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Linear Quaternionic Equations and Their Systems

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Abstract. The general linear quaternionic equation with one unknown and systems of linear quaternionic equations with two unknown are solved. Examples of equations and their systems are considered.

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1. Introduction

In this work we give a new method of solution of the general linear quaternionic equation and of some systems of quaternionic equations with two unknowns. Now we shall observe briefly some earlier results on this topic.

In the article [9] they solve linear equations of the forms $ax = xb$ and $ax = \bar{x}b$ in the real Cayley–Dickson algebras (quaternions, octonions, sedenions), and establish a form of roots of such equations. The author of the paper [6] classifies solutions of the quaternionic equation $ax + xb = c$. The empty set, one or two points or a plane are obtained as the sets of its solutions. In the article [1] they consider equations of the forms $ax = xb$ and $ax = \bar{x}b$ for some generalizations of quaternions and octonions. In [3] they solve by matrix method the general linear quaternionic equation of the form (12) and linear quaternionic systems of some type. Also in the paper [7] they describe solution of the general linear quaternionic equation (12). In the article [4] they describe the set of solutions of the equation $x\alpha = x + \beta$. In [5] systems of linear quaternionic equations have been considered and Cramer's rules for right and left quaternionic systems of linear equations have been obtained.

In contrast to the papers [7], [3], in this article other method is given, which is simpler, namely, "method of rearrangements" to solve general linear quaternionic equations (12) and to solve systems of quaternionic equations with two unknowns. This method allows to reduce simply any equation with the left and right coefficients to a system of four equations. For example, in

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the article [8] quadratic equations are reduced, and in this paper the general linear equation is reduced.

We have no affair with matrix representations of quaternions and operations on matrices, and we work with algebraic expressions. Thus, equations of the form (1) are easily represented as a system (4). Moreover as for equation (12) we rewrite it instantly as an equivalent system of equations (13) (four linear equations with four real unknowns). Such method is effective for solution of linear equations and of their systems. While, e.g., solution of system (5) from [3] demands huge matrix calculations (see example 5.3 in the paper [3]).

Moreover we solve some certain linear equations and their systems with two unknowns as examples.

2. Notations of the Paper

For a quaternion we use the standard notations:

$$x = x_0 + x_1 i + x_2 j + x_3 k, \quad x_0, x_1, x_2, x_3 \in \mathbb{R},$$

where for the imaginary units i, j, k the following equalities are true: $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$. Note also that we deal with only real quaternions, i.e., their components are real; we use the word "quaternion" only for real one and denote the system of all real quaternions by $\mathbb{H}$. The number $\mathrm{Sc}(x) := x_0$ is referred to as the scalar part of the quaternion x; $x_1 i + x_2 j + x_3 k$ is called the vector part of x and is denoted by $\mathrm{Vec}(x)$ or $\vec{x}$; $x_0 - x_1 i - x_2 j - x_3 k$ is called the conjugate number with respect to x and is denoted by $\bar{x}$; $|x| := \sqrt{x\bar{x}} = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}$ is called the modulus of x and is denoted by $|x|$; we shall denote by $\overrightarrow{\mathbb{H}}$ the set $\{x \in \mathbb{H} : \mathrm{Sc}(x) = 0\}$. Let $y = y_0 + y_1 i + y_2 j + y_3 k$, $y_0, y_1, y_2, y_3 \in \mathbb{R}$, then the real number $(x, y) := x_0 y_0 + x_1 y_1 + x_2 y_2 + x_3 y_3$ is referred to as the scalar product of x and y. According to the introduced definitions $|\vec{x}| = \sqrt{x_1^2 + x_2^2 + x_3^2}$, $\overrightarrow{(\vec{x})} = \vec{x}$.

We shall agree that top index in brackets denote corresponding coordinates of coefficients of equations, that is $a = a^{(0)} + a^{(1)} i + a^{(2)} j + a^{(3)} k$, where $a^{(0)}, a^{(1)}, a^{(2)}, a^{(3)} \in \mathbb{R}$ (though for unknowns we use subindices).

3. The Linear Equations and Systems with Two Addends

Consider a linear quaternionic equation of the form:

$$\alpha x \beta + \gamma x \delta = \rho, \tag{1}$$

where $\{\alpha, \beta, \gamma, \delta, \rho\} \subset \mathbb{H}$.

Consider Lemma 2 of [8]: for any $a, b \in \mathbb{H}$ the following equality is true:

$$ab = ba - 2\vec{b}\vec{a} - 2(\vec{a}, \vec{b}). \tag{2}$$

Then

$$\alpha x\beta = \alpha(\beta x - 2\vec{\beta}\vec{x} - 2(\vec{\beta},\vec{x})) = \alpha\beta x - 2\alpha\vec{\beta}\vec{x} - 2\alpha(\vec{\beta},\vec{x})$$
$$= \alpha\beta(x_0+\vec{x}) - 2\alpha\vec{\beta}\vec{x} - 2\alpha(\vec{\beta},\vec{x}) = \alpha\beta x_0 + (\alpha\beta - 2\alpha\vec{\beta})\vec{x} - 2\alpha(\vec{\beta},\vec{x}).$$

Similarly

$$\gamma x\delta = \gamma\delta x_0 + (\gamma\delta - 2\gamma\vec{\delta})\vec{x} - 2\gamma(\vec{\delta},\vec{x}).$$

From the last two equalities, equation (1) is equivalent to the following:

$$\begin{aligned}
&(\alpha\beta+\gamma\delta)x_0 + \left[(\alpha\beta - 2\alpha\vec{\beta} + \gamma\delta - 2\gamma\vec{\delta})i - 2\alpha\beta^{(1)} - 2\gamma\delta^{(1)}\right]x_1\\
&\quad + \left[(\alpha\beta - 2\alpha\vec{\beta} + \gamma\delta - 2\gamma\vec{\delta})j - 2\alpha\beta^{(2)} - 2\gamma\delta^{(2)}\right]x_2\\
&\quad + \left[(\alpha\beta - 2\alpha\vec{\beta} + \gamma\delta - 2\gamma\vec{\delta})k - 2\alpha\beta^{(3)} - 2\gamma\delta^{(3)}\right]x_3\\
&=: \chi x_0 + \varphi x_1 + \psi x_2 + \phi x_3\\
&= \left[\chi^{(0)} + \chi^{(1)}i + \chi^{(2)}j + \chi^{(3)}k\right]x_0 + \left[\varphi^{(0)} + \varphi^{(1)}i + \varphi^{(2)}j + \varphi^{(3)}k\right]x_1\\
&\quad + \left[\psi^{(0)} + \psi^{(1)}i + \psi^{(2)}j + \psi^{(3)}k\right]x_2 + \left[\phi^{(0)} + \phi^{(1)}i + \phi^{(2)}j + \phi^{(3)}k\right]x_3\\
&= \rho.
\end{aligned} \tag{3}$$

Then equation (1) is equivalent to the system

$$\begin{cases}
\chi^{(0)}x_0 + \varphi^{(0)}x_1 + \psi^{(0)}x_2 + \phi^{(0)}x_3 = \rho^{(0)},\\
\chi^{(1)}x_0 + \varphi^{(1)}x_1 + \psi^{(1)}x_2 + \phi^{(1)}x_3 = \rho^{(1)},\\
\chi^{(2)}x_0 + \varphi^{(2)}x_1 + \psi^{(2)}x_2 + \phi^{(2)}x_3 = \rho^{(2)},\\
\chi^{(3)}x_0 + \varphi^{(3)}x_1 + \psi^{(3)}x_2 + \phi^{(3)}x_3 = \rho^{(3)},
\end{cases} \tag{4}$$

so that the problem has been reduced to solution of a system of real linear equation.

As the next problem, consider a system of linear quaternionic equations of the form

$$\begin{cases}
ayb + cxd = e,\\
fyg + \gamma x\delta = h,
\end{cases} \tag{5}$$

where $a \neq 0, b \neq 0, c, d, e, f, g, \gamma, \delta, h$ are given quaternions, x, y are unknown.

We multiply the first equation by conjugate $\bar{a}$ on the left and by $\bar{b}$ on the right:

$$|a|^2 y|b|^2 + \bar{a}cxd\bar{b} = \bar{a}e\bar{b}.$$

Then we multiply this equation by $\frac{1}{|a|^2|b|^2}$:

$$y = \frac{1}{|a|^2|b|^2}\bar{a}e\bar{b} - \frac{1}{|a|^2|b|^2}\bar{a}cxd\bar{b}. \tag{6}$$

Let us substitute the last expression into the second equation of system (5):

$$\frac{1}{|a|^2|b|^2}f\bar{a}e\bar{b}g - \frac{1}{|a|^2|b|^2}f\bar{a}cxd\bar{b}g + \gamma x\delta = h.$$

The last equation is of the form (1) with

$$\alpha := -\frac{1}{|a|^2|b|^2} f\bar{a}c, \ \ \beta := d\bar{b}g, \ \ \rho := h - \frac{1}{|a|^2|b|^2} f\bar{a}e\bar{b}g.$$

Solving the received equation we find x. Then from equation (6), we calculate y.

4. Examples

4.1. *Solve in* $\mathbb{H}$ *the following equation:*

$$(i-2k)xj + (1-j)x(i+k) = 1 + 4i + 5j + 6k.$$

This equation is of the form (1). Here $\alpha\beta = \alpha\vec{\beta} = (i-2k)j = 2i+k$, $\gamma\delta = \gamma\vec{\delta} = (1-j)(i+k) = 2k$, $\alpha\beta^{(1)} = \alpha\beta^{(3)} = \gamma\delta^{(2)} = 0$, $\alpha\beta^{(2)} = i-2k$, $\gamma\delta^{(1)} = \gamma\delta^{(3)} = 1-j$. Then from (3) we have:

$$(2i+3k)x_0 - jx_1 + (i+2k)x_2 + (1+4j)x_3 = 1 + 4i + 5j + 6k,$$

that is equivalent to the system

$$\begin{cases} x_3 = 1, \\ 2x_0 + x_2 = 4, \\ -x_1 + 4x_3 = 5, \\ 3x_0 + 2x_2 = 6. \end{cases}$$

The last system has one solution: $x_0 = 2,\ x_1 = -1,\ x_2 = 0, x_3 = 1$, then $x = 2 - i + k$.

4.2. *For which* $\vartheta \in \mathbb{H}$ *has the equation*

$$x\vartheta - \vartheta x = 1 \tag{7}$$

no root?

We apply the formula (2) to the product $x\vartheta$. Then equation (7) is equivalent to the equation

$$\vartheta x - 2\vec{\vartheta}\vec{x} - 2(\vec{\vartheta}, \vec{x}) - \vartheta x = 1,$$

or in an equivalent form

$$-2\left(\vec{\vartheta}\vec{x} + (\vec{\vartheta}, \vec{x})\right) = 1. \tag{8}$$

Now we find the scalar part of the left side of equalities (8), we get:

$$\begin{aligned} \mathrm{Sc}\left(\vec{\vartheta}\vec{x} + (\vec{\vartheta}, \vec{x})\right) &= \mathrm{Sc}(\vec{\vartheta}\vec{x}) + (\vec{\vartheta}, \vec{x}) \\ &= -\vartheta^{(1)}x_1 - \vartheta^{(2)}x_2 - \vartheta^{(3)}x_3 + \vartheta^{(1)}x_1 + \vartheta^{(2)}x_2 + \vartheta^{(3)}x_3 = 0. \end{aligned} \tag{9}$$

So, equation (7) has no root for any $\vartheta \in \mathbb{H}$, because the scalar part of the left side of equation (7) is equal to 0, and the scalar part of the right side is equal to 1. A particular case of such equation is described in [2].

4.3. *Solve in* $\mathbb{H}$ *the following system:*

$$\begin{cases} iyj + jxi = k, \\ kyi + ixk = 1 + i. \end{cases} \tag{10}$$

This is a system of the form (5). We find $\alpha := -\frac{1}{|a|^2|b|^2} f\bar{a}c = kij = -1$, $\beta := d\bar{b}g = -iji = -j$, $\rho := h - \frac{1}{|a|^2|b|^2} f\bar{a}e\bar{b}g = 1 + i - kikji = 1 + i - j$. Then we can calculate x from the following equation of type (1):

$$xj + ixk = 1 + i - j.$$

Then from (3) we get:

$$-2x_2 - 2ix_3 = 1 + i - j.$$

This equation is equivalent to the system

$$\begin{cases} -2x_2 = 1, \\ -2x_3 = 1, \\ 0 = -1, \\ 0 = 0. \end{cases}$$

The last system has no solution, and thus system (10) has no solution.

4.4. *Solve in* $\mathbb{H}$ *the following system:*

$$\begin{cases} iyj + jxi = k, \\ kyi + ixk = 1 + i + j. \end{cases} \tag{11}$$

This is a system of the form (5). Using data from the example 4.3 we get the equation allowing to find x:

$$-2x_2 - 2ix_3 = 1 + i.$$

Then $x = x_0 + x_1 i - \frac{1}{2}j - \frac{1}{2}k$, for any $x_0, x_1 \in \mathbb{R}$. From formula (6), we get: $y = ikj - ij(x_0 + x_1 i - \frac{1}{2}j - \frac{1}{2}k)ij = (1 + x_0) - x_1 i + \frac{1}{2}j - \frac{1}{2}k$ for any $x_0, x_1 \in \mathbb{R}$.

Then the solution of the system (11) has the form

$$\begin{cases} x = x_0 + x_1 i - \frac{1}{2}j - \frac{1}{2}k, \\ y = (1 + x_0) - x_1 i + \frac{1}{2}j - \frac{1}{2}k, \end{cases}$$

for any $x_0, x_1 \in \mathbb{R}$.

5. The General Linear Equation with One Unknown and Systems of Equations with Two Unknowns

Consider the general linear quaternionic equation:

$$\sum_{s=1}^{N} \alpha_s x \beta_s = \rho. \tag{12}$$

Similarly to section 3, we can write the equation to which (12) is equivalent:

$$
\begin{aligned}
&Ax_0+\Big[(A-2B)\,i-2C_1\Big]x_1+\Big[(A-2B)\,j-2C_2\Big]x_2\\
&+\Big[(A-2B)\,k-2C_3\Big]x_3=\rho,
\end{aligned}
\tag{13}
$$

where $A := \sum\limits_{s=1}^{N} \alpha_s\beta_s$, $B := \sum\limits_{s=1}^{N} \alpha_s\vec{\beta}_s$, $C_q := \sum\limits_{s=1}^{N} \alpha_s\beta_s^{(q)}$, $q = 1,2,3$. If all $\beta_s \in \overrightarrow{\mathbb{H}}$, then $A = B$.

We denote

$$
\begin{aligned}
A &:= A^{(0)}+A^{(1)}i+A^{(2)}j+A^{(3)}k,\\
(A-2B)\,i-2C_1 &:= \lambda=\lambda^{(0)}+\lambda^{(1)}i+\lambda^{(2)}j+\lambda^{(3)}k,\\
(A-2B)\,j-2C_2 &:= \mu=\mu^{(0)}+\mu^{(1)}i+\mu^{(2)}j+\mu^{(3)}k,\\
(A-2B)\,k-2C_3 &:= \nu=\nu^{(0)}+\nu^{(1)}i+\nu^{(2)}j+\nu^{(3)}k.
\end{aligned}
\tag{14}
$$

Then equation (12) is equivalent to the system

$$
\begin{cases}
A^{(0)}x_0+\lambda^{(0)}x_1+\mu^{(0)}x_2+\nu^{(0)}x_3=\rho^{(0)},\\
A^{(1)}x_0+\lambda^{(1)}x_1+\mu^{(1)}x_2+\nu^{(1)}x_3=\rho^{(1)},\\
A^{(2)}x_0+\lambda^{(2)}x_1+\mu^{(2)}x_2+\nu^{(2)}x_3=\rho^{(2)},\\
A^{(3)}x_0+\lambda^{(3)}x_1+\mu^{(3)}x_2+\nu^{(3)}x_3=\rho^{(3)}.
\end{cases}
\tag{15}
$$

Proposition 1. *If $x = x_0 + \vec{x}$ is a root of equation* (12)*, then:*

1. *$(-x)$ is a root of equation* (12) *if and only if $\rho = 0$;*
2. *the conjugate $\bar{x}$ is a root of equation* (12) *if and only if $Ax_0 = \rho$.*

The proof follows from (13).

Consider any system of linear quaternionic equations of the form

$$
\begin{cases}
ayb+\sum\limits_{p=1}^{n} c_p x d_p = e,\\
\sum\limits_{m=1}^{r} f_m y g_m+\sum\limits_{t=1}^{\ell} \gamma_t x \delta_t = h,
\end{cases}
\tag{16}
$$

where $a \neq 0, b \neq 0, c_p, d_p, e, f_m, g_m, \gamma_t, \delta_t, h$ are given quaternions, x, y are unknown.

From the first equation we can find y as an expression with x and substitute it into the second equation, then for x we have an equation of the form (12) with $N := rn + \ell$. From (15) we find x, and then we find y.

Let the following system of general form be given:

$$
\begin{cases}
\sum\limits_{\tau=1}^{v} a_\tau y b_\tau+\sum\limits_{p=1}^{n} c_p x d_p = e,\\
\sum\limits_{m=1}^{r} f_m y g_m+\sum\limits_{t=1}^{\ell} \gamma_t x \delta_t = h.
\end{cases}
\tag{17}
$$

In order to solve system (17), we write every addend $\sum\limits_{\tau=1}^{v} a_\tau y b_\tau$, $\sum\limits_{p=1}^{n} c_p x d_p$, $\sum\limits_{m=1}^{r} f_m y g_m$, $\sum\limits_{t=1}^{\ell} \gamma_t x \delta_t$ in the form similar to (13):

$$\sum_{\tau=1}^{v} a_\tau y b_\tau = \widetilde{A} y_0 + \left[\left(\widetilde{A} - 2\widetilde{B}\right) i - 2\widetilde{C}_1\right] y_1 + \left[\left(\widetilde{A} - 2\widetilde{B}\right) j - 2\widetilde{C}_2\right] y_2 + \left[\left(\widetilde{A} - 2\widetilde{B}\right) k - 2\widetilde{C}_3\right] y_3,$$

where $\widetilde{A} := \sum\limits_{\tau=1}^{v} a_\tau b_\tau$, $\widetilde{B} := \sum\limits_{\tau=1}^{v} a_\tau \vec{b}_\tau$, $\widetilde{C}_q := \sum\limits_{\tau=1}^{v} a_\tau b_\tau^{(q)}$, $q = 1, 2, 3$,

$$\sum_{p=1}^{n} c_p x d_p = \widehat{A} x_0 + \left[\left(\widehat{A} - 2\widehat{B}\right) i - 2\widehat{C}_1\right] x_1 + \left[\left(\widehat{A} - 2\widehat{B}\right) j - 2\widehat{C}_2\right] x_2 + \left[\left(\widehat{A} - 2\widehat{B}\right) k - 2\widehat{C}_3\right] x_3,$$

where $\widehat{A} := \sum\limits_{p=1}^{n} c_p d_p$, $\widehat{B} := \sum\limits_{p=1}^{n} c_p \vec{d}_p$, $\widehat{C}_q := \sum\limits_{p=1}^{n} c_p d_p^{(q)}$, $q = 1, 2, 3$,

$$\sum_{m=1}^{r} f_m y g_m = A' y_0 + \left[\left(A' - 2B'\right) i - 2C_1'\right] y_1 + \left[(A' - 2B') j - 2C_2'\right] y_2 + \left[(A' - 2B') k - 2C_3'\right] y_3,$$

where $A' := \sum\limits_{m=1}^{r} f_m g_m$, $B' := \sum\limits_{m=1}^{r} f_m \vec{g}_m$, $C_q' := \sum\limits_{m=1}^{r} f_m g_m^{(q)}$, $q = 1, 2, 3$,

$$\sum_{t=1}^{\ell} \gamma_t x \delta_t = A'' x_0 + \left[\left(A'' - 2B''\right) i - 2C_1''\right] x_1 + \left[(A'' - 2B'') j - 2C_2''\right] x_2 + \left[(A'' - 2B'') k - 2C_3''\right] x_3,$$

where $A'' := \sum\limits_{t=1}^{\ell} \gamma_t \delta_t$, $B'' := \sum\limits_{t=1}^{\ell} \gamma_t \vec{\delta}_t$, $C_q'' := \sum\limits_{t=1}^{\ell} \gamma_t \delta_t^{(q)}$, $q = 1, 2, 3$.

We shall introduce notations similar to those of (14), then each of the equations of system (17) will be equivalent to a system of four equations with eight unknowns. Thus, the system (17) with two quaternionic unknowns x, y is equivalent to a system of eight equations with eight real unknowns $x_\eta, y_\eta,\ \eta = 0, 1, 2, 3$.

6. Examples

6.1. *Solve in* $\mathbb{H}$ *the following system:*

$$\begin{cases} y - ixj - jx(i + j) = 0, \\ kyj + (i - k)yi + jxk + kxj = 23. \end{cases} \tag{18}$$

This is a system of the form (16). From the first equation we find y and substitute it into the second equation. Then we have the following equation:

$$-jx + ix(1-k) + (1+j)xk + (i+k)x(-1-k) + jxk + kxj = 23.$$

We calculate coefficients of the equality (13) for the case of (18):

$$A = -j + i(1-k) + (1+j)k + (i+k)(-1-k) + jk + kj = 1 + i + j,$$
$$B = -j - ik + (1+j)k - (i+k)k + jk + kj = 1 + i + j + k,$$
$$C_1 = C_2 = 0, \ C_3 = 1 - 2i + j - k.$$

Then to find x we use the following system (obtained by (14), (15)):

$$\begin{cases} x_0 + x_1 + x_2 = 23, \\ x_0 - x_1 + 2x_2 + 3x_3 = 0, \\ x_0 - 2x_1 - x_2 - x_3 = 0, \\ x_1 - x_2 + x_3 = 0. \end{cases}$$

It has one solution: $x_0 = 13, \ x_1 = 7, \ x_2 = 3, \ x_3 = -4$. That is $x = 13 + 7i + 3j - 4k$. Then $y = ixj + jx(i+j) = -13 + i - 17j - 4k$.

6.2. *Solve in* $\mathbb{H}$ *the following system:*

$$\begin{cases} iyj + jyi + kxi + ixk = 0, \\ (1+i)y(1+j) + (1+j)y(1+i) + \\ +(1+k)x(1+i) + (1+i)x(1+k) = 2k. \end{cases} \tag{19}$$

This is a system of the form (17).

1) For the first two addends of the first equation we calculate coefficients: $\widetilde{A} = \widetilde{B} = 0$, $\widetilde{C}_1 = j$, $\widetilde{C}_2 = i$, $\widetilde{C}_3 = 0$.
2) For the third and fourth addends of the first equation we calculate coefficients: $\widehat{A} = \widehat{B} = 0$, $\widehat{C}_1 = k$, $\widehat{C}_2 = 0$, $\widehat{C}_3 = i$.
3) For the first two addends of the second equation we calculate coefficients: $A' = 2 + 2i + 2j$, $B' = i + j$, $C_1' = 1 + j$, $C_2' = 1 + i$, $C_3' = 0$.
4) For the third and fourth addends of the second equation we calculate coefficients: $A'' = 2+2i+2k$, $B'' = i+k$, $C_1'' = 1+k$, $C_2'' = 0$, $C_3'' = 1+i$.

Then similarly to (13), we have a system with two equations:

$$\begin{cases} -2jy_1 - 2iy_2 - 2kx_1 - 2ix_3 = 0, \\ (2+2i+2j)y_0 + (-2+2i-2j)y_1 + (-2-2i+2j)y_2 + 2ky_3 \\ +(2+2i+2k)x_0 + (-2+2i-2k)x_1 + 2jx_2 + (-2-2i+2k)x_3 = 2k. \end{cases}$$

From the last system we get the following system with eight real unknowns:

$$\begin{cases} y_2 + x_3 = 0, \\ y_1 = x_1 = 0, \\ y_0 - y_1 - y_2 + x_0 - x_1 - x_3 = 0, \\ y_0 + y_1 - y_2 + x_0 + x_1 - x_3 = 0, \\ y_0 - y_1 + y_2 + x_2 = 0, \\ y_3 + x_0 - x_1 + x_3 = 1. \end{cases}$$

The solution of the last system and accordingly of the system (19) is the following:

$$\begin{cases} x = (1 - x_3 - y_3) + (1 - y_3)j + x_3 k, \\ \qquad y = (x_3 + y_3 - 1) - x_3 j + y_3 k \end{cases}$$

for any $x_3, y_3 \in \mathbb{R}$.

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