Equational maps of subdirectly-closed abstract classes of algebras

Let \( i = 1, 2 \), \( \mathcal{L}_i \) an infinitary algebraic signature, \( K_i \) a class of \( \mathcal{L}_i \)-algebras and \( C_i (S_i) \) the category constituted by members of \( K_i \) as objects and by (respectively, surjective) homomorphisms between them as morphisms.

An \textit{equational map} from \( K_1 \) to \( K_2 \) \cite{1} is any \( e : K_1 \to K_2 \) preserving carriers such that:

- for each \( f \in \mathcal{L}_2 \) of rank \( r \) such that \( r > 0 \) whenever \( \mathcal{L}_1 \) has no constant, there is an \( \mathcal{L}_1 \)-term \( t(\bar{x}) \), where \( \bar{x} \) is a sequence of pairwise-distinct variables of length \( r \), such that, for every \( A \in K_1 \), \( t^A = f^{e(A)} \);

- in case \( \mathcal{L}_1 \) has no constant, it holds that, for each constant \( c \in \mathcal{L}_2 \), there is an \( \mathcal{L}_1 \)-term \( s(x) \) with a single variable \( x \) such that, for every \( A \in K_1 \) and all \( a \in A \), \( s^A(a) = c^{e(A)} \).

Next, \( K_1 \) and \( K_2 \) are said to be \textit{rationally equivalent} \cite{2} provided there are mutually-inverse equational maps from \( K_1 \) to \( K_2 \) and from \( K_2 \) to \( K_1 \).

**Theorem 1** Suppose \( K_1 \) is an abstract class closed under formation of subdirect products of non-empty systems. Then, equational maps from \( K_1 \) to \( K_2 \) are exactly object components of those functors from \( C_1 (S_1) \) to \( C_2 \) (respectively, \( S_2 \)) which commute with forgetful set functor.

**Corollary 1** Assume both \( K_1 \) and \( K_2 \) are abstract classes closed under formation of subdirect products of non-empty systems. Then, \( K_1 \) and \( K_2 \) are rationally equivalent iff there is an isofunctor between \( C_1 (S_1) \) and \( C_2 \) (respectively, \( S_2 \)) commuting with forgetful set functor.

The particular cases of Theorem 1 and Corollary 1 not involving the categories \( S_1 \) and \( S_2 \) are proved in \cite{1} and \cite{2}, respectively, for hereditary multiplicative abstract classes.

Neither Theorem 1 nor Corollary 1 can be extended to multiplicative abstract classes. For instance, when \( K_1 \) is the class of all bounded distributive lattices having complement and \( K_2 \) is the variety of all Boolean algebras, the equational map from \( K_2 \) to \( K_1 \) that assigns complement-less reducts to Boolean algebras is the object component of an isofunctor between \( C_2 \) and \( C_1 \) commuting with forgetful set functor whereas there is no equational map from \( K_1 \) to \( K_2 \) because the only unary polynomial operations of any bounded distributive lattice are the diagonal and the constants zero and unit while the complement operation of any non-trivial Boolean algebra is neither diagonal nor constant.
