On Filters of a Module

Let $R$ be a ring and let $M$ be a left $R$-module. Let $F(M)$ be some non-empty collection of submodules of $M$.

Consider the following conditions:

1. $L \in F(M), L \leq N \leq M \implies N \in F(M)$;
2. $L \in F(M), f \in \text{End}(M) \implies (L : f)_m \in F(M)$;
3. $N, L \in F(M) \implies N \cap L \in F(M)$;
4. $N \in F(M), N \in \text{Gen}(M), L \leq N \leq M \land \forall g \in \text{End}(M)_N : (L : g)_M \in F(M) \implies L \in F(M)$;
5. $N, K \in F(M), N \in \text{Gen}(M) \implies t_{(K \subseteq M)} \in F(M)$.

**Proposition 1.** If $F(M)$ satisfies (1), (2), (4) then it satisfies (5).

**Proposition 2.** Let $M$ be a left $R$-module and $K$ be a fully invariant submodule of $M$. Then $U = \{L|K \leq L \leq M \text{ satisfies } (1), (2), (3)\}$.

**Proposition 3.** Let $M$ be a left $R$-module and $K$ be a fully invariant submodule of $M$ such that $t_{(K \subseteq M)}(K) = K$. Then $U = \{L|K \leq L \leq M\}$ satisfies (1), (2), (3), (4).

**Proposition 4.** Let $M$ be a semisimple left $R$-module with a unique homogeneous component and let $M = \bigoplus_{i \in I} M_i$, where $M_i$ simple for each $i \in I$. Then

(i) All preradical (radical) filters of $M$ are trivial iff $\text{Card}(I) < \infty$;
(ii) Every non-trivial preradical (radical) filter of $M$ is of the form

$$M_p := \{L|L \leq M, \text{Card}(M/L) < p\},$$

where $p$ is an infinite cardinal number such that $p \leq \text{Card}(I)$;
(iii) $M_p$ is a preradical (radical) filter of $M$ for each infinite cardinal number $p$ such that $p \leq \text{Card}(I)$. 