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Static Lower Bound for Basket Pay-Off Generated by Risk Vector with Normal Marginals

Consider a random vector $\underline{X} = (X_1, \ldots, X_n)$ of which we can only observe the marginal distributions. We suppose that all the marginals are normal, but the total vector is not necessarily normal. Our goal is to construct an optimal static lower bound for $g_B(\underline{X}, K) = (S - K)_+$ where $S = X_1 + \ldots + X_n$, in terms of random variables (rv's) $g_C(X_i, K_i) = (X_i - K_i)_+$ and $g_P(X_i, K_i) = (K_i - X_i)_+$, respectively. We interpret X_i's as financial or actuarial risks. We mention the paper of Hobson et al. (2005) who studied a lower bound for basket options of two components, which is a special case of our problem for nonnegative rv's X_i's and $n = 2$. A related problem of upper bound for basket options is investigated, e.g., in *Rüschendorf* (2005), p.34, where comonotonic random vectors of asset prices are involved (see [1] for the definition of a comonotonic vector). Below we present the main our result.

Let μ_i and σ_i^2 be the fixed values of mean and variance of X_i, $i = 1, \ldots, n$, σ_m^2 be the largest of those variances, $\sigma = (\sigma_m - \sum_{i \neq m} \sigma_i)_+$ and $M = \mu_1 + \ldots + \mu_n$. If $\sigma > 0$ then for all nonrandom vectors $\underline{x}$, it holds:

$$g_B(\underline{x}, K) \geq g_C(x_m, K_m) - \sum_{i \neq m} g_P(x_i, K_i) =: g_{SR}^+(\underline{x}),$$

moreover

$$\min \mathbf{E} g_B(\underline{X}, K) = \mathbf{E} g_B(\underline{X}^*, K) = \mathbf{E} g_{SR}^+(\underline{X}^*).$$

Hereafter minimum is taken over all possible vectors $\underline{X}$ with fixed normal marginals, and $\underline{X}^* = (-\sigma_1\gamma, -\sigma_2\gamma, \ldots, +\sigma_m\gamma, -\sigma_{m+1}\gamma, \ldots, -\sigma_n\gamma)$ with $\gamma \sim N(0,1)$, and $K_m =$
$= \sigma_m \sigma^{-1}(K - M) + \mu_m$, $K_i = -\sigma_i \sigma^1 (K - M) + \mu_i$, $i \neq m$.

If $\sigma = 0$ and $M \leq K$ then $g_B(\underline{x}, K) \geq 0$ and $\min \mathbf{E} g_B(\underline{X}, K) = 0$. Thus, in this case an optimal lower bound is equal to zero.

Finally, if $\sigma = 0$ and $M > K$ then for all nonrandom $\underline{x}$, it holds:

$$g_B(\underline{x}, K) \geq \sum_{i=1}^{n} g_C(x_i, -Kn^{-1}) =: g_{SR}^0(\underline{x}),$$

moreover

$$\min \mathbf{E} g_B(\underline{X}, K) = M - K = \mathbf{E} g_{SR}^0(\underline{X}).$$

Due to put-call parity, the function $g_{SR}^0(\underline{x})$ can be expressed as an algebraic sum of put and call pay-offs. Therefore, in this case the rv $g_{SR}^0(\underline{X})$ provides an optimal lower bound.

Also, for an arbitrage-free market with one underlying asset, we show that under a mild regularity assumption, the subsequent values of the asset price form a comonotonic random vector only under a deterministic linear relationship.

Keywords: Gaussian risks, Optimal lower bounds, Basket options, Arbitrage-free market, Comonotonic random vector, Linear relationship.

[1] D. Hobson, P. Laurence and T.-H. Wang. Static - arbitrage optimal subreplicating strategies for basket options //Insurance: Mathematics and Economics. — 2005. — **37**, p.553-572.

[2] L. Ruschendorf. Stochastic ordering of risks, influence of dependence, and a.s. constructions // N. Balakrishnan, I.G. Bairamov and O.L. Gebizlioglu (Eds.), Advances on Models, Characterizations and Applications, Chapman and Hall/CRC, Boca Raton. — 2005.