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Solving a System of Linear Two-Sided Matrix Equations

We examine a system of linear homogeneous two-sided matrix equations in two unknown square matrices that appear on the different sides of the matrix coefficients. The proposed method for solving the system is essentially based on the preliminary reduction of certain matrices that are specially constructed from the coefficients of the equations to the canonical form by the so-called $(m,\ n)$ - block diagonal similarity transformations. To this end, yet another important problem is solved, namely, the classification of matrices of a certain class up to $(m,\ n)$ -block diagonal similarity. Hence, the method developed here can be applied to certain classification problems in linear algebra.

Let C by the field of complex numbers. For convenience, we write the block matrices $\left\| \begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array} \right\|$, $\left\| \begin{array}{c|c} 0 & C \\ \hline D & 0 \end{array} \right\|$ with the rectangular blocks A , B , C and D as $[A/B]$, $[D\backslash C]$ respectively.

Consider the system of equations

$$XA_1 - B_1Y = 0 \ , \ B_2X - YA_2 = 0 \ (*)$$

in two unknowns X and Y . Here, the coefficients A_1 and B_1 are $(m \times n)$ - matrices; the coefficients A_2 and B_2 are $(n \times m)$ -matrices; and the unknowns X and Y are square matrices of orders m and n , respectively.We are interested in solutions to this system in which both components are nonsingular.

Definition . Square matrices A and B of order $m+n$ are said to be $(m,\ n)$ -block diagonally similar if they are similar in the conventional sense, i.e., $TAT^{-1} = B$, and the similarity matrix T is block diagonal: $T = T_1 \oplus T_2$, where T_1 and T_2 are square matrices of orders m and n , respectively.

In relation to the analysis of these systems, a canonical form is found for a certain class of matrices and the block diagonal similarity transformations [1].

Theorem 1. *System (*) has nonsingular solutions if and only if matrices* $[A_2\backslash A_1]$ *and* $[B_2\backslash B_1]$ *, which are formed by the coefficients of this system, have the same canonical form with respect to the* $(m,\ n)$ *- block diagonal similarity.*

Theorem 2. *Let* $[A_2\backslash A_1]$ *and* $[B_2\backslash B_1]$ *, by the matrices formed by the coefficients of system (*),and let* $[P_1/P_2]$ *and* $[T_1/T_2]$ *transform to the canonical form* $[A_2\backslash A_1]$ *and* $[B_2\backslash B_1]$ *, respectively. Then, a (nonsingular) solution to this system can be calculated by* $(X,\ Y) = (T_1^{-1}P_1,\ T_2^{-1}P_2)$.

Theorem 3. *Consider the system of matrix equations (*). Assume that*

$$[P_1/P_2]\ [A_2\backslash A_1]\ [P_1/P_2]^{-1} = [I_2\backslash I_1]\ ,\ \ [T_1/T_2]\ [B_2\backslash B_1]\ [T_1/T_2]^{-1} = [I_2\backslash I_1]\ ,$$

where $[I_2 \backslash I_1]$ *is the canonical form of the matrices* $[A_2 \backslash A_1]$ *and* $[B_2 \backslash B_1]$ *with respect to the* $(m,\ n)$ *-block diagonal similarity. Then, the general solution to system* (*) *is given by* $X = T_1^{-1} Q_1 P_1$, $Y = T_2^{-1} Q_2 P_2$. *Here,* Q_1 and Q_2 *are arbitrary square matrices of orders* m *and* n *, respectively, such that the matrix* $[Q_1 / Q_2]$ *commutes with the canonical matrix* $[I_2 \backslash I_1]$.

[1] Shavarovskii B.Z. Solving a System of Linear Homogeneous Two-Sided Matrix Equations in Two Unknowns. // Computational Mathematics and Mathematical Physics. **44**. No. 11. 2004. P. 1853 - 1866.