The Radon-Nikodym-type theorem for Maharam traces

Let $M$ be an von Neumann algebra, let $S(M)$ be the $*$-algebra of all measurable operators affiliated with to $M$ and let $t(M)$ be the topology of convergence locally in measure in $S(M)$ ([1], III, §3.5). Let $A$ be an arbitrary commutative von Neumann algebra and let $\Phi$ be the faithful normal $S(A)$-valued trace on $M$. An operator $x \in S(M)$ is said to be $\Phi$-integrable if there exists a sequence $\{x_n\} \subset M$ such that $x_n \xrightarrow{t(M)} x$ and $\Phi(|x_n - x_m|) \xrightarrow{t(A)} 0$ as $n, m \to \infty$. In this case, there exists $\hat{\Phi}(x) \in S(A)$ such that $\Phi(x_n) \xrightarrow{t(A)} \hat{\Phi}(x)$. Let $L^1(M, \Phi)$ be the set of all $\Phi$-integrable operators and denote $\|x\|_\Phi = \hat{\Phi}(|x|)$ for every $x \in L^1(M, \Phi)$. Then $(L^1(M, \Phi), \| \cdot \|_\Phi)$ is a $(bo)$-complete lattice normed space.

The trace $\Phi$ possesses the Maharam property if for any $x \in M_+$, $0 \leq f \leq \Phi(x) \in S(A)$ there exists a positive $y \leq x$ such that $\Phi(y) = f$. A faithful normal $S(A)$-valued trace $\Phi$ with the Maharam property is called Maharam trace (compare with [2], III, 3.4.1).

**Theorem 1.** $L^1(M, \Phi)$ is a Banach-Kantorovich space if and only if $\Phi$ is a Maharam trace.

Let $\Phi$ be a $S(A)$-valued Maharam trace on $M$ and let $\Psi$ be a normal $S(A)$-valued trace on $M$. We denote by $s(a)$ the support of an element $a \in S(A)$. A trace $\Psi$ is called absolutely continuous with respect to $\Phi$ if $s(\Psi(p)) \leq s(\Phi(p))$ for all projections $p \in M$.

The next theorem is a noncommutative version of the Radon-Nikodym-type theorem for Maharam traces (compare with [2], VI, 6.1.11).

**Theorem 2.** Let $\Phi$ be a $S(A)$-valued Maharam trace on a von Neumann algebra $M$. If $\Psi$ is a normal $S(A)$-valued trace on $M$ absolutely continuous with respect to $\Phi$, then there exists an operator $y \in L^1_+(M, \Phi) \cap S(Z(M))$ such that

$$\Psi(x) = \hat{\Phi}(yx)$$

for all $x \in M$ (here $Z(M)$ is the center of $M$).
