Independent sets and partitions of infinite graphs

Let $\Gamma(V, E)$ be a connected graph with the set of vertices $V$ and the set of edges $E$, $d$ be the path metric on $V$, $B(v, r) = \{x \in V : d(v, x) \leq r\}$ be the ball of radius $r$ with the center $v$. A subset $A \subseteq V$ is of finite index if there exists $m$ such that $B(A, m) = V$. In this case, we define the index of $A$ by $\text{ind} A = \min\{m : B(A, m) = V\}$.

Let $\Gamma(V, E)$ be a graph, $r$ be a natural number such that $|V| \geq r$. By [1, Theorem 1], there exists a partition $V = V_1 \cup V_2 \cup \ldots \cup V_r$ such that $\text{ind} V_i \leq r - 1$ for each $i \in \{1, \ldots, r\}$.

**Theorem 1.** For every infinite graph $\Gamma(V, E)$, there exists a partition $V = V_1 \cup V_2 \cup \ldots$ such that each subset $V_i$ is of finite index.

**Theorem 2.** Let $\Gamma(V, E)$ be a graph, $m$ be a natural number, $\kappa$ be a cardinal. If $|B(v, m)| \geq \kappa$ for every $v \in V$, then $V$ can be partitioned in $\kappa$ subsets of index $\leq 3m$.

**Theorem 3.** Let $\Gamma(V, E)$ be an infinite graph, $\kappa$ be a limit cardinal, $|V| = \kappa$. Suppose that, for each $\kappa' < \kappa$, there exists a natural number $m$ such that $|B(v, m)| \geq \kappa'$ for every $v \in V$. Then $V$ can be partitioned in $\kappa$ subsets of finite index.

**Theorem 4.** Let $\Gamma(V, E)$ be an infinite graph, $\kappa$ be an infinite cardinal such that $|B(v, 1)| \geq \kappa$ for each $v \in V$. Then $V$ can be partitioned in $k$ subsets of index 1.

For proofs of above theorems, we used the independent subsets of graphs and some its generalizations.