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On some corner problems in Hele-Shaw flows

A Hele-Shaw cell is a device for investigating the two-dimensional flow of viscous fluids in a narrow gap between two parallel plates. Through the similarity in their governing equations, Hele-Shaw flows are linked to saturated flows in porous media. One of the main sources of interest in this type of flows is the oil industry. The fluid movement is driven by injection or suction of material, which typically happens through a part of the boundary of the container or a hole (or holes) inside it. Problems with injection lead to expanding fluid regions and are also called blowing problems. One such problem was studied by Richardson in connection with an application in the plastic industry, injection moulding. In this model fluid is injected into an infinite cell by an off-centre point source. On the contrary, suction problems lead to contracting fluid regions. As has been shown earlier, in the case of zero surface tension (ZST) suction problems in contrast to blowing problems are ill-posed. The one phase version of the problems has been intensively studied for half a century. The more complete review on the results to the problems are represented by S.Howison (http://people.maths.ox.ac.uk/ howison/).

At the present moment, the investigations of the Hele-Shaw flows with singularities generate intense interest. It is possible to sort two main problems under these researches. The first type singular problems deal with the singularities located on the initial shape of the free boundary in the case of injection (well-posed) ZST problems. The second type problems connection with the ill-posed (suction) problem. Note that, in the case of ZST, the singularities on the free boundary generate for finite times even the case of smooth initial free boundary curve.

One of the questions into the fist type problem is existence of smooth solutions for singular initial data as well as the description the shape of the moving boundary in the case of one-phase ZST injection problem. Here we has constructed the appropriate weighted classes where the solution exists and obtained the sufficiently condition on initial data when there is "waiting time" phenomenon.

As for the second type problems, the solutions of the Hele-Shaw equations, for the suction flow with inviscid surrounding fluid, are known to form finite-time singularities before the fluid interface reaches the sink. These singularities are generally in the form of $\frac{3}{2}$-power cusps. From the analytical point of view, very little is known about the Hele-Shaw solutions in the presence of surface tension. There is a local existence result by Duchon and Robert, and a global in time for initial data close to a circle by Constantin and Pugh. Tiam shows that singularity formation is inevitable if the center of the viscous blob is not at the sink. However, the type of singularity is still unknown. The singularity could be caused by the interface reaching the sink or by other means. It is natural to use the knowledge of the ZST solutions to study the asymptotic effects of surface tension

as a perturbation parameter. However,a perturbation analysis is difficult due to the ill posedness of the underlying ZST problem and to the singular nature of the perturbation. Instead, Howison propose an asymptotic model in which small surface tension would cause the interface in the neighborhood of the cusp to propagate rapidly as a narrow jet, analogous to a thin crack. However, this so-called "crack" model relies on the notion of a self-similar steady-state solution whose existence is unknown. Thus, the effects of very small surface tension past the cusp time remain unclear. The questions: how surface tension regularizes the cusped flows and what the limiting form of this regularization is as surface tension tends to zero in the case of bounded or unbounded domain with a point sink located in the origin, have been intensively studied numerically and analytically by Ben Amar, Ceniceros, King, Cummings, Hou, Si, Tian.

We studied the suction problem with small positive surface tension in the unbounded domain with the flow driven from infinity rather than from the origin; specifically,

$$p \sim -r^{\pi/\varphi} \cos(\pi\theta/\varphi), \quad as \quad r \to \infty.$$

We have constructed a family of solutions to the ZST Hele-Shaw free boundary problem which model the intermediate asymptotic wedge-type blow-up behaviour of the nonzero surface tension (NZST) suction problem. After studying the Schwarz function that governs the evolution of solutions to the NZST Hele-Shaw problem, we hypothesized that a single member of our family of ZST solutions can be the limit of the NZST solution as the surface tension goes to zero ($\gamma \to 0^+$): that for which the zero in the derivative of the Schwarz function is repeated. We claim that this is the unique solution that is selected in the limit $\gamma \to 0^+$.

This work is joint with B.V.Bazaliy, J.R.King and L.J.Cummings.
