Determination of the minor coefficient in a parabolic equation in a free boundary domain

In the domain $\Omega_T = \{(x, t) : h_1(t) < x < h_2(t), 0 < t < T\}$, where $h_1 = h_1(t), \ h_2 = h_2(t)$ are unknown functions, we consider the inverse problem for the parabolic equation

$$u_t = a(x, t)u_{xx} + b(x, t)u_x + c(t)u + f(x, t), \ (x, t) \in \Omega_T,$$

with unknown coefficient $c = c(t)$, initial condition

$$u(x, 0) = \varphi(x), \ x \in [h_1(0), h_2(0)],$$

boundary conditions

$$u(h_1(t), t) = \mu_1(t), \ u(h_2(t), t) = \mu_2(t), \ t \in [0, T],$$

and overdetermination conditions

$$\int_{h_1(t)}^{h_2(t)} u(x, t)dx = \mu_3(t), \ \int_{h_1(t)}^{h_2(t)} xu(x, t)dx = \mu_4(t), \ \int_{h_1(t)}^{h_2(t)} x^2u(x, t)dx = \mu_5(t), \ t \in [0, T],$$

where $h_1(0) = h_{01}$ is given.

By change of the variables $y = \frac{x-h_1(t)}{h_2(t)-h_1(t)}, \ t = t$ we reduce the problem (1)-(4) to the problem with unknown functions $(c(t), h_1(t), h_2(t), v(y, t))$, $h_3(t) = h_2(t) - h_1(t)$, $v(y, t) = u(yh_3(t) + h_1(t), t)$ in the domain $Q_T = \{(y, t) : 0 < y < 1, 0 < t < T\}$.

We establish conditions of local existence and uniqueness of solution to the problem (1)-(4).

**Theorem.** Suppose that the following conditions hold:

1) $a \in C^2,0(R \times [0, T]), \ b, f \in C^{1,0}(R \times [0, T]), \ \varphi \in C^1[h_{01}, \infty), \ \mu_i \in C^1[0, T], \ i = 1,5$;

2) $0 < a_0 \leq a(x, t) \leq a_1, \ |a_x(x, t)| \leq a_2, \ |b(x, t)| \leq b_0, \ 0 < f(x, t) \leq f_0, \ |f_x(x, t)| \leq f_1, \ \varphi(0) \leq \varphi(x) \leq \varphi_1, \ x \in [h_{01}, \infty), \ \mu_i(t) > 0, \ i = 1,3, \ t \in [0, T]$;

3) $\varphi(h_{01}) = \mu_1(0), \ \varphi(h_2(0)) = \mu_2(0)$.

Then we can indicate a number $T_0, 0 < T_0 \leq T$, depending on the data that there exists a unique solution $(c, h_1, h_3, v) \in C[0, T_0] \times (C^1[0, T_0])^2 \times C^{2,1}(Q_{T_0}) \cap C^{1,0}(\overline{Q}_{T_0}), \ h_3(t) > 0, t \in [0, T_0]$ to the problem (1)-(4).

The proof of the theorem is based on Shauder fixed-point theorem and the properties of Volterra integral equations of the second kind.