Morera-type Theorems in the Unit Disk

Let $D = \{z \in \mathbb{C} : |z| < 1\}$ be the hyperbolic plane; group $G = \{(a \ b) : |a|^2 - |b|^2 = 1\}$ acts on $D$ by mappings $g \circ z = \frac{az + b}{b \bar{z} + \pi}$. Iwasawa decomposition is of the form $G = KAN$, where $K = \text{SO}(2)$, $A = \{a_t = (\cosh t \sinh t) : t \in \mathbb{R}\}$, $N = \{n_s = (\frac{1 + is}{1 - is}, \frac{1 + is}{1 - is}) : s \in \mathbb{R}\}$.

For a fixed compact $A \subset D$ consider

$$\int_{\partial(gA)} f(z) \, dz = 0, \quad \forall g \in G$$

(1)

The problem posed is to decide, whether any $f \in C(D)$ that satisfies (1), is necessarily a holomorphic function. It is not true in general case. From the results of M.L. Agranovsky [1] it follows, in particular, that for $f \in L^2(D)$ equation (1) implies holomorphicity of $f$.

Precise condition for the case of cyrcle was found by V.V. Volchkov [2].

Let us define $K_\alpha = \{z = a_\alpha n_s \circ 0 : 0 \leq s \leq 1, \ 0 \leq t \leq \alpha\}$, and $Q_\alpha = \{z = n_s a_t \circ 0 : 0 \leq s \leq \alpha, \ 0 \leq t \leq 1\}$ ($\alpha > 0$). In the present work the author found the precise conditions for nonholomorphic function with vanishing integrals along $\partial(gK_\alpha)$ (or $\partial(gQ_\alpha)$) to exist (for some $\alpha > 0$, and all $g \in NA$).

For $l \in \mathbb{N}$, and $z \in D$ define $\eta_l(z) = \frac{l(1-|z|^2) + iz(1-\bar{z})}{l(1-|z|^2) + (1+|z|^2 - 2|z|\bar{z})}$, $m_l(z) = 1 - |z|^2$, $M_f(z) = \int_{-1}^{1} \int_{-1}^{1} |(f \cdot m_l)(n_s a_\alpha \circ z)| \, du \, dv$.

**Theorem 1.** Let $f \in C(D)$, $\forall z \in D$ $M_f(\eta_l(z)) = o(1/l)$ as $l \to +\infty$, $l \in \mathbb{N}$;

$$\int_{\partial(gK_\alpha)} f(z) \, dz = \int_{\partial(gQ_\alpha)} f(z) \, dz = 0, \quad \forall g \in NA.$$

If $\alpha_1/\alpha_2 \notin \mathbb{Q}$, or $M_f(\eta_l(z))$, $M_f(z) = o(1)$ as $z \to 1$ along the geodesic lines, then $f(z)$ is holomorphic in $D$.

**Theorem 2.** Let $f \in C(D)$, $\forall z \in D$ $M_f(\lambda_l(z)) = o(1)$ as $l \to +\infty$, $l \in \mathbb{N}$;

$$\int_{\partial(gQ_\alpha)} f(z) \, dz = \int_{\partial(gK_\alpha)} f(z) \, dz = 0, \quad \forall g \in NA.$$

If $\alpha_1/\alpha_2 \notin \mathbb{Q}$, or $M_f(z) = o(1)$ as $z \to 1$ along the horocycles, then $f(z)$ is holomorphic in $D$.
