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Harmonic transfinite diameter, Chebyshev constants and capacities for compact sets in $\mathbb{R}^{p+2}$

The coincidence of the transfinite diameter $d(K)$, Chebyshev constant $\tau(K)$ and capacity $c(K)$ is one of fundamental results in the classical geometric function theory (Fekete, Szegö, Erdösz et al). Some analogues of this result in $\mathbb{C}^n$ were considered by Leja, Zakharyuta, Rumely. Here we discuss the problem on comparing similar characteristics connected with harmonic functions of several real variables. This talk is based on joint results with N. Skiba [3, 4].

Let $\Sigma := \{(\theta_j) \in \mathbb{R}^p : 1 > \theta_1 > \ldots > \theta_{p-1} > |\theta_p| \geq 0\}$; $\mathcal{M}_n$ the set of all $m = (m_j) \in \mathbb{Z}^p$ such that $n = m_0 \geqslant m_1 \geqslant \ldots \geqslant |m_p| \geqslant 0$ and r_k be defined by the formula $r_k = \sqrt{t_{k+1}^2 + t_{k+2}^2 + \cdots + t_{p+2}^2}$, where $k = 0, 1, \ldots, p$. Then the functions

$$H(n,m;t) =: \left(\frac{t_{p+1}}{r_p} + i\frac{t_{p+2}}{r_p}\right)^{m_p} r_p^{m_p} \prod_{k=0}^{p-1} r_k^{m_k - m_{k+1}} C_{m_k - m_{k+1}}^{m_{k+1} + \frac{p}{2} - \frac{k}{2}} \left(\frac{t_{k+1}}{r_k}\right) \tag{1}$$

where $t = (t_j) \in \mathbb{R}^{p+2}$, $m \in \mathcal{M}_n$, $n = 0, 1, \ldots$, form the complete system of orthogonal harmonic polynomials in the harmonic L_2-Hardy space; here $C_n^{\nu}(x)$ is the Gegenbauer (ultraspheric) polynomial of degree n and order ν (see, e.g., [1]).

The standard system of harmonic polynomials $e_i = e_i(t)$, $i \in N$, is obtained from the system (1) by normalizing and enumerating in the lexicographic order with respect to the indices $n(i), m_1(i), \ldots, m_k(i), \ldots, m_p(i)$; in particular, this provides that the degree sequence $n(i)$ is non-decreasing.

In what follows K is a compact set in $\mathbb{R}^{p+2}$. The *harmonic transfinite diameter* of K is determined by the formula

$$d^h(K) = \limsup_{n\to\infty} \left(\sup\left\{|\det\left(e_\mu(\xi_\nu)\right)_{\mu,\nu=1}^{r(n)}| : (\xi_j) \in K^{r(n)}\right\}\right)^{\frac{1}{l(n)}}, \tag{2}$$

where $l(n) = \sum\limits_{\nu=1}^{n} \nu \cdot s(\nu)$, $r(n) = \sum\limits_{\nu=0}^{n} s(\nu)$ and $s(\nu)$ is the number of all harmonic polynomials (1) of degree n.

Let $\theta(i) = (\theta_k(i)) =: \left(\frac{m_k(i)}{n(i)}\right)_{k=1}^{p} \in \Sigma$. The *directional harmonic Chebyshev constant of K in the direction* $\theta \in \Sigma$ is the number

$$\tau^h(K,\theta) = \limsup_{\theta(i)\to\theta} (M_i)^{1/n(i)}, \quad \theta = (\theta_k) \in \Sigma, \tag{3}$$

where $M_i := \inf\{\max_{t\in K} |p(t)| : p(t) = e_i(t) + \sum\limits_{j=1}^{i-1} c_j e_j(t)\}$, $i \in \mathbb{N}$.

The *principal harmonic Chebyshev constant* can be defined as the integral geometric mean of the directional Chebyshev constants (3):

$$\tau^h(K) = \exp\left\{ \frac{1}{\lambda(\Sigma)} \int\limits_{\Sigma} \ln \tau^h(K, \theta)\, d\lambda(\theta) \right\},$$

here λ is the probability equidistributed measure on Σ.

The equality $d^h(K) = \tau^h(K)$ and existence of a usual limit in (2) is proved under some quite wide assumptions about K.

We introduce also *Lh-capacities* based on the notions of the *Lh*-Green functions defined in [2, 5, 6] and compare them with the above characteristics.

Some explicit formulas for all these characteristics are given for both prolate and oblate spheroids.

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