Yuri Maturin (Drohobych State Pedagogical University, Drohobych, Ukraine)

On Filters of a Module

Let R be a ring and let M be a left R-module. Let $F(M)$ be some non-empty collection of submodules of M .

Consider the following conditions:

(1) $L \in F(M), L \leq N \leq M \Longrightarrow N \in F(M)$;

(2) $L \in F(M), f \in End(M) \Longrightarrow (L : f)_m \in F(M)$;

(3) $N, L \in F(M) \Longrightarrow N \cap L \in F(M)$;

(4) $N \in F(M), N \in Gen(M), L \leq N \leq M \wedge \forall g \in End(M)_N : (L : g)_M \in F(M) \Longrightarrow L \in F(M)$;

(5) $N, K \in F(M), N \in Gen(M) \Longrightarrow t_{(K \subseteq M)} \in F(M)$.

Proposition 1. If $F(M)$ satisfies (1), (2), (4) then it satisfies (5).

Proposition 2. Let M be a left R-module and K be a fully invariant submodule of M . Then $U = \{L | K \leq L \leq M$ satisfies (1), (2), (3).

Proposition 3. Let M be a left R-module and K be a fully invariant submodule of M such that $t_{(K \subseteq M)}(K) = K$. Then $U = \{L | K \leq L \leq M\}$ satisfies (1), (2), (3), (4).

Proposition 4. Let M be a semisimple left R-module with a unique homogeneous component and let $M = \bigoplus_{i \in I} M_i$, where M_i simple for each $i \in I$. Then

(i) All preradical (radical) filters of M are trivial iff $Card(I) < \infty$;

(ii) Every non-trivial preradical (radical) filter of M is of the form

$$M_p := \{L | L \leq M, Card(M/L) < p\},$$

where p is an infinite cardinal number such that $p \leq Card(I)$;

(iii) M_p is a preradical (radical) filter of M for each infinite cardinal number p such that $p \leq Card(I)$.