On a Heyde characterization theorem for discrete Abelian groups

A lot of studies were devoted to characterizations of a Gaussian distribution on the real line. Specifically, in 1970 Heyde proved the following theorem.

**Heyde theorem** ([1, §13.4.1]). Let $\xi_j, j = 1, 2, ..., n, n \geq 2,$ be independent random variables. Let $\alpha_j, \beta_j$ be nonzero constants such that $\beta_i \alpha_i^{-1} \pm \beta_j \alpha_j^{-1} \neq 0$ for all $i \neq j$. If the conditional distribution of $L_2 = \beta_1 \xi_1 + \cdots + \beta_n \xi_n$ given $L_1 = \alpha_1 \xi_1 + \cdots + \alpha_n \xi_n$ is symmetric then all random variables $\xi_j$ are Gaussian.

Let $X$ be a locally compact separable Abelian metric group, Aut$(X)$ the set of topological automorphisms of $X$. Let $\xi_j, j = 1, 2, ..., n, n \geq 2,$ be independent random variables with values in $X$ and distributions $\mu_j$. Consider the linear forms $L_1 = \alpha_1 \xi_1 + \cdots + \alpha_n \xi_n$ and $L_2 = \beta_1 \xi_1 + \cdots + \beta_n \xi_n$, where $\alpha_j, \beta_j \in$ Aut$(X)$ such that $\beta_i \alpha_i^{-1} \pm \beta_j \alpha_j^{-1} \in$ Aut$(X)$ for all $i \neq j$. Formulate the following problem.

**Problem 1.** Describe groups $X$ for which the symmetry of the conditional distribution of the linear form $L_2$ given $L_1$ implies that all distributions $\mu_j$ are either Gaussian or belong to a class of distributions that can be considered as a natural analogue of the class of Gaussian distributions.

Problem 1 have not been solved, nevertheless it was studied in different important subclasses of the class of locally compact Abelian groups. In [2] Problem 1 was completely solved in the class of finite Abelian groups, and then in [3] it was solved in the class of countable discrete Abelian groups. For these classes of groups the class of idempotent distributions can be regarded as a natural analogue of the class of Gaussian distributions. In both cases a corresponding class of group can be easily described. It consists of groups containing no elements of order two.

Formulate now the following general problem.

**Problem 2.** Let $X$ be a locally compact separable Abelian metric group. Assume that the conditional distribution of the linear form $L_2$ given $L_1$ is symmetric. Describe possible distributions $\mu_j$.

Problem 2 was solved in the class of finite Abelian groups in [4]. We solve Problem 2 in the class of countable discrete Abelian groups.


