Formally self-adjoint quasi-differential operators

Let \([a, b]\) be a closed interval, \(m \in \mathbb{N}\). We denote by \(W_2^{[m]}\) the space of functions \(y(x) \in L_2([a, b], \mathbb{C}) =: L_2\) such that following Shin-Zettl quasi-differential expressions exist a.e. on \([a, b]\):
\[
D_0 y := y, \quad D_k y := D(D_{k-1} y) + \sum_{s=0}^{k-1} p_{k,s}(x) D_s y, \quad D := -i \frac{d}{dx}, \quad k = 1, m,
\]
where \(p_{k,s}(x) = p_{m-s,m-k}(x) \in L_1([a, b], \mathbb{C}), s = 0, k-1\). They contain classical differential expressions and many others [3]. We consider in Hilbert space \(L_2\) the minimal operator \(L_{\min} y := D_m y, \quad \text{Dom}(L_{\min}) := \{ y \in W_2^{[m]} : D_k y(a) = D_k y(b) = 0, k = 0, m-1\}\). It is a closed densely defined symmetric operator in \(L_2\) with deficiency index \((m, m)\).

**Theorem 1** ([1]). Triplet \((\mathcal{C}^n, \Gamma_1, \Gamma_2)\), where \(\Gamma_1, \Gamma_2\) are mappings from \(W_2^{[m]}\) into \(\mathcal{C}^n\) such that:
\[
\begin{align*}
\Gamma_1 y & := i \left( D_{2n-1} y(a), \ldots, D_n y(a), -D_{2n-1} y(b), \ldots, -D_n y(b) \right), \\
\Gamma_2 y & := (D_0 y(a), \ldots, D_{n-1} y(a), D_0 y(b), \ldots, D_{n-1} y(b)), \quad \text{for } m = 2n, \text{ and} \\
\Gamma_1 y & := i \left( D_{2n} y(a), \ldots, D_{n+1} y(b), i D_n y(b) + D_n y(a) \right), \\
\Gamma_2 y & := (D_0 y(a), \ldots, D_{n+1} y(b), \left( -\frac{1}{2} + i \right) D_n y(b) + \left( 1 - \frac{1}{2} i \right) D_n y(a)), \quad \text{for } m = 2n + 1,
\end{align*}
\]

is a space of boundary values for symmetric operator \(L_{\min}\).

Theorem together with results of [2, Ch.3, §1] imply

**Theorem 2** ([1]). Restriction of \(L_{\max}\) on the set of functions \(y(x) \in W_2^{[m]}\) satisfying homogeneous boundary condition
\[
(K - I) \Gamma_1 y + i (K + I) \Gamma_2 y = 0, \quad (1)
\]
where \(K\) is a unitary operator in \(\mathcal{C}^n\), is a self-adjoint extension \(L_K\) of \(L_{\min}\). Inversely, for every self-adjoint extension \(\tilde{L}\) such unitary operator \(K\) in \(\mathcal{C}^n\) that \(\tilde{L} = L_K\) exists. This relation between unitary operators \(K\) and self-adjoint extensions is bijective.

Different characterization of self-adjoint extensions of quasi-differential operators, based on classical GKN theory, may be found in [3]. However, our approach has advantage of bijectivity of parametrization. Its another important advantage is that formula (1) describes all maximal dissipative extensions, if \(K\) is contraction in \(\mathcal{C}^n\). Also in [1] we use this approach to describe all generalized resolvents of \(L_{\min}\).

These results were obtained together with V. A. Mikhailets [1].