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On $*$-representation of polynomial semilinear relations.

Pairs of self-adjoint linear operators in a Hilbert space which satisfy semilinear relations arise in different problems of mathematics and physics (see, for example [1, 2]) and were studied in [5, 7] and others.

Following [5, 7] we will say, that a pair of bounded selfadjoint operators (A, B) in a Hilbert space H satisfy semilinear relation if

$$\sum_{i=1}^{m} f_i(A)\, B\, g_i(A) = h(A),$$

where $f_i(t)$, $g_i(t)$, $h(t)$ are polynomials on $\mathbb{R}$ for all $i = \overline{1, m}$. A pair of operators (A, B), which satisfy the last equation are called representation of semilinear relation.

With any semilinear relation the following two objects are associated:

- a characteristic function, which is a polynomial of two variables on $\mathbb{R}^2$:

$$P_n(t, s) := \sum_{i=1}^{m} f_i(t) g_i(s) = \sum_{0 \le i+j \le n} a_{ij}\, t^i s^j, \quad (t\,, s\,) \in \mathbb{R}^2, a_{ij} \in \mathbb{R}.$$

  Here we assume, that for the characteristic function $P_n(t, s) = 0$ iff $P_n(s, t) = 0$;

- a simple non-oriented graph Γ, for which the set of vertices is $\mathbb{R}$ and a vertex $t \in \mathbb{R}$ is connected with a vertex $s \in \mathbb{R}$ if and only if $P_n(t, s) = 0$.

The structure of irreducible representations depends crucially on the structure of connected components of the graph Γ. Applying well-known facts about $*$-representations of graphs, we see that the problem of unitary description of all $*$-representations for pairs of operators witch satisfy semilinear relations is not $*$-wild (see [5]) if and only if all connected components of the corresponding graph are of the forms:  .

We show that even for polynomial semilinear relations arbitrarily complicated graphs can arise: for any finite simple non-oriented graph Γ we construct a polynomial characteristic function such that Γ is its graph.

We list graphs which arise as connected components of characteristic functions of degree one and two. To do it, we construct a dynamical system on the set of zeroes of polynomial such that connected components are described in terms of orbits of the dynamical system.

As one could expect, only few of them are $*$-tame, so it is natural to study pairs (A, B) satisfying semilinear relations with additional conditions.

We introduce the condition of G-orthoscalarity and show that the description of G-orthoscalar pairs satisfying semilinear relation with quadratic characteristic function is

$*$-tame. The G-orthoscalarity condition is closely related to the orthoscalarity condition in representations of graphs (see [3, 4]). On the other hand, the Fairlie algebra [1, 2, 6] generated by self-adjoint elements $a = a^*$, $b = b^*$ and two semilinear relations $[a, [a, b]_q]_{q^{-1}} = -b$, $[b, [b, a]_q]_{q^{-1}} = -a$, $(q \in \mathbb{T} \cup \mathbb{R}\backslash\{0\})$ gives an important example of semilinear relation with quadratic characteristic function and G-orthoscalarity condition.

[1] D.B. Fairlie, *Quantum deformation of $SU(2)$*, J. Phys. A: Math. and Gen. **23** (1990), no. 5, 183–187.

[2] M. Jimbo, *Quantum R-matrix to the generalized Toda system: an algebraic approach*, Lect. Nones in Phys. **246** (1986), 335–361.

[3] S.A. Kruglyak, V.I. Rabanovich, Yu.S. Samoilenko, *On sums of projections*, Funktsional. Anal. i Prilozhen. **36** (2002), no. 3, 20–35, (Russian); translation in Funct. Anal. Appl. **36** (2002), no. 3, 182–195.

[4] S.A. Kruglyak, A.V. Roiter, *Locally scalar representations of graphs in the category of Hilbert spaces*, Funktsional. Anal. i Prilozhen. **39** (2005), no. 2, 13–30, (Russian); translation in Funct. Anal. Appl. **39** (2005), no. 2, 91–105.

[5] V.L. Ostrovskyi, Yu.S. Samoilenko, *Introduction to the theory of representations of finitely presented $*$-algebras. I. Representations by bounded operators*, Rev. Math. Math. Phys, **11**, Harwood Academic Publishers, Amsterdam, 1999.

[6] L.B. Turowska, Yu.S. Samoilenko, *Semilinear relations and $*$-representations of deformations of $so(3)$*, Quantum groups and quantum spaces, Banach center publications, Inst. of Math. Polish Acad. of Sc., Warszava **40** (1997), 21–40.

[7] L.B. Turowska, Yu.S. Samoilenko, V.S. Shul'man, *Semilinear relations and their $*$-representation*, Methods Funct. Anal. Topology **2** (1996), no. 1, 55–111.