Justification of c-Number Substitution in Bogolyubov’s Superfluidity Theory Hamiltonian

In 1947 Bogolyubov introduced the model Hamiltonian of the superfluidity theory \([1, 2]\),
\[
\hat{H}_B^\Lambda = \sum_{k \in \Lambda^*} \epsilon_k \hat{a}_k^\dagger \hat{a}_k + \frac{1}{2V} \sum_{k \neq 0} \nu(k) (\hat{a}_k^\dagger \hat{a}_{-k} \hat{a}_0^2 + \text{c.c.}) + \frac{\hat{a}_0^\dagger \hat{a}_0}{V} \sum_{k \neq 0} \nu(k) + \nu(0) \hat{a}_k^\dagger \hat{a}_k + \frac{\nu(0)}{2V} \hat{a}_0^\dagger \hat{a}_0^2.
\]
Here \(\hat{a}_k^\dagger, \hat{a}_k\) are the usual boson creation (annihilation) operators, \(\epsilon_k = |k|^2/(2m)\) is the one-particle energy spectrum of free bosons in the modes \(k \in \Lambda^*\) (we propose \(\hbar = 1\)), \(\nu(k)\) is the Fourier transform of the interaction pair potential \(\Phi(x)\). We suppose that \(\Phi(x) = \Phi(|x|) \in L^1(\mathbb{R}^3)\) and \(\nu(k)\) is a real function with a compact support such that \(0 \leq \nu(k) = \nu(-k) \leq \nu(0)\) for all \(k \in \mathbb{R}^3\). Bogolyubov proposed to use a c-number substitution for the most relevant operators in this Hamiltonian, namely \(\hat{a}_0^\dagger/\sqrt{V} \rightarrow \bar{c}, \ \hat{a}_0/\sqrt{V} \rightarrow c\), where \(c \in \mathbb{C}\).


Here the validity of substituting a c-number for the \(k = 0\) mode operators \(\hat{a}_p^\#\) is established rigorously for the Bogolyubov’s Hamiltonian, thereby verifying one aspect of Bogolyubov’s 1947 theory. Our analysis confirms an assertion, that if the system is stable after the c-number substitution, then so is the original one (in contrast to the papers [5, 6]). We have shown that if the potential \(\nu(k)\) in the Bogolyubov model satisfies the condition \(2\nu(0) \geq (2\pi)^{-3} \int_{\mathbb{R}^3} \epsilon_k^{-1} \nu^2(k)\), then there exists the domain of stability on the phase diagram \(\{0 < \mu \leq \mu^*, 0 \leq \theta \leq \theta_0(\mu)\}\), where the nontrivial solution of the self-consistency equation takes place (\(\mu\) is the chemical potential, \(\theta\) is the temperature). In this domain there is the non-zero Bose condensate. At the boundary \(\theta = \theta_0(\mu)\) of this domain the Bose condensate density equals \(\rho_0 = \mu/\nu(0)\). In this case the quasi-particles spectrum of the Bogolyubov Hamiltonian \(E_k = \sqrt{\epsilon_k (\epsilon_k + 2\rho_0 \nu(k))}\) has a gapless type and the famous criterion of superfluidity \(\min_k (E_k/|k|) > 0\) holds.