A Blaschke-type bound for a class of unbounded analytic functions

The Blaschke-type condition for zeros $z_n$ of an analytic and bounded in the unit disk function has the form

$$\sum_{|z_n|<1} (1 - |z_n|) < \infty.$$ 

There are many generalizations to unbounded functions (M.M. Djrbashian, F.A. Shamoyan, and many others). Corresponding weights always depend on $(1 - |z|)$.

Now let $E$ be a closed subset of the unit circle $\{ |\zeta| = 1 \}$ such that for some real $\alpha$ the integral

$$\int_0^1 \frac{\text{mes}\{ \zeta : |\zeta| = 1, \text{dist}(\zeta, E) < t \}}{t^{\alpha+1}} dt$$

converges, and $f$ be an analytic function in the unit disk such that $|f(0)| = 1$ and $|f(z)| \leq \exp(D\text{dist}(z, E)^{-q})$. We find the condition for zeros $z_n$ in the form

$$\sum_n (1 - |z_n|) \text{dist}(z_n, E)^{(q-\alpha)+} \leq D C(q, E).$$

The natural setting is the class of subharmonic functions $v$ and their Riesz measures (generalized Laplacians) $\mu = (1/2\pi)\Delta v$. The results are optimal.

Application

Let $U$ be a unitary operator in the Hilbert space with the spectrum in the proper closed subset $E$ of the unit circle, and $T$ be a contraction linear operator such that $U - T$ belongs to the Schatten — von Neumann class $S_q$, $q > 0$. Then the sum over all eigenvalues $z_n$ of $T$ in the unit disk

$$\sum_{|z_n|<1} (1 - |z_n|) \text{dist}(z_n, E)^{(q-\alpha)+}$$

is finite.