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On the order of exponential growth of solutions of linear difference and partial differential equations in Banach space

Let the following difference boundary problem

$$\Delta_1^{P_1}\ldots\Delta_n^{P_n}y-\sum A_{q_1\ldots q_n}\Delta_1^{q_1}\ldots\Delta_n^{q_n}y=f\ , \tag{1}$$

$$y\big|_{0\le t_j<P_j\delta_j}=f_j(t_1,\ldots,t_n),\quad j=\overline{1,n}, \tag{2}$$

in the region $0\le t_1,\ldots,t_n<\infty$. Here

$$\Delta_j y=\frac{1}{\delta_j}\left[y(t_1,\ldots,t_{j-1},t_j+\delta_j,t_{j+1},\ldots,t_n)-y(t_1,\ldots,t_n)\right],\ \delta_j>0\,,\ j=\overline{1,n},$$

the functions $y=y(t_1,\ldots,t_n)$, $f=f(t_1,\ldots,t_n)$ are continuous vector functions whose values lie in some complex Banach space X; $A_{q_1\ldots q_n}=A_{q_1\ldots q_n}(t_1,\ldots,t_n)$ denotes families of bounded periodic linear operators which act in X .

The first term in (1)-(2) is the highest order term: $p_i\ge q_i$; $\sum p_j\ge\sum q_j$.

Let

$$E_\alpha=\left\{f(t_1,...,t_n)\left|\overline{\lim_{t_1+...+t_n\to\infty}}\right\|f(t_1,...,t_n)\|_x\exp(-\alpha-\varepsilon)(t_1+...+t_n)=0,\forall\varepsilon>0\right\}$$

We select a subspace from E_α denoted by B_α, $-\infty<\alpha<+\infty$, which consists of function satisfying the condition

$$\sup_{0\le t_1,\ldots,t_n<\infty}\left(\|f(t_1,\ldots,t_n)\|_x\exp[-\alpha(t_1,\ldots,t_n)]\right)<\infty\,.$$

For any $f\in B_\alpha$ the solution y belongs to some E_β for β sufficiently large. Let $\chi(\alpha)$ denote the greatest lower bound of such β.

We have the following

Theorem. There exists an α_0 such that $\chi(\alpha)=\alpha_0$ for $\alpha\le\alpha_0$ and $\chi(\alpha)=\alpha$ for $\alpha>\alpha_0$.

The boundary problem (1)-(2) is investigated by use of the methods developed in [1].

Consider the boundary problem

$$\begin{cases}\dfrac{\partial^n y}{\partial t_1\ldots\partial t_n}-\sum A_j\dfrac{\partial^{n-1}y}{\partial t_1\ldots\partial t_{j-1}\partial t_{j+1}\ldots\partial t_n}-\sum A_{ij}\dfrac{\partial^{n-1}y}{\partial t_1\ldots\partial t_{j-1}\partial t_{j+1}\ldots\partial t_n}-\\ \qquad -\ldots-A_{12\ldots n}y=f(t_1,...,t_n),0\le t_j<\infty,\,j=2,3,...n, & (3)\\ \quad y(0,t_2,...,t_n)=y(t_1,0,t_2,...,t_n)=...=y(t_1,t_2,...,t_{n-1},0)=0 & (4)\end{cases}$$

in the region $0\le t_1,\ldots,t_n<\infty$.

Here $y(t_1,t_2,...,t_n)$, $f(t_1,t_2,...,t_n)$ are continuous vector functions whose values lie in some complex Banach space X; $A_j=A_j(t_1,...,t_n)$, $A_{ij}=A_{ij}(t_1,...,t_n)$, …, $A_{12\ldots n}=A_{12\ldots n}(t_1,...,t_n)$ are families of bounded linear compact operators which act in X.

Let

$$E_{\alpha}=\left\{ f(t_1,...,t_n)\Big| \overline{\lim_{t_1+...+t_n\to\infty}} \| f(t_1,...,t_n)\|_x \exp(-\alpha-\varepsilon)(t_1+...+t_n)=0, \forall\varepsilon>0 \right\}.$$

The totality of solutions y is covered by E_{β} for β sufficiently large if f ranger over $B_{\alpha}\subset E_{\alpha}$; B_{α} is a Banach space with respect to the norm

$$\| f\|_{B_{\alpha}} = \sup_{0\le t_1,\ldots,t_n<\infty} \left(\| f(t_1,\ldots,t_n)\|_x \exp[-\alpha(t_1,\ldots,t_n)]<\infty \right).$$

Denote by $\inf\beta=\chi(\alpha)$ and is colled exponential characteristic of problem (3) – (4). [2]

We have the following

Theorem. There exists an $(-\infty<)\alpha_0\le\beta_0\le\gamma_0(<+\infty)$ such that $\chi(\alpha)=\beta_0$ for $\alpha\le\alpha_0$; $\chi(\alpha)=\alpha$ for $\alpha\ge\gamma_0$; $\chi(\alpha)$ is increasing function on (α_0,γ_0).

For problem (3) - (4) with periodic coefficients we get $\alpha_0=\beta_0=\gamma_0$ and $\chi(\alpha)=\max(\alpha,\alpha_0)$. Here β_0 is highest order and γ_0 is general order of associated uniform problem [3].

[1] Rutman M.A. Boundedness of the solutions of certain linear partial difference equations, Dokl. Akad. Nauk. SSSR 159 (1964), 273-275 = Soviet Math. Dokl. 5 (1964), 1480-1482. MR 31 #1484.

[2] Orlik L.K. || J. Diff.Equations.-Vol.25. – №10 (1989). – p. 1819 – 1820.

[3] Orlik L.K. Exponential characteristic of hyperbolic partial differential equation in Banach space, /Internathional Conference: Modern Analysis and Applications (MMA 2007) dedicated to the centenary of Mark Krein, Odessa, Ukraine, april 9-14, 2007, P. 106-107./