Nonlocal effects in homogenization of $p(x)$–Laplacian in perforated domains

We study the asymptotic behaviour, as $\varepsilon \to 0$, of $u^\varepsilon$ solutions to a nonlinear elliptic equation with nonstandard growth condition in domains containing a grid–type microstructure $F^\varepsilon$ that is concentrated in an arbitrary small neighborhood of a given hypersurface $\Gamma$. Mainly:

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\begin{align*}
-\text{div}(|\nabla u^\varepsilon|^{p^\varepsilon(x)-2} \nabla u^\varepsilon) + |u^\varepsilon|^{\sigma(x)-2} u^\varepsilon &= g(x) \quad \text{in } \Omega^\varepsilon; \\
u^\varepsilon &= A^\varepsilon \text{ on } \partial F^\varepsilon; \\
u^\varepsilon &= 0 \text{ on } \partial\Omega; \\
\int_{\partial F^\varepsilon} |\nabla u^\varepsilon|^{p^\varepsilon(x)-2} \frac{\partial u^\varepsilon}{\partial \nu} \, ds &= 0,
\end{align*}
$$

where $\varepsilon > 0$; $\Omega^\varepsilon = \Omega \setminus F^\varepsilon$ is a perforated domain in $\mathbb{R}^n$ $(n \geq 2)$ with $\Omega$ being a bounded Lipschitz domain and $F^\varepsilon$ being an open connected subset in $\Omega$ like a net that is concentrated near a hypersurface $\Gamma \subseteq \Omega$; $A^\varepsilon$ is an unknown constant; the growth functions $p^\varepsilon$ and $\sigma$ satisfy some natural conditions and $g$ is a given function. Equations of such a type are known as $p^\varepsilon(x)$–Laplacian equations with a nonstandard growth condition. Without any periodicity assumption for a large range of perforated domains and by means of the variational homogenization technique [1], we find the global behavior of $\{u^\varepsilon, A^\varepsilon\}$ as $\varepsilon \to 0$ which is described by the following non–local problem

$$
\begin{align*}
-\text{div} \left(|\nabla u|^{p_0(x)-2} \nabla u\right) + |u|^{\sigma(x)-2} u &= g(x) \quad \text{in } \Omega \setminus \Gamma; \\
u &= 0 \text{ on } \partial\Omega; \quad [u]^\pm_\Gamma = 0, \\
\left[|\nabla u|^{p_0(x)-2} \frac{\partial u}{\partial \nu}\right]^\pm_\Gamma &= c'(u, x-u - A) \quad \text{and } \int_{\Gamma} c'(u, x-u - A) \, dS = 0,
\end{align*}
$$

where $\lim_{\varepsilon \to 0} A^\varepsilon = A < +\infty$ is an unknown constant, $c(x, u)$ is a given function, $c'_u$ denotes the partial derivative of the function $c$. The general result is illustrated with a periodic example.