains a non-tied element. Then the datum $(\tilde{X}, l, \approx)$ is admissible.

**Example 4.10.** Let $\tilde{X}$ be any curve and $\tilde{x}_1 \neq \tilde{x}_2 \in \tilde{X}$ be two distinct points. Define a weight function $\mathbb{P}^1 \to \mathbb{N}$ by the rule:

$$l(\tilde{x}) = \begin{cases} 2 & \text{if } \tilde{x} = \tilde{x}_1 \\ 1 & \text{otherwise.} \end{cases}$$

Let $\approx$ be given by the rule: $(\tilde{x}_1, 1) \approx (\tilde{x}_2, 1)$. Then the datum $(\tilde{X}, l, \approx)$ is not admissible.

### 4.3. Non-commutative nodal curves of tame representation type.

In this subsection we recall, following the paper [21], the description of those non-commutative projective nodal curves $\mathcal{X} = (X, \mathcal{A})$ for which the category $\mathcal{VB}(\mathcal{X})$ of vector bundles (i.e. of locally projective coherent $\mathcal{A}$-modules) has tame representation type. Let $\tilde{X} = \tilde{X}_1 \sqcup \cdots \sqcup \tilde{X}_r$ be a disjoint union of $r$ projective lines. We choose homogeneous coordinates on each component $\tilde{X}_k$ and define points $\tilde{o}_k, \tilde{o}_k^+, \tilde{o}_k^- \in \tilde{X}_k$ setting: $\tilde{o}_k := (1 : 1), \tilde{o}_k^+ := (0 : 1)$ and $\tilde{o}_k^- := (1 : 0)$. Assume that $(\tilde{X}, l, \approx)$ is an admissible datum defining a non-commutative nodal curve $\mathcal{X}$. For each $1 \leq k \leq r$, let $\Sigma_k \subset \tilde{X}_k$ be the corresponding set of special points.

- If $|\Sigma_k| = 2$, we may without loss of generality assume that $\Sigma_k = \{\tilde{o}_k^+, \tilde{o}_k^-\}$.
- Similarly, if $|\Sigma_k| = 3$, we assume that $\Sigma_k = \{\tilde{o}_k, \tilde{o}_k^+, \tilde{o}_k^-\}$.

The following result is proved in [21].

**Theorem 4.11.** Let $\mathcal{X}$ be a non-commutative projective curve. Then $\mathcal{VB}(\mathcal{X})$ has tame representation type if and only if the following conditions are satisfied.

- $\mathcal{X}$ is Morita equivalent to a commutative elliptic curve, i.e. $\tilde{X}$ is an elliptic curve, while $l$ and $\approx$ are trivial.
- $\mathcal{X}$ is the rational non-commutative nodal curve attached to an admissible datum $(\tilde{X}, l, \approx)$ such that $\tilde{X} = \tilde{X}_1 \sqcup \cdots \sqcup \tilde{X}_r$ is a disjoint union of $r$ projective lines, whereas $(l, \approx)$ satisfies the following conditions:
  - For any $1 \leq k \leq r$ we have: $|\Sigma_k| \leq 3$.
  - If $|\Sigma_k| = 3$ then we additionally have: the set $\Pi_{\tilde{x}}$ contains special elements for precisely one point $\tilde{x} \in \Sigma_k$ (say, for $\tilde{x} = \tilde{o}$), whereas for the remaining two points of $\Pi_{\tilde{x}}$ (say, for $\tilde{o}^\pm$) we have: $l(\tilde{o}^\pm) = 2$.

**Definition 4.12.** Consider the pair $(\vec{p}, \vec{q})$, where

$$\vec{p} = (p_1^+, p_1^-), \ldots, (p_t^+, p_t^-) \in (\mathbb{N}^2)^t \quad \text{and} \quad \vec{q} = (q_1, \ldots, q_s) \in \mathbb{N}^s$$

for some $t, s \in \mathbb{N}_0$ (either of this tuples may be empty). Let

$$\tilde{X} := \tilde{X}_1 \sqcup \cdots \sqcup \tilde{X}_t \sqcup \tilde{X}_{t+1} \sqcup \cdots \sqcup \tilde{X}_{t+s}$$
be a disjoint union of $t + s$ projective lines. We define the weight function $\tilde{X} \to \mathbb{N}$ by the following rules

- For each $1 \leq k \leq t$ we put: $l(\tilde{o}^+_{k}) = p^+_k$.
- For each $1 \leq k \leq s$ we put: $l(\tilde{o}^+_{t+k}) = 2$ and $l(\tilde{o}^+_{t+k}) = q_k$.

Let $\approx$ be a relation on the set $\left(\Pi_{\tilde{o}_1^+} \cup \Pi_{\tilde{o}_1^-}\right) \cup \cdots \cup \left(\Pi_{\tilde{o}_t^+} \cup \Pi_{\tilde{o}_t^-}\right)$, satisfying the conditions of Definition 4.12. If $(\tilde{p}, \tilde{q}, \approx)$ is admissible and $\text{wt}$ is a compatible weight, we denote by $\mathcal{X}(\tilde{p}, \tilde{q}, \approx)$ the corresponding non-commutative nodal rational projective curve. Since the weight $\text{wt}$ does not imply the derived category, we often omit it and write $\mathcal{X}(\tilde{p}, \tilde{q}, \approx)$.

One can rephrase Theorem 4.11 in the following way.

**Theorem 4.13.** The category $\text{VB}(\mathcal{X})$ of vector bundles on a non-commutative projective curve $\mathcal{X}$ is representation tame if and only if $\mathcal{X}$ is either a commutative elliptic curve or a non-commutative nodal curve $\mathcal{X}(\tilde{p}, \tilde{q}, \approx)$, where $(\tilde{p}, \tilde{q}, \approx)$ is an admissible datum as in Definition 4.12.

**Remark 4.14.** Let $\tilde{p} = ((2, 2), (2, 2))$, $\tilde{q}$ be void and $\approx$ be given by the following rule: $(\tilde{o}_1^+, 1) \approx (\tilde{o}_k^-, 1)$ for $k = 1, 2$ and $(\tilde{o}_2^+, 2) \approx (\tilde{o}_2^-, 2)$. Then the central curve $X$ of the corresponding non-commutative nodal curve $\mathcal{X}(\tilde{p}, \approx)$ is given by the following Cartesian diagram:

$$
\begin{array}{ccc}
\text{Spec}(\kappa) \cup \text{Spec}(\kappa) & \xrightarrow{i} & E \sqcup E \\
\downarrow & & \downarrow \\
\text{Spec}(\kappa) & \xrightarrow{\iota} & X
\end{array}
$$

where $E \cong \overline{V(u^2 - u^3 - u^2)} \subset \mathbb{P}^2$ is a plane nodal cubic and the image of $i$ of the singular set of $E \sqcup E$. Note that the arithmetic genus of $X$ is two. In fact, the central curve of a tame non-commutative nodal curve can have arbitrary high arithmetic genus.

**4.4. Remarks on stacky cycles of projective lines.** In this subsection, let $\kappa$ be an algebraically closed field of characteristic zero.

**Example 4.15.** Let $E$ be a plane nodal cubic and $\mathbb{P}^1 \xrightarrow{\nu} E$ be its normalization. Let us choose homogeneous coordinates $(z : w)$ on $\mathbb{P}^1$ in such a way that $\nu^{-1}(s) = \{\tilde{o}^+, \tilde{o}^-, \hat{\tilde{o}}\}$, where $s$ is the singular point of $E$ and $\tilde{o}^+ = (0 : 1)$ and $\tilde{o}^- = (1 : 0)$. Consider the action of the cyclic group $G := \langle \rho | \rho^2 = e \rangle$ on $\mathbb{P}^1$ given by the formula $(z : w) \mapsto (\zeta z : w)$, where $\zeta$ is a primitive $n$-th root of unity. It is clear that the action of $G$ on $\mathbb{P}^1$ descends to an action of $G$ on $E$ such that $E' := E/G \cong E$. Let $A := \mathcal{O}_{E'} \ast G$ be the sheaf on $E'$ defined by the following rule:

$$
U \mapsto \Gamma(\pi^{-1}(U), \mathcal{O}_E) \ast G \quad \text{for any open } U \subseteq E',
$$
where \( E \xrightarrow{\pi} E' \) is the projection map. Then \( \mathcal{A} \) is a sheaf of nodal orders on the projective curve \( E' \). For any \( x \in E' \setminus \{s\} \), the order \( \mathcal{A}_x \) is maximal, whereas \( \tilde{\mathcal{A}}_s \) is the nodal order given by (32).

The following result is obvious.

**Lemma 4.16.** The category \( \text{Coh}^G(E) \) of \( G \)-equivariant sheaves on \( E \) is equivalent to the category \( \text{Coh}(\mathcal{E}) \), where \( \mathcal{E} = (E', \mathcal{A}) \).

**Remark 4.17.** The non-commutative nodal curve \( \mathcal{E} \) admits the following description. Consider the length function \( \mathbb{P}^1 \xrightarrow{l} \mathbb{N} \) given by the rule:

\[
l(\tilde{x}) = \begin{cases} n & \text{if } \tilde{x} \in \{\tilde{\partial}^+, \tilde{\partial}^-\} \\
1 & \text{otherwise}.
\end{cases}
\]

We then define the relation \( \approx \) on the set \( \Pi \) setting \( (\tilde{\partial}^+, k) \approx (\tilde{\partial}^-, n - k) \) for \( 1 \leq k \leq n \), where we replace 0 by \( n \). It is easy to see that the datum \( (\mathbb{P}^1, l, \approx) \) is admissible. Using Theorem 4.2 one can conclude that \( \mathcal{E} \) and the non-commutative nodal curve corresponding to the datum \( (\mathbb{P}^1, l, \approx) \) are Morita equivalent.

**Example 4.18.** Again, let \( E = \overline{V(v^2 - u^3 - u^2)} \subset \mathbb{P}^2 \) be a plane nodal cubic. Consider the involution \( E \xrightarrow{\tau} E \) given by the rule \( (u, v) \mapsto (u, -v) \). Let \( E \xrightarrow{\pi} E/G \cong \mathbb{P}^1 \) be the projection map, where \( G = \langle \tau \rangle \cong \mathbb{Z}_2 \). Next, let \( \mathcal{A} := \mathcal{O}_E * G \) and \( \mathcal{E} = (\mathbb{P}^1, \mathcal{A}) \) be the corresponding non-commutative nodal curve. Again, the categories \( \text{Coh}^G(E) \) and \( \text{Coh}(\mathcal{E}) \) are equivalent. Let us choose homogeneous coordinates on \( \mathbb{P}^1 \) in such a way that \( \pi^{-1}(\tilde{\partial}^+) \) is the singular point of \( E \) and \( \pi^{-1}(\tilde{\partial}^-) \) is its infinite point. Consider the length function \( \mathbb{P}^1 \xrightarrow{l} \mathbb{N} \) given by the rule:

\[
l(\tilde{x}) = \begin{cases} 2 & \text{if } \tilde{x} = \tilde{\partial}^- \\
1 & \text{otherwise}.
\end{cases}
\]

We define the relation \( \approx \) on the set \( \Pi \) by setting \( (\tilde{\partial}^+, 1) \approx (\tilde{\partial}^+, 1) \). Obviously, the datum \( (\mathbb{P}^1, l, \approx) \) is admissible. According to Lemma 3.25 and Theorem 4.2 the datum \( (\mathbb{P}^1, l, \approx) \) defines a non-commutative nodal curve, which is Morita equivalent to \( \mathcal{E} \).

**Example 4.19.** In this example, we give a description of stacky cycles of projective lines used in the paper of Lekili and Polishchuk [35] in the language of non-commutative nodal curves. Let \( r \in \mathbb{N}, \vec{n} = (n_1, \ldots, n_r) \in \mathbb{N}^r \) and \( \vec{c} = (c_1, \ldots, c_r) \in \mathbb{N}^r \) be such that \( \gcd(n_k, c_k) = 1 \) for any \( 1 \leq k \leq r \). Let \( E_r \) be a cycle of \( r \) projective lines and \( \bar{X} \xrightarrow{\pi} E_r \) be its normalization. Then \( \bar{X} = \bar{X}_1 \sqcup \cdots \sqcup \bar{X}_r \) is a disjoint union of \( r \) projective lines. Let \( \{o_1, \ldots, o_r\} \) be the set of singular points of \( E_r \), where we choose their labeling in such a way that \( \pi^{-1}(o_k) = \{\tilde{o}_k^-, \tilde{o}_{k+1}^+\} \), where \( \tilde{o}_k^- = (1 : 0) \in \bar{X}_k \) and \( \tilde{o}_{k+1}^+ = (0 : 1) \in \bar{X}_{k+1} \). The completion of the local ring of \( E_r \) at each point \( o_k \) is isomorphic to the commutative nodal ring \( k[u, v]/(uv) \). For any \( 1 \leq k \leq n \), consider the action of the cyclic group \( G_{n_k} = \langle \rho | \rho^{n_k} = e \rangle \) on \( D = k[u, v]/(uv) \) given by the rule

\[
\begin{align*}
\rho \circ u &= \zeta_k u \\
\rho \circ v &= \zeta_k^v v,
\end{align*}
\]
where \( \zeta_k \) is some primitive \( n_k \)-th root of 1. The category of coherent sheaves \( \text{Coh}(E) \) on a stacky cycle of projective lines \( E := E_r(\vec{n}, \vec{c}) \) is an abelian category satisfying the following property: the category \( \text{Tor}(E) \) of finite length objects of \( \text{Coh}(E) \) splits into a direct sum of blocks:

\[
\text{Tor}(E) \cong \bigcup_{x \in E_r} \text{Tor}_x(E),
\]

where \( \text{Tor}_x(E) \) is equivalent to the category of finite length \( k[w] \)-modules if \( x \in E_r \) smooth and to the category of finite length \( D \ast G_{n_k} \)-modules if \( x = o_k \), where the action of \( G_{n_k} \) is given by the rule \((53)\).

The stacky cycle of projective lines \( E_r(\vec{n}, \vec{c}) \) can be understood as an appropriate cyclic gluing of weighted projective lines \( \mathbb{P}^1(n_1, n_2), \ldots, \mathbb{P}^1(n_r-1, n_r), \mathbb{P}^1(n_r, n_1) \). Now let us proceed with a formal definition of \( E_r(\vec{n}, \vec{c}) \) viewed as a non-commutative nodal curve. As above, let \( \tilde{X} \) be a disjoint union of \( r \) projective lines. Consider the length function \( \tilde{X} \xrightarrow{l} \mathbb{N} \) given by the rule:

\[
l(\tilde{x}) = \begin{cases} 
n_k & \text{if } \tilde{x} \in \{\tilde{o}_k^+, \tilde{o}_k^+\} 
1 & \text{otherwise.}
\end{cases}
\]

Let \( \Pi_k^\pm = \Pi_k^\mp \). It is convenient to use the identification

\[
\Pi_k^- = \{1, \ldots, \tilde{n}_k\} = \Pi_k^+ \cap \mathbb{N},
\]

given by the rule: \((\tilde{o}_k^-, j) = \tilde{j} = (\tilde{o}_{k+1}^+, j)\) for \(1 \leq j \leq n_k\). We have a bijection

\[
\Pi_k^- \xrightarrow{\tau_k} \Pi_{k+1}^+, \quad \tilde{j} \mapsto c_k \cdot \tilde{j}.
\]

Let \( \approx \) be a relation on the set \( \Pi \), given by the rule: \((\tilde{o}_k^-, j) \approx (\tilde{o}_{k+1}^+, \tau_k(j))\). Note that the set \( \Pi \) does not contain reflexive elements, hence \( \Pi^I = \Pi \) in this case.

Next, we claim that the datum \((\tilde{X}, l, \approx)\) is admissible. Indeed, let \( m := n_1 \ldots n_r \). Then we can define a compatible weight function \( \Pi \xrightarrow{\text{wt}} \mathbb{N} \) by the following rules:

\[
\text{wt}(\tilde{x}, j) = \begin{cases} 
\frac{m}{n_k-1} & \text{if } \tilde{x} = \tilde{o}_k^+ \text{ and } j \leq n_k-1, \\
\frac{m}{n_k} & \text{if } \tilde{x} = \tilde{o}_k^- \text{ and } 1 \leq j \leq n_k, \\
m & \text{otherwise.}
\end{cases}
\]

Then the stacky cycle of projective lines \( E_r(\vec{n}, \vec{c}) \) can be defined as the non-commutative nodal curve corresponding to the datum \((\tilde{X}, l, \approx, \text{wt})\). Recall that, up to Morita equivalence, the non-commutative nodal curve \( E_r(\vec{n}, \vec{c}) \) does not depend on the choice of the weight function \( \text{wt} \); see Theorem \(4.2\).
4.5. Auslander curve of a non-commutative nodal curve.

Definition 4.20. Let $X = (X, A)$ be a non-commutative nodal curve and $\mathcal{H}$ be the hereditary cover of $A$. The conductor ideal sheaf $\mathcal{C}$ is defined as follows: for any open subset $U \subseteq X$ we put:

$$\Gamma(U, \mathcal{C}) := \left\{ f \in \Gamma(U, A) \mid fg \in \Gamma(U, A) \text{ for all } g \in \Gamma(U, \mathcal{H}) \right\}.$$  

Lemma 4.21. The following results are true.

- The canonical morphism of $O_X$-modules $\mathcal{C} \to \text{Hom}_A(\mathcal{H}, A)$ given on the level of local sections by the rule (36) is an isomorphism.
- $\mathcal{C}$ is a sheaf of two-sided ideals both in $A$ and $\mathcal{H}$.
- The $\kappa$-algebras $\tilde{A} := \Gamma(X, A/\mathcal{C})$ and $\tilde{H} := \Gamma(X, \mathcal{H}/\mathcal{C})$ are finite dimensional and semisimple.

Proof. All statements follow directly from the corresponding local statements; see Proposition 3.29 and Corollary 3.30. □

Definition 4.22. For a nodal curve $X = (X, A)$ as above, we call the sheaf of orders

$$B := \begin{pmatrix} A & H \\ C & \mathcal{H} \end{pmatrix}$$

the Auslander sheaf of $A$. The corresponding non-commutative nodal curve $Y := (X, B)$ is called the Auslander curve of $X$.

Consider the following idempotent sections $e := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $f := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in \Gamma(X, B)$.

Then we get the following sheaves of locally projective left $B$-modules:

$$P := Be \cong \begin{pmatrix} A \\ C \end{pmatrix} \text{ and } Q := Bf \cong \begin{pmatrix} \mathcal{H} \\ \mathcal{H} \end{pmatrix}.$$  

We denote $P^\vee := \text{Hom}_B(P, B) \cong eB$ and $Q^\vee := \text{Hom}_B(Q, B) \cong fB$.

Similarly to the local case, we introduce the following functors.

- An exact functor $G := \text{Hom}_B(P, -) \cong P^\vee \otimes_B -$ from $\text{QCoh}(Y)$ to $\text{QCoh}(X)$.
- Similarly, we have an exact functor $\tilde{G} = \text{Hom}_B(Q, -) \cong Q^\vee \otimes_B -$ from $\text{QCoh}(Y)$ to $\text{QCoh}(\tilde{X})$.
- We have functors $F := P \otimes_A -$ and $H := \text{Hom}_A(P^\vee, -)$ from $\text{QCoh}(X)$ to $\text{QCoh}(Y)$.
- Similarly, we have functors $\tilde{F} := Q \otimes \mathcal{H} -$ and $\tilde{H} := \text{Hom}_H(Q^\vee, -)$ from $\text{QCoh}(\tilde{X})$ to $\text{QCoh}(\tilde{Y})$.
- Let $D_G, D_{\tilde{G}}, LF, L\tilde{F}, RH$ and $R\tilde{H}$ be the corresponding derived functors.

Remark 4.23. It is clear that all functors $G, \tilde{G}, F, \tilde{F}, H$ and $\tilde{H}$ can be restricted to the corresponding subcategories of coherent sheaves. The functor $\tilde{F}$ is exact and transforms locally projective $\mathcal{H}$-modules to locally projective $B$-modules.
All statements and proofs of Theorem 2.4 and Corollary 3.38 can be generalized to the global setting in a straightforward way. In particular, we have the following results.

**Theorem 4.24.** Let \( X = (X, A) \) be a non-commutative nodal curve as in Definition 4.7, \( \tilde{X} = (X, H) \) be its hereditary cover and \( Y = (X, B) \) be the corresponding Auslander curve. Then the following results are true.

- We have: \( \text{gl.dim}(\text{Coh}(Y)) = 2 \).
- We have a recollement diagram

\[
\begin{array}{c}
\text{D}^b(\tilde{A}\text{-mod}) \xrightarrow{I} \text{D}^b(\text{Coh}(Y)) \xrightarrow{\text{DG}} \text{D}^b(\text{Coh}(\tilde{X})).
\end{array}
\]

Here, the exact functor \( I \) is determined by the rule \( I(\tilde{A}) = S \), where \( S \) is given by the locally projective resolution

\[
0 \rightarrow \left( \begin{array}{c} C \\ C \end{array} \right) \rightarrow \left( \begin{array}{c} A \\ C \end{array} \right) \rightarrow S \rightarrow 0.
\]

In particular, we have a semi-orthogonal decomposition

\[
\text{D}^b(\text{Coh}(Y)) = \left\langle \text{Im}(I), \text{Im}(\tilde{L}F) \right\rangle = \left\langle \text{D}^b(\tilde{A}\text{-mod}), \text{D}^b(\text{Coh}(\tilde{X})) \right\rangle.
\]

- Moreover, we have the following commutative diagram of categories and functors:

\[
\begin{array}{c}
\text{D}^b(\text{Coh}(\tilde{X})) \xrightarrow{\nu^*} \text{D}^b(\text{Coh}(Y)) \xrightarrow{\text{LG}} \text{D}^b(\text{Coh}(X)) \xrightarrow{E} \text{Perf}(X)
\end{array}
\]

where \( \text{Perf}(X) \) is the perfect derived category of coherent sheaves on \( X \), \( E \) is the canonical inclusion functor, \( \text{LG} \) and \( \tilde{L}F \) are fully faithful, \( \text{DG} \) is an appropriate localization functor and \( \nu^* \) is the functor induced by the “normalization map” \( \tilde{X} \xrightarrow{\nu} X \).

5. **Tilting on rational non-commutative nodal projective curves**

5.1. **Tilting on projective hereditary curves.** We begin with a brief description of the standard tilting bundle on a weighted projective line due to Geigle and Lenzing [23] reexpressed in the language of non-commutative hereditary curves. Let \( \tilde{X} = \mathbb{P}^1 \) and \( \tilde{X} \xrightarrow{l} \mathbb{N} \) be any length function. Let us fix any weight function \( \Pi \xrightarrow{\text{wt}} \mathbb{N} \) compatible with \( l \). As usual, we put

\[
\tilde{p}(\tilde{x}) := (\text{wt}(\tilde{x}, 1), \ldots, \text{wt}(\tilde{x}, l(\tilde{x}))) \quad \text{for any} \quad \tilde{x} \in \tilde{X}.
\]

Let \( m := |\tilde{p}(\tilde{x})| \) for some (hence for any) point \( \tilde{x} \in \tilde{X} \) and \( \tilde{H}_{\tilde{x}} := H(O_{\tilde{x}}, \tilde{p}(\tilde{x})) \subseteq \text{Mat}_m(O_{\tilde{x}}) \) be the standard hereditary order, defined by the vector \( \tilde{p}(\tilde{x}) \). Next, let \( \tilde{Q}_{(\tilde{x},1)}, \ldots, \tilde{Q}_{(\tilde{x},l(\tilde{x}))} \)
be the standard indecomposable projective left $\widetilde{H}_x$-modules, i.e. we have a direct sum decomposition

$$\widetilde{H}_x \cong \widetilde{Q}_{(x,1)}^{\oplus \text{wt}(x,1)} \oplus \cdots \oplus \widetilde{Q}_{(x,l(x))}^{\oplus \text{wt}(x,l(x))}.$$ 

Here, we have:

$$\widetilde{Q}_{(x,1)} := \begin{pmatrix} O_{\widetilde{x}} \\ \vdots \\ O_{\widetilde{x}} \end{pmatrix} \cong \begin{pmatrix} m_{\widetilde{x}} \\ \vdots \\ m_{\widetilde{x}} \end{pmatrix} =: \widetilde{Q}'_{(x,1)}.$$ 

Note that there is a chain of embeddings of $H_x$-modules $\widetilde{Q}'_{(x,1)} \subset \widetilde{Q}_{(x,l(x))} \subset \cdots \subset \widetilde{Q}_{(x,1)}$.

Let $\mathcal{H} := \mathcal{H}(l, \text{wt})$ be the sheaf of hereditary orders on $\widetilde{X}$ defined by $(l, \text{wt})$ and $\overline{\mathcal{X}} = (\widetilde{X}, \mathcal{H})$ the corresponding non-commutative hereditary curve. Recall that $\mathcal{H}$ is a subsheaf of the sheaf of maximal orders $\text{Mat}_m(O_{\widetilde{X}})$ such that $\mathcal{H}_x = \widetilde{H}_x$ for any $\widetilde{x} \in \widetilde{X}$. First of all, note that we have an exact fully faithful functor

$$\text{Coh}(\widetilde{X}) \xrightarrow{\mathcal{F}} \text{Coh}(\overline{\mathcal{X}}), \ E \mapsto \begin{pmatrix} E \\ \vdots \\ E \end{pmatrix},$$

which transforms locally free sheaves on $\widetilde{X}$ into locally projective $\widetilde{H}$-modules. We put:

$$\tilde{L} := \mathcal{F}(O_{\widetilde{X}}) = \begin{pmatrix} O_{\widetilde{X}} \\ \vdots \\ O_{\widetilde{X}} \end{pmatrix}.$$ 

Next, for any $\tilde{x} \in \widetilde{X}$ such that $l(\tilde{x}) \geq 2$ and $2 \leq i \leq l(\tilde{x})$, we have a locally projective $\mathcal{H}$-module $\tilde{L}_{(\tilde{x},i)}$ uniquely determined by the following properties:

- The sheaf $\tilde{L}_{(\tilde{x},i)}$ is a subsheaf of $\tilde{L}$.
- For any $\tilde{y} \in \widetilde{X}$ we have:

$$\left(\tilde{L}_{(\tilde{x},i)}\right)_{\tilde{y}} = \begin{cases} \tilde{L}_{\tilde{y}} & \text{if } \tilde{y} \neq \tilde{x} \\ \text{if } \tilde{y} = \tilde{x}. \end{cases}$$

For a convenience of notation, we also define for any point $\tilde{x} \in \widetilde{X}$ the locally projective subsheaf $\tilde{L}_{(\tilde{x},1)}$ of $\tilde{L}$ defined by the following condition on the stalks:

$$\left(\tilde{L}_{(\tilde{x},1)}\right)_{\tilde{y}} = \begin{cases} \tilde{L}_{\tilde{y}} & \text{if } \tilde{y} \neq \tilde{x} \\ \text{if } \tilde{y} = \tilde{x}. \end{cases}$$

It is clear that

$$\tilde{L}_{(\tilde{x},1)} := \mathcal{F}(O_{\widetilde{X}}(-\tilde{x})) = \begin{pmatrix} O_{\widetilde{X}}(-\tilde{x}) \\ \vdots \\ O_{\widetilde{X}}(-\tilde{x}) \end{pmatrix}.$$
Note that $\tilde{\mathcal{L}}(\tilde{x}',1) \cong \tilde{\mathcal{L}}(\tilde{x}'',1)$ for any $\tilde{x}', \tilde{x}'' \in \tilde{X}$. Abusing the notation, we shall denote this sheaf by $\tilde{\mathcal{L}}(-1)$. Next, let
\begin{equation}
\Phi := \{ \tilde{x} \in \tilde{X} \mid l(\tilde{x}) \geq 2 \} =: \{ \tilde{x}_1, \ldots, \tilde{x}_r \}
\end{equation}
be the set of weighted points of $\tilde{X}$. Let us choose some homogeneous coordinates $(z : w)$ on $\tilde{X}$. Then we have a pair of distinguished sections $z, w \in \text{Hom}_{\tilde{X}}(\mathcal{O}_{\tilde{X}}(-1), \mathcal{O}_{\tilde{X}})$ vanishing, respectively at the points $\tilde{\sigma}^+ := (0 : 1)$ and $\tilde{\sigma}^- := (1 : 0)$. Let $\tilde{x}_i = (\lambda_i : \mu_i)$ and $l_i = l(\tilde{x}_i)$ for $1 \leq i \leq r$.

The following theorem is a restatement of the classical result of Geigle and Lenzing; see [23] Proposition 4.1.

\begin{theorem}
The locally projective $\mathcal{H}$-module
\begin{equation}
\tilde{T} := (\tilde{\mathcal{L}} \oplus \tilde{\mathcal{L}}(-1)) \oplus \bigoplus_{\tilde{x} \in \Phi} \tilde{\mathcal{L}}(\tilde{x}, i)
\end{equation}
is a tilting object in the derived category $D^b(\text{Coh}(\tilde{X}))$ (called, in what follows, the standard tilting bundle on $\tilde{X}$) and the corresponding algebra $\Gamma := (\text{End}_{\tilde{X}}(\tilde{T}))^0$ is isomorphic to the Ringel canonical algebra $\Gamma = \Gamma((\tilde{x}_1, l_1), \ldots, (\tilde{x}_r, l_r))$ which is the path algebra of the following quiver\footnote{In this picture the leftmost vertex corresponds to the sheaf $\mathcal{L}$, the rightmost one corresponds to the sheaf $\mathcal{L}(-1)$, while the internal vertices of the $k$-th branch correspond to the sheaves $\tilde{\mathcal{L}}(\tilde{x}_k, i)$ for a special point $\tilde{x}_k$ and $l_k = l(\tilde{x}_k)$.}

\begin{equation}
\begin{CD}
\circ @>u_11>> \circ @>u_12>> \circ @>u_21>> \circ @>>u_22>>& \cdots \circ @>>u_{r1}>>& \circ \\
@. @. @. @. @. @. @. @. \\
\circ @>u_{11}>> \circ @>u_{12}>> \circ @>u_{21}>> \circ @>>u_{22>>& \cdots \circ @>>u_{r2}> & \circ \\
@. @. @. @. @. @. @. @. \\
\circ @>u_{11}>> \circ @>u_{12}>> \circ @>u_{21}>> \circ @>>u_{22>>& \cdots \circ @>>u_{r2}> & \circ \\
\end{CD}
\end{equation}

modulo the relations
\begin{equation}
u_{il_1} \ldots u_{il} = \lambda_i w - \mu_i z \quad \text{for} \quad 1 \leq i \leq r.
\end{equation}

In other words, the derived functor
\begin{equation}
\text{RHom}_{\tilde{X}}(\tilde{T}, -) : D^b(\text{Coh}(\tilde{X})) \longrightarrow D^b(\Gamma\text{-mod})
\end{equation}
is an equivalence of triangulated categories.
Remark 5.2. In what follows, we shall call the arrows $u_{ij}$ in the quiver (63) for $1 \leq i \leq r$, $1 \leq j \leq l_i$ essential, whereas the arrows $z$ and $w$ will be called redundant. Note that the relations (63) defining the canonical algebra $\Gamma((\bar{x}_1, l_1), \ldots, (\bar{x}_r, l_r))$ generate an admissible ideal if and only if the set $\Phi$ is empty (in this case, the canonical algebra is the path algebra of the Kronecker quiver).

Note also that we can formally add to the set $\Phi$ any point $\bar{x} \in \bar{X}$ of length one, which correspond to a formal addition of another redundant arrow and does not change the corresponding canonical algebra. We will use this procedure in the study of nodal curves.

Example 5.3. Let $\lambda \in \mathbb{k} \setminus \{0, 1\}$, $\Phi := \{(0 : 1), (1 : 0), (1 : 1), (\lambda : 1)\}$ and $l(\bar{x}) = 2$ for any $\bar{x} \in \Phi$. Then the corresponding canonical algebra (63) is the tubular algebra (1) from the introduction.

5.2. Tilting on non-commutative rational projective nodal curves. We begin with an admissible datum $(\bar{X}, l, \approx)$, where $\bar{X} = \bar{X}_1 \sqcup \cdots \sqcup \bar{X}_r$ is a smooth rational projective curve (each $\bar{X}_i \simeq \mathbb{P}^1$). Let $w$ be any compatible weight function, $\bar{X} = (X, \mathcal{A})$ be the corresponding non-commutative nodal curve, $\bar{X} = (X, \mathcal{H}) = \bar{X}_1 \sqcup \cdots \sqcup \bar{X}_r$ be its hereditary cover and $\bar{Y} = (X, \mathcal{B})$ be the corresponding Auslander curve.

Theorem 5.4. Let $\bar{T}_i$ be the standard tilting bundle on $\bar{X}_i$ defined by (62), $\bar{T} := \bar{T}_1 \oplus \cdots \oplus \bar{T}_r$ and $T := \bar{F}(\bar{T}) = \begin{pmatrix} T_1 \ & \ \ & T_r \end{pmatrix}$. Consider the complex $X^\bullet := T \oplus S[-1]$ in the derived category $D^b(\text{Coh}(\bar{Y}))$, where $S$ is the torsion sheaf defined by the short exact sequence (68). Then $X^\bullet$ is a tilting complex in $D^b(\text{Coh}(\bar{Y}))$. In particular, if $\Lambda := (\text{End}_{D^b(\bar{Y})}(X^\bullet))^\circ$ then the derived categories $D^b(\text{Coh}(\bar{Y}))$ and $D^b(\Lambda\text{-mod})$ are equivalent.

Proof. The fact that $X^\bullet$ generates the derived category $D^b(\text{Coh}(\bar{Y}))$ follows from the recollement diagram (57) and the facts that $\bar{T}$ generates $D^b(\text{Coh}(\bar{X}))$ and $\bar{A}$ generates $D^b(\bar{A}\text{-mod})$. Since the functors $I$ and $L\bar{F}$ are fully faithful, we have:

$$\text{Ext}_Y^i(S, S) = 0 = \text{Ext}_Y^i(T, T)$$

for $i \geq 2$. Since the functor $L\bar{F}$ is left adjoint to $DG$ and $DG(S) = 0$, we have:

$$\text{Ext}_Y^i(T, S) \cong \text{Hom}_{D^b(\bar{X})}(\bar{T}, DG(S)[i]) = 0 \quad \text{for all } i \in \mathbb{Z}.$$ 

Finally, for any $i \in \mathbb{Z}$ we have: $\text{Ext}_Y^i(S, T) \cong \Gamma(X, \text{Ext}^i_{\bar{Y}}(S, T))$. Since $S$ is torsion and $T$ is locally projective, we have: $\text{Hom}_{\bar{Y}}(S, T) = 0$. It follows from the exact sequence (58) that $\text{Ext}_{\bar{Y}}(S, T) = 0$ for $i \geq 2$. Therefore, $\text{Hom}_{D^b(\bar{Y})}(X^\bullet, X^\bullet[i]) = 0$ for $i \neq 0$ and

$$\Lambda := (\text{End}_{D^b(\bar{Y})}(X^\bullet))^\circ \cong \begin{pmatrix} \Gamma & 0 \\ W & A \end{pmatrix},$$

where $\Gamma = (\text{End}_{\bar{X}}(\bar{T}))^\circ$, $\bar{A} = (\text{End}_{\bar{Y}}(S))^\circ$ and $W := \Gamma(X, \text{Ext}^1_{\bar{Y}}(S, T))$ (viewed as an $A\Gamma$-bimodule).
**Corollary 5.5.** Let \((\widetilde{X}, l, \approx)\) be an admissible datum, where \(\widetilde{X}\) is a disjoint union of projective lines. Then we have the following commutative diagram of categories and functors:

\[
\begin{array}{ccc}
D^b(\text{Coh}(\widetilde{X})) & \xrightarrow{\nu_*} & D^b(\text{Coh}(X)) \\
\downarrow^{\text{LF}} & & \downarrow^{\text{LF}} \\
\downarrow^{\text{DG}} & & \downarrow^{\text{T}} \\
D^b(\text{Coh}(\mathbb{Y})) & \xrightarrow{E} & D^b(\Lambda\text{-mod})
\end{array}
\]

in which \(T\) is an exact equivalence of triangulated categories, \(\text{LF}\) and \(\text{LF}^\circ\) are fully faithful exact functors, \(E\) is the canonical inclusion, \(\text{DG}\) is an appropriate Verdier localization functor and \(\nu_*\) is induced by the forgetful functor \(\text{Coh}(\widetilde{X}) \rightarrow \text{Coh}(X)\) (normalization).

**5.3. Tame non-commutative nodal curves and tilting.** We are especially interested in studying those finite dimensional \(\mathbb{k}\)-algebras \(\Lambda\) arising in the diagram \((66)\) for which the derived category \(D^b(\Lambda\text{-mod})\) has tame representation type. Since \(D^b(\Lambda\text{-mod})\) contains the category of vector bundles \(\text{VB}(X)\) as a full subcategory, the non-commutative nodal curve \(X\) has to be vector bundle tame, i.e. of the form \(X(\vec{p}, \vec{q}, \approx)\), where \((\vec{p}, \vec{q}, \approx)\) is a datum from Definition 4.12; see Theorem 4.13.

In this subsection we are going to elaborate one step further an explicit description of the corresponding algebras \(\Lambda(\vec{p}, \vec{q}, \approx)\).

**Definition 5.6.** Let us start with a pair of tuples

\[\vec{p} = ((p_1^+, p_1^-), \ldots, (p_r^+, p_r^-)) \in (\mathbb{N}^2)^r \quad \text{and} \quad \vec{q} = (q_1, \ldots, q_s) \in \mathbb{N}^s,\]

where \(r, s \in \mathbb{N}_0\) (either of this tuples is allowed to be empty).

For any \(1 \leq i \leq r\), let \(\Xi_i^\pm := \{x_{i,1}^\pm, \ldots, x_{i,p_i}^\pm\}\) and

\[
\Gamma(p_i^+, p_i^-) = \begin{array}{c}
\xrightarrow{x_{i,1}^+} x_{i,2}^+ \ldots \xrightarrow{x_{i,p_i}^+} \\
\xleftarrow{x_{i,1}^-} x_{i,2}^- \ldots \xleftarrow{x_{i,p_i}^-}
\end{array}
\]

\[(67)\]
Next, for any $1 \leq j \leq s$, let $\Xi_j := \{w_{j,1}, \ldots, w_{j,q_j}\}$ and

\begin{equation}
\Gamma(2,2,q_j) = \text{[diagram]}
\end{equation}

modulo the relation $v_j^+ u_j^+ + v_j^- u_j^- + w_{j,q_j} \ldots w_{j,1} = 0$.

Let $\approx$ be a symmetric relation on the set

$\Xi := (\Xi_1^+ \cup \Xi_1^-) \cup \cdots \cup (\Xi_s^+ \cup \Xi_s^-) \cup (\Xi_0^+ \cup \cdots \cup \Xi_0^-)$

such that for any $\xi \in \Xi$ there exists at most one $\xi' \in \Xi$ such that $\xi \approx \xi'$. Then the datum $(\vec{p}, \vec{q}, \approx)$ defines a finite dimensional $k$-algebra $\Lambda = \Lambda(\vec{p}, \vec{q}, \approx)$ which is obtained from the disjoint union of quivers with relations $\Gamma(p_i^+, p_i^-)$ and $\Gamma(2,2,q_j)$ by the following combinatorial procedure.

- For any pair of tied elements $\varrho' \approx \varrho''$ of $\Xi$, we add a new vertex and two arrows ending in it:

\begin{equation}
\text{[diagram]}
\end{equation}

The new arrows satisfy the following zero relations: $\varrho' \varrho' = 0 = \varrho'' \varrho''$.

- For each reflexive element $\varrho \in \Xi$, we add two new vertices and two arrows ending in each new vertex:

\begin{equation}
\text{[diagram]}
\end{equation}

The new arrows satisfy the following zero relations: $\vartheta_\pm \varrho = 0$.

\textbf{Remark 5.7.} In the case when $s = 0$ (i.e. when the tuple $\vec{q}$ is void) the algebra $\Lambda$ is skew-gentle [24]. If additionally $\xi \not\approx \xi$ for all $\xi \in \Xi$, then the algebra $\Lambda$ is gentle [2]. We also refer to [14] for a survey of results on the derived categories of gentle and skew-gentle algebras.
Theorem 5.8. Let $\mathbb{X} = \mathbb{X}(\bar{\varphi}, \bar{\psi}, \approx)$ be the non-commutative nodal curve attached to an admissible datum $(\bar{\varphi}, \bar{\psi}, \approx)$ from Definition 4.12. Let $Y$ be the Auslander curve of $X$ and $\Lambda = \Lambda(\bar{\varphi}, \bar{\psi}, \approx)$ be the finite dimensional algebra from Definition 5.6. Then the following results hold:

- The derived categories $D^b(\text{Coh}(Y))$ and $D^b(\Lambda - \text{mod})$ are equivalent.
- Moreover, $D^b(\text{Coh}(Y))$ and $D^b(\text{Coh}(X))$ have tame representation type.

Proof. According to Theorem 5.4, there exists a tilting complex $X^\bullet := T \oplus S[-1]$ in the derived category $D^b(\text{Coh}(Y))$ such that

$$\tilde{\Lambda} := (\text{End}_{D^b(Y)}(X^\bullet))^\circ \cong \left( \begin{array}{c} \Gamma \\ W \\ A \end{array} \right),$$

Note, that $\Gamma \cong (\Gamma(2, 2), \Gamma(2, 2), \ldots, \Gamma(2, 2)) \times (\Gamma(2, 2, q_1), \ldots, \Gamma(2, 2, q_s))$. Next, $\tilde{A}$ is a product of several copies of the semisimple algebras $\mathbb{K}$ and $\mathbb{K} \times \mathbb{K}$. Namely, each pair $\omega', \omega'' \in \Pi$ of tied elements gives a factor $\mathbb{K}$, whereas each reflexive element $\omega \in \Pi$ gives a factor $\mathbb{K} \times \mathbb{K}$. Taking into account the description of the space $W = \Gamma(X, Ext^1_S(T))$ viewed as right $\Gamma$-module given by Lemma 3.40 we can conclude that actually $\tilde{\Lambda} = \Lambda$, giving the first statement.

Since the derived category $D^b(\Lambda - \text{mod})$ is representation tame (it can be deduced as in [12]), the derived category $D^b(\text{Coh}(Y))$ is representation tame too. Since $D^b(\text{Coh}(X))$ can be obtained as a Verdier localization of $D^b(\text{Coh}(Y))$ (see Theorem 4.24), one can conclude that $D^b(\text{Coh}(X))$ is representation tame as well. □

6. Tilting exercises with some tame non-commutative nodal curves

In this section we are going to study in more details several special cases of the setting of Corollary 5.5.

6.1. Elementary modifications. We are going to introduce two “elementary modifications”, which allow to replace the algebra $\Lambda = \Lambda(\bar{\varphi}, \bar{\psi}, \approx)$ by a derived-equivalent algebra.

Lemma 6.1. Any fragment of $\Lambda$ of the form \[ \theta' \theta'' \] can be replaces by the fragment

$$\theta_2' \theta_1' = 0, \theta_1'' \theta_2'' = 0$$

Proof. Let $j$ be the common target of the arrows $\vartheta'$ and $\vartheta''$, $i'$ be the source of $\vartheta'$ and $i''$ be the source of $\vartheta''$. Consider the complex

$$T_j := (\ldots \rightarrow 0 \rightarrow P_j \xrightarrow{(\vartheta', \vartheta'')} P_i' \oplus P_i'' \rightarrow 0 \rightarrow \ldots),$$

Another approach to establish the representation tameness of $D^b(\text{Coh}(X))$ is given in [22].
where the underlined term of $T_*$ is located in the zero degree. Let $\Omega$ be the set of vertices of the quiver of the algebra $\Lambda$. Then $T := T_j \oplus (\oplus_{i \in \Omega \setminus \{j\}} P_i)$ is a tilting object of $D^b(\Lambda\text{-mod})$. Let $\Gamma := (\text{End}_{D^b(\Lambda)}(T))^\circ$. Then on the level of quivers and relations we get precisely the transformation described in the statement of Lemma. □

Example 6.2. Let $\Lambda$ be the path algebra of the following quiver

(71)

 modulo the relations: $u_i x_i = 0$ for $i \in \{1, 3\}$ and $v_j y_j = 0$ for $j \in \{2, 3\}$.

Making an elementary transformation at both bullets, we get a derived equivalent algebra $\Gamma$, which is the path algebra of the following quiver

(72)

 modulo the relations: $x_1^{(2)} y_2^{(1)} = x_2 y_1^{(1)} = x_3^{(2)} y_3^{(1)} = y_3^{(2)} x_1^{(1)} = 0$.

Lemma 6.3. Any fragment of $\Lambda$ of the form [70] can be replaced by the fragment

Proof. Let $j_\pm$ be the target of $\vartheta_\pm$ and $i$ be their common source. Consider the complexes

$T_{j_\pm} := (\ldots \rightarrow 0 \rightarrow P_{j_\pm} \xrightarrow{\vartheta_{j_\pm}} P_i \rightarrow 0 \rightarrow \ldots)$.

Again, let $\Omega$ be the set of vertices of the quiver of the algebra $\Lambda$. Then

$T := (T_{j_+} \oplus T_{j_-}) \oplus (\oplus_{i \in \Omega \setminus \{j_+, j_-\}} P_i)$

is a tilting object in $D^b(\Lambda\text{-mod})$. If $\Gamma := (\text{End}_{D^b(\Lambda)}(T))^\circ$, then on the level of quivers and relations the passage from $\Lambda$ to $\Gamma$ gives the desired elementary transformation. □
Example 6.4. Let $\Lambda$ be the path algebra of the following quiver

subject to the relations: $u_ix_i = 0$ for all $1 \leq i \leq 3$, $v_2y_2 = 0$ and $v_1^\pm y_1 = 0$.

Performing the elementary transformations at all bullets, we get a derived equivalent algebra $\Gamma$, given as the path algebra of the following quiver

subject to the relations:

$$x_2^{(2)}x_1^{(1)} = x_1^{(2)}x_2^{(1)} = x_3^{(2)}x_2^{(1)} = y_2^{(2)}y_3^{(1)} = 0 \quad \text{and} \quad y_1^{12}y_1^{11} = y_1^{12}y_1^{11}.$$}

6.2. Degenerate tubular algebra. Let $E = V(zy^2 - x^2(x - z)) \subset \mathbb{P}^2$ be a plane nodal cubic and $G = \langle \tau \rangle \cong \mathbb{Z}_2$, where $E \xrightarrow{\tau} E$ is the involution given by the rule $(x : y : z) \mapsto (x : -y : z)$. Then the category $\text{Coh}^G(E)$ of $G$-equivariant coherent sheaves on $E$ is equivalent to the category of coherent sheaves on the non-commutative nodal curve $\mathcal{E} = \mathbb{X}(\vec{p}, \vec{q}, \approx)$ described in Example 4.18. Recall that the vector $\vec{p}$ is void, $\vec{q} = (1)$ and $(\tilde{o}, 1) \approx (\tilde{o}, 1)$. Then the corresponding algebra $\Lambda = \Lambda(\vec{p}, \vec{q}, \approx)$ is the path algebra of the
following quiver

modulo the relations \(x_2x_1 + y_2y_1 + w = 0\) and \(u_\pm w = 0\). Note that the corresponding ideal in the path algebra is not admissible and the arrow \(w\) is redundant. Applying the elementary transformation from Lemma 6.3 to the arrow \(w\), we end up with the path algebra \(T\) of the following quiver

modulo the relations \(b_1a_1 + b_2a_2 + b_3a_3 = 0\) and \(b_1a_1 = b_4a_4\), i.e. the degenerate tubular algebra from Introduction. Since the derived categories \(D^b(\Lambda\text{-mod})\) and \(D^b(T\text{-mod})\) are equivalent, the commutative diagram of categories and functors [3] is a special case of the setting from Corollary 5.5.

Let \(S\) be the path algebra of the following quiver

modulo the following set of relations:

\[d_\pm a_\pm = b_\mp c_\pm\quad \text{and} \quad b_\pm a_\pm = d_\mp p a_\mp,\]

i.e. any two paths with the same source and target are equal.

**Proposition 6.5.** The derived categories \(D^b(\Lambda\text{-mod})\) and \(D^b(S\text{-mod})\) are equivalent.
Proof. Let $A$ be the path algebra of the affine Dynkin quiver

(77)

and $M$ be the left $A$-module corresponding to the representation

(78)

Then we have: $T \cong \begin{pmatrix} k & 0 \\ M & A \end{pmatrix}$, i.e. $T$ is a so-called one-point extension of the algebra $A$ by the left $A$-module $M$. Next, consider the following left $A$-modules:

$$V_1 = \begin{pmatrix} 0 & 0 \\ k & 1 \end{pmatrix} \quad V_2 = \begin{pmatrix} 0 & 0 \\ k & 1 \end{pmatrix}$$

$$V_3 = \begin{pmatrix} k & 0 \\ (\frac{1}{0}) & (0) \end{pmatrix} \quad V_4 = \begin{pmatrix} k & 0 \\ (\frac{1}{0}) & (0) \end{pmatrix}$$

and

$$V_5 = \begin{pmatrix} 0 & k \\ 1 & 1 \end{pmatrix}$$

Then $V := V_1 \oplus V_2 \oplus V_3 \oplus V_4 \oplus V_5$ is a tilting module in $D^b(A\text{-mod})$ and

$$T := \operatorname{RHom}_A(T, -) : D^b(A\text{-mod}) \rightarrow D^b(B\text{-mod})$$

is an equivalence of triangulated categories, where $B := (\operatorname{End}_A(V))^\circ$. Note that $B$ is isomorphic to the path algebra of the following quiver

(79)
modulo the relations: \( b_\pm u_\pm = d_\pm u_\mp \). It is not difficult to see that \( T(M) \cong N \), where \( N \) is the following representation of the quiver (79):

\[
\begin{array}{cccc}
0 & k & 1 & k \\
& k & 1 & 1 \\
& & k & k \\
\end{array}
\]

Observe that \( S \cong \begin{pmatrix} k & 0 \\ N & B \end{pmatrix} \). The statement follows now a result of Barot and Lenzing [5, Theorem 1] on derived equivalences of one-point extensions. □

**Remark 6.6.** According to [32, Theorem 8.1.10 and Exercise 8.1], the derived category \( D^b(S\text{-mod}) \) is equivalent to the derived category of constructible sheaves \( D^b_\Sigma(\text{ConSh}(S^2)) \) on the two-dimensional real sphere \( S^2 \) with respect to the stratification \( \Sigma \) described in the following picture:

![Stratification Diagram](image)

Putting together all results obtained in this subsection, we get the following commutative diagram of triangulated categories and exact functors:

\[
\begin{array}{cccc}
D^b(T\text{-mod}) & T & D^b_\Sigma(\text{ConSh}(S^2)) & P \\
& & & D^b(\text{Coh}^G(E)) \\
E & & & \text{Perf}^G(E) \\
\end{array}
\]

(80)

where \( I \) is the canonical inclusion functor, \( E \) is a fully faithful functor, \( T \) is an equivalence of categories and \( P \) is an appropriate localization functor. It would be quite interesting to give an interpretation of this result in terms of the homological mirror symmetry in the spirit of the approach of [38].

### 6.3. A purely commutative application of non-commutative nodal curves.

Again, let \( E = V(z y^2 - x^2(x - z)) \subset \mathbb{P}^2 \) be a plane nodal cubic. Consider the action of the cyclic group \( G \cong \mathbb{Z}_2 \) from Example 4.15. As an application of the technique of non-commutative nodal curves, we give a direct proof of the following known result; see [18, Example 1.4].
**Proposition 6.7.** Let \( \bar{E} \) be the cycle of two projective lines. Then the derived categories \( D^b(\text{Coh}(E)) \) and \( D^b(\text{Coh}^G(E)) \) are equivalent.

**Proof.** Let \( E \) be the non-commutative nodal curve from Example 4.15 (i.e. the categories \( \text{Coh}(E) \) and \( \text{Coh}^G(E) \) are equivalent). Let \( Y \) (respectively, \( \bar{Y} \)) be the Auslander curve of \( E \) (respectively, of \( \bar{E} \)). Let \( K := \text{Ker}(D^b(\text{Coh}(X)) \xrightarrow{P} D^b(\text{Coh}(E))) \) and \( \bar{K} := \text{Ker}(D^b(\text{Coh}(\bar{X})) \xrightarrow{\bar{P}} D^b(\text{Coh}(\bar{E}))) \) be the kernels of the corresponding localization functors from Corollary 5.5. We are going to construct an equivalence of triangulated categories \( D^b(\text{Coh}(X)) \xrightarrow{E} D^b(\text{Coh}(\bar{X})) \), which induces a commutative diagram of categories and functors.

\[
\begin{array}{c}
K \\
\downarrow \\
D^b(\text{Coh}(Y)) \\
\downarrow P \\
D^b(\text{Coh}(E))
\end{array} \xrightarrow{E} \begin{array}{c}
\bar{K} \\
\downarrow \\
D^b(\text{Coh}(\bar{Y})) \\
\downarrow \bar{P} \\
D^b(\text{Coh}(\bar{E}))
\end{array}
\]

where all horizontal arrows are equivalences of triangulated categories.

Let \( D^b(\text{Coh}(Y)) \xrightarrow{T} D^b(\Lambda\text{-mod}) \) and \( D^b(\text{Coh}(\bar{Y})) \xrightarrow{T} D^b(\bar{\Lambda}\text{-mod}) \) be the equivalences of triangulated categories, where the algebras \( \Lambda \) and \( \bar{\Lambda} \) are the algebras corresponding, respectively, to \( Y \) and to \( \bar{Y} \) as in Corollary 5.5. Recall that

\[
\Lambda = \begin{pmatrix} a_+ & 2 & - \\ 3 & b_- & c_+ \\ -2 & a_- & - \\ d_+ & 4 & 5 \\ - & - & b_\pm a_\mp = 0 \text{ and } d_\pm c_\mp = 0
\end{pmatrix}
\]

whereas

\[
\bar{\Lambda} = \begin{pmatrix} u_+ & v_+ & 3_+ & - \\ 2_+ & w_+ & - \\ 4_+ & z_+ & 5_+ \\ - & - & v_\pm u_\mp = 0 \text{ and } w_\pm z_\mp = 0
\end{pmatrix}
\]
Consider the third gentle algebra
\[
\Gamma = \begin{array}{ccc}
1 & 2 & 3_+ \\
\uparrow & \downarrow & \rightarrow \\
u_+ & v_+ & c_+ \\
\downarrow & \uparrow & \downarrow \\
u_- & v_- & c_- \\
& d_+ & 4 \downarrow \\
& \rightarrow & \rightarrow \\
& d_- & 5 \\
3_- & & \\
\end{array}
\]

\(v_\pm u_\mp = 0 \) and \(d_\pm c_\mp = 0\).

We construct now a pair of equivalences of triangulated categories:

\[D^b(\Lambda\text{-mod}) \xrightarrow{T_1} D^b(\Gamma\text{-mod}) \xleftarrow{T_2} D^b(\tilde{\Lambda}\text{-mod}).\]

- The first equivalence \(T_1\) is just the elementary modification from Lemma 6.1, applied to the third vertex.
- The second equivalence \(T_2\) is given by the tilting complex

\[X^* := S_1[-2] \oplus S_2[-1] \oplus P_{3_+} \oplus P_{3_-} \oplus P_4 \oplus P_5.\]

The image of the localizing subcategory \(K \subset D^b(\text{Coh}(\tilde{Y}))\) in \(D^b(\Lambda\text{-mod})\) under the tilting equivalence \(T\) is the triangulated envelope \(\langle X_+, X_-, Y_+, Y_- \rangle\), where

\[X_\pm = (\ldots \rightarrow 0 \rightarrow P_5 \xrightarrow{d_\pm} P_4 \xrightarrow{c_\mp} P_{2_+} \rightarrow 0 \rightarrow \ldots)\]

and \(Y_\pm = (\ldots \rightarrow 0 \rightarrow P_3 \xrightarrow{b_\mp} P_{2_\pm} \xrightarrow{a_\pm} P_{1_\pm} \rightarrow 0 \rightarrow \ldots)\). One can check that \(T_1(X_\pm) \cong S_{3_\pm}\), whereas \(T_1(Y_\pm) \cong Z_\pm\), where

\[Z_\pm = (\ldots \rightarrow 0 \rightarrow P_3 \xrightarrow{v_\mp} P_{2_\pm} \xrightarrow{a_\pm} P_{1_\pm} \rightarrow 0 \rightarrow \ldots).\]

In an analogous way one can check that the image of the localizing subcategory \(\tilde{K}\) under the chain of equivalences of derived categories

\[D^b(\text{Coh}(\tilde{Y})) \xrightarrow{T} D^b(\tilde{\Lambda}\text{-mod}) \xrightarrow{T_2} D^b(\Gamma\text{-mod})\]

is again the triangulated category \(\langle S_{3_+}, S_{3_-}, Z_+, Z_- \rangle\). It proves the proposition. \(\square\)

6.4. **Tilting on Zhelobenko’s non-commutative cycles of projective lines.** For \(n \in \mathbb{N}\), let \(E = E_{2n}\) be a cycle of \(2n\) projective lines. It is convenient to label the irreducible components of \(E\) by the natural numbers \(\{1, 2, \ldots, 2n\}\). Let \(\tilde{E} \xrightarrow{\nu} E\) be the normalization of \(E\), \(\tilde{O} := \nu_*(O_E)\) and \(C := \text{Ann}_E(\tilde{O}/O) \cong \text{Hom}_E(\tilde{O}, O)\) be the corresponding conductor ideal sheaf. In this particular case, \(C\) is just the ideal sheaf of the set \(\{o_1, o_2, \ldots, o_{2n}\}\) of the singular points of \(E\). Let \(E = (E, B)\) be the Auslander curve of \(E\), where \(B = \left(\begin{array}{c}
O \\
C \\
\tilde{O}
\end{array} \right)\). Recall that for any \(1 \leq k \leq 2n\) we have: \(\tilde{B}_{0k} \cong B\), where
\textit{B} is the Auslander order (23). Let \( S^\pm_k \) be the simple torsion sheaf on \( \mathbb{E} \) supported at the singular point \( o_k \) which corresponds to the vertex \( \pm \) of the quiver (24). Note that

\begin{equation}
\text{Ext}^p_E(S^\pm_k, S^\pm_k) \cong \left\{ \begin{array}{ll} k & \text{if } p = 0 \\
0 & \text{otherwise.} \end{array} \right.
\end{equation}

Let \( e := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) in \( \Gamma(E, \mathcal{B}) \), where \( 1 = 1' + 1'' \) is the decomposition of the identity section corresponding to the decomposition \( \tilde{\mathcal{O}} = \tilde{\mathcal{O}}' \oplus \tilde{\mathcal{O}}'' \) of \( \tilde{\mathcal{O}} \) in the direct sum of “even” and “odd” components. Let

\[ \mathcal{A} := \text{Hom}_\mathcal{B}(\mathcal{B}e) \cong \begin{pmatrix} \mathcal{O} & \tilde{\mathcal{O}}' \\ \mathcal{C} & \tilde{\mathcal{O}}'' \end{pmatrix} \]

and \( \tilde{\mathcal{A}} := (E, \mathcal{A}) \) be the corresponding non-commutative nodal curve. Note that for any \( 1 \leq k \leq 2n \) we have: \( \tilde{\mathcal{A}}_{o_k} \cong A \), where \( A \) is the Zhelobenko order (21), so we call \( \tilde{\mathcal{A}} \) Zhelobenko’s non-commutative cycle of projective lines. Next, we have a splitting \( \mathcal{C} = \mathcal{C}' \oplus \mathcal{C}'' \), where \( \mathcal{C}' \subset \tilde{\mathcal{O}}' \) (respectively, \( \mathcal{C}'' \subset \tilde{\mathcal{O}}'' \)). Under these notations, the following sequences of sheaves are exact:

\[ 0 \rightarrow \mathcal{C}' \rightarrow \mathcal{O} \rightarrow \tilde{\mathcal{O}}'' \rightarrow 0 \quad \text{and} \quad 0 \rightarrow \mathcal{C}'' \rightarrow \mathcal{O} \rightarrow \tilde{\mathcal{O}}' \rightarrow 0. \]

Next, let

\[ \mathcal{I} := \text{Im} (\mathcal{B}e \otimes_{\mathcal{B}e} \mathcal{B} \xrightarrow{\text{mult}} \mathcal{B}) = \begin{pmatrix} \mathcal{O} & \tilde{\mathcal{O}}' \oplus \tilde{\mathcal{O}}'' \\ \mathcal{C} & \tilde{\mathcal{O}}' \oplus \tilde{\mathcal{O}}'' \end{pmatrix} \]

and \( \mathcal{B} := \mathcal{B}/\mathcal{I} \). Then we have:

\[ \tilde{\mathcal{O}}' := \Gamma(E, \mathcal{O}/\mathcal{C}) \cong \Gamma(E, \mathcal{B}) \cong \prod_{k=1}^{2n} k \times \cdots \times k. \]

\textbf{Proposition 6.8.} We have a recollement diagram

\begin{equation}
\text{D}^b(\tilde{\mathcal{O}}''\text{-mod}) \quad \xrightarrow{\text{D}^b(\text{Coh}(\mathcal{E}))} \quad \text{D}^b(\text{Coh}(\mathcal{A})).
\end{equation}

In particular, there exists an equivalence of triangulated categories:

\begin{equation}
\text{D}^b(\text{Coh}(\mathcal{A})) \hookrightarrow \langle S^+_1, S^-_2, \ldots, S^+_2n_{-1}, S^-_{2n} \rangle \subset (\text{Coh}(\mathcal{E})).
\end{equation}

\textbf{Proof.} It is a consequence of the corresponding local statement (see Theorem 2.4) combined with the fact (following from (83)) that the functor

\[ \text{D}^b(\tilde{\mathcal{O}}''\text{-mod}) \rightarrow \text{D}^b(\text{Coh}(\mathcal{E})) = \langle S^+_1, S^-_2, \ldots, S^+_2n_{-1}, S^-_{2n} \rangle \]

is an equivalence of triangulated categories. \( \square \)

\textbf{Theorem 6.9.} Let \( \Upsilon = \Upsilon_n \) be the gentle algebra given by (9). Then the derived categories \( \text{D}^b(\text{Coh}(\mathcal{A})) \) and \( \text{D}^b(\Upsilon\text{-mod}) \) are equivalent.
Proof. Let \( D^b(\text{Coh}(E)) \xrightarrow{\tau} D^b(\Lambda\text{-mod}) \) be the tilting equivalence from Corollary 5.5. Recall from [13, Section 5.2] that \( \Lambda = \Lambda_{2n} \) is the path algebra of the following quiver

modulo the relations \( b^+_k a^-_k = 0 = c^+_k d^-_k \) for all \( 1 \leq k \leq n \). For any \( 1 \leq k \leq n \), consider the complexes

\[
A^+_k := (\ldots \rightarrow 0 \rightarrow P_{\phi_k^+} \xrightarrow{c^+_k} P_{\gamma_k} \xrightarrow{d^+_k} P_{\delta_k} \rightarrow 0 \rightarrow \ldots).
\]

Then we have:

\[
\begin{cases}
T(S^+_k) \cong A^+_k & 1 \leq k \leq n, \\
T(S^-_k) \cong A^-_k & 1 \leq k \leq n.
\end{cases}
\]

For any \( 1 \leq k \leq n \), consider the following objects of \( D^b(\Lambda\text{-mod}) \):

\[
B_k = (\ldots \rightarrow 0 \rightarrow P_{\phi_{k-1}^+} \oplus P_{\phi_k} \xrightarrow{(b_k^+ a_k^- b_k^+ a_k^-)} P_{\beta_k} \rightarrow 0 \rightarrow \ldots),
\]

\[
C_k = (\ldots \rightarrow 0 \rightarrow P_{\phi_{k-1}^+} \oplus P_{\phi_k} \xrightarrow{(b_k a_k^- b_k a_k^+)} P_{\alpha_k} \rightarrow 0 \rightarrow \ldots)
\]
and

\[ H := \bigoplus_{k=1}^{n} (P_{\gamma k} \oplus P_{\delta k} \oplus B_k \oplus C_k). \]

It is not difficult to check that \( \text{Hom}_{D^b(A)}(H, H[p]) = 0 \) and \( H \) generates the triangulated category \( \langle A_1^+, A_1^-, \ldots, A_n^+, A_n^- \rangle \). Hence \( H \) is a tilting object of the latter category. Moreover, one can show that \( (\text{End}_{D^b(A)}(H))^\circ \cong \Upsilon \). Summing up, we have a chain of equivalences of triangulated categories

\[ D^b(\text{Coh}(A)) \rightarrow \langle A_1^+, A_1^-, \ldots, A_n^+, A_n^- \rangle \rightarrow D^b(\Upsilon \text{-mod}), \]

which yields the desired statement.

For any \( n \in \mathbb{N} \), consider the graded gentle algebra \( \Theta = \Theta_n \), given as the path algebra of the following quiver

\[ (86) \]

modulo the relations \( b_k w_k = 0 = w_k a_k \) for all \( 1 \leq k \leq n \), where the grading is given by the rule \( \text{deg}(a_k) = \text{deg}(b_k) = 0 \), whereas \( \text{deg}(w_k) = 1 \).

**Proposition 6.10.** The triangulated categories \( D^b(\Upsilon \text{-mod}) \) and \( D^b(\Theta) \) are equivalent, where \( D^b(\Theta) \) denotes the derived category of \( \Theta \) viewed as a differential graded category with trivial differential. As a consequence, the triangulated categories \( D^b(\text{Coh}(A)) \) and \( D^b(\Theta) \) are equivalent, too.

**Proof.** Let \( S \) be a Serre functor of the derived category \( D^b(\text{Coh}(E)) \). In [16, Lemma 5.2] it was observed that \( S(S_k^\pm) \cong S_k^\pm[2] \) for any \( 1 \leq k \leq 2n \). Moreover, for \( p \in \mathbb{N}_0 \) and \( \varepsilon, \delta \in \{+, -\} \) we have:

\[
\text{Ext}^p_E(S_k^\varepsilon, S_k^\delta) = \begin{cases} 
\mathbb{C} & \text{if } p = 0 \text{ and } \varepsilon = \delta \\
0 & \text{if } p = 2 \text{ and } \varepsilon \neq \delta.
\end{cases}
\]

In other words, for any \( 1 \leq k \leq 2n \), the pair of objects \( S_k^+, S_k^- \) forms a generalized 2-spherical collection. Let \( T_k : D^b(\text{Coh}(E)) \rightarrow D^b(\text{Coh}(E)) \) be the corresponding Seidel-Thomas twist functor. According to [16, Proposition 2.10] (see also [9, Theorem 2], [14, Remark 2.5] and [11]), the functor \( T_k \) is an auto-equivalence of \( D^b(\text{Coh}(E)) \). For any \( 1 \leq k, l \leq 2n \) we have:

\[
T_k(S_l^\pm) \cong \begin{cases} 
S_l^\pm & \text{if } l \neq k \\
S_l^\pm[2] & \text{if } l = k.
\end{cases}
\]
It follows that the composition $\tilde{T} := T_1 \circ T_3 \circ \cdots \circ T_{2n-1}$ induces an equivalence of triangulated categories

$$D^b(\mathcal{Y}\text{-mod}) \longrightarrow \langle S^+_1, S^-_2, \ldots, S^+_n, S^-_{2n} \rangle \cong \langle S^-_1, S^+_2, \ldots, S^-_{2n-1}, S^+_n \rangle.$$  

Let $\tilde{A}_k := (\ldots \rightarrow 0 \rightarrow P_{\phi_k}^{b_k} \rightarrow P_{\gamma_k}^{a_k} \rightarrow P_{\alpha_k} \rightarrow 0 \rightarrow \ldots)$. Then we have:

$$\mathcal{T}(S_{2k-1}) \cong \tilde{A}_k^-$$  

for all $1 \leq k \leq n$. As a consequence, the categories $D^b(\mathcal{Y}\text{-mod})$ and $\langle A^-_1, A^-_2, \ldots, A^-_n, \tilde{A}_n \rangle$ are equivalent.

For any $1 \leq k \leq n$, consider the following object in $D^b(\Lambda\text{-mod})$:

$$\begin{align*}
X_k &= (\ldots \rightarrow 0 \rightarrow P_{\beta_k} \rightarrow P_{\alpha_k} \rightarrow 0 \rightarrow \ldots) \\
Y_k &= (\ldots \rightarrow 0 \rightarrow P_{\gamma_k} \rightarrow P_{\delta_k} \rightarrow 0 \rightarrow \ldots) \\
U_k &= (\ldots \rightarrow 0 \rightarrow P_{\phi_k}^{b_k} \rightarrow P_{\beta_k+1} \rightarrow 0 \rightarrow \ldots) \\
V_k &= (\ldots \rightarrow 0 \rightarrow P_{\phi_k}^{c_k} \rightarrow P_{\gamma_k} \rightarrow 0 \rightarrow \ldots).
\end{align*}$$  

One can show that $\mathcal{G} := \bigoplus_{k=1}^n (X_k \oplus Y_k \oplus U_k \oplus V_k)$ generates the orthogonal category $\langle A^-_1, A^-_2, \ldots, A^-_n, \tilde{A}_n \rangle$. Moreover, we have an isomorphism of graded algebras $\Theta \cong (\text{Ext}^*_{D^b(\Lambda)}(\mathcal{G}))^\circ$.

A result of Bardzell (see [4, Theorem 4.1]) allows to write down a minimal resolution of $\Theta$ viewed as a module over its enveloping algebra $\Theta^e := \Theta \otimes_k \Theta^e$. From the explicit form of this resolution one can conclude that $\text{gl.dim} \Theta = 3$. Moreover, one can show that in the category of graded left $\Theta^e$-modules the following vanishing is true:

$$\text{Ext}^3_{\text{gr}(\Theta^e)}(\Theta, \Theta(-1)) = 0.$$  

A result of Kadeishvili [31] implies that the algebra $\Theta$ is intrinsically formal, i.e. that any minimal $A_{\infty}$-structure on $\Theta$ is equivalent to the trivial one. According to Keller’s work [32], the categories $\langle A^-_1, A^-_2, \ldots, A^-_n, A^e_n \rangle$ and $D^b(\Theta)$ are equivalent, which implies the statement. \hfill \Box

**Remark 6.11.** It follows from Bardzell’s resolution [4, Theorem 4.1] that the third Hochschild cohomology $HH^3(\mathcal{Y})$ is non-vanishing. Assume that $\Gamma$ is a finite dimensional algebra, which is derived equivalent to $\mathcal{Y}$. According to a result of Schröer and Zimmermann [45], $\Gamma$ is a gentle algebra. Rickard’s derived Morita theorem [42] implies that $HH^3(\Gamma) \cong HH^3(\mathcal{Y}) \neq 0$ (see for instance [34, Section 2.4] for a detailed argument). It follows from [29, Section 1.5] that $\text{gl.dim}(\Gamma) \geq 3$. In particular, the gentle algebra $\mathcal{Y}$ can not be derived equivalent to a gentle algebra $\Lambda(\tilde{p}, \approx)$ from Definition 5.6. As a consequence, a Zhelobenko’s non-commutative cycle of projective lines $\Lambda$ is not derived equivalent to the Auslander curve of a non-commutative projective nodal curve.
References


