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CONSTRUCTIVE DESCRIPTION OF MONOGENIC FUNCTIONS IN A HARMONIC ALGEBRA OF THE THIRD RANK

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By using analytic functions of a complex variable, we give a constructive description of monogenic functions that take values in a commutative harmonic algebra of the third rank over the field of complex numbers. We establish an isomorphism between algebras of monogenic functions in the case of transition from one harmonic basis to another.

The efficiency of methods of the theory of analytic functions of a complex variable in the investigation of plane potential fields inspires mathematicians to develop analogous methods for space fields. These methods can be based on mappings of Banach algebras.

In [1 – 3], commutative associative Banach algebras were constructed such that twice Gâteaux differentiable functions with values in these algebras have components satisfying the three-dimensional Laplace equation.

Let $\mathbb{A}$ be a commutative associative Banach algebra (over the field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$) with basis $\{e_k\}_{k=1}^{n}$, $3 \leq n \leq \infty$. If the basis elements e_1, e_2, and e_3 satisfy the condition

$$e_1^2 + e_2^2 + e_3^2 = 0, \tag{1}$$

then, by virtue of the equality

$$\Delta_3 \Phi := \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = \Phi''(\zeta)\left(e_1^2 + e_2^2 + e_3^2\right),$$

every twice Gâteaux differentiable function $\Phi(\zeta)$ of a variable $\zeta = xe_1 + ye_2 + ze_3$, $x, y, z \in \mathbb{R}$, with values in the algebra $\mathbb{A}$ satisfies the three-dimensional Laplace equation

$$\Delta_3 \Phi = 0,$$

i.e., it is a *monogenic potential* [3, p. 30].

Following [1–3], we call a triple of vectors e_1, e_2, e_3 satisfying relation (1) a *harmonic triple,* and an algebra $\mathbb{A}$ that contains a harmonic triple a *harmonic algebra.*

In [1–3], all harmonic bases in third-rank algebras over the field $\mathbb{C}$ were described and it was proved that harmonic three-dimensional algebras over the field $\mathbb{R}$ do not exist. Some four-dimensional harmonic algebras over the field $\mathbb{R}$ were constructed in [4]. An infinite-dimensional harmonic algebra over the field $\mathbb{R}$ was constructed in [3, 5].

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In the present paper, we consider the harmonic algebra $\mathbb{A}_3$ [2, 3], whose basis (note that it is not harmonic) consists of the identity 1 of the algebra and of elements ρ_1 and ρ_2 that satisfy the multiplication rules

$$\rho_1^2 = \rho_2, \qquad \rho_1\rho_2 = \rho_2^2 = 0. \tag{2}$$

By virtue of the fact that, similarly to complex potentials of plane fields, monogenic potentials form a functional algebra in the domain of definition, the algebra $\mathbb{A}_3$ contains a collection of monogenic potentials no less than the set of holomorphic functions in the algebra $\mathbb{C}$, and there is a no lesser collection of tools for their construction. In Theorem 1.7 in [3], monogenic potentials were explicitly constructed in the form of principal extensions of holomorphic functions of a complex variable to the algebra $\mathbb{A}_3$.

In what follows, we give a constructive description of all monogenic potentials in the algebra $\mathbb{A}_3$ using analytic functions of a complex variable. We also establish an isomorphism between the algebras of monogenic potentials $\Phi(\zeta)$ of a variable $\zeta = xe_1 + ye_2 + ze_3$, $x, y, z \in \mathbb{R}$, under the variation in the harmonic basis $\{e_1, e_2, e_3\}$ in the algebra $\mathbb{A}_3$.

1. Constructive Description of Monogenic Functions in the Algebra $\mathbb{A}_3$

It was shown in Theorem 1.6 in [3] that harmonic bases in the algebra $\mathbb{A}_3$ are the bases $\{e_1, e_2, e_3\}$ whose decompositions in the basis $\{1, \rho_1, \rho_2\}$ have the form

$$\begin{aligned} e_1 &= 1, \\ e_2 &= n_1 + n_2\rho_1 + n_3\rho_2, \\ e_3 &= m_1 + m_2\rho_1 + m_3\rho_2, \end{aligned} \tag{3}$$

where n_k and m_k, $k = 1, 2, 3$, are complex numbers that satisfy the system of equations

$$\begin{aligned} 1 + n_1^2 + m_1^2 &= 0, \\ n_1 n_2 + m_1 m_2 &= 0, \\ n_2^2 + m_2^2 + 2(n_1 n_3 + m_1 m_3) &= 0, \\ n_2 m_3 - n_3 m_2 &\neq 0, \end{aligned} \tag{4}$$

and at least one number in each of the pairs (n_1, n_2) and (m_1, m_2) is different from zero. Moreover, multiplying the elements of harmonic bases of the form (3) by arbitrary invertible elements of the algebra, one can obtain all harmonic bases in the algebra $\mathbb{A}_3$ [3, p. 29].

In the algebra $\mathbb{A}_3$, we consider the linear span $E_3 := \{\zeta = xe_1 + ye_2 + ze_3 : x, y, z \in \mathbb{R}\}$ generated by the vectors $e_1 = 1$, e_2, and e_3. We associate a subset S of the three-dimensional space $\mathbb{R}^3$ with the set

$$S_\zeta = \{\zeta = xe_1 + ye_2 + ze_3 : (x, y, z) \in S\} \quad \text{in} \quad E_3.$$

A continuous function $\Phi : \Omega_\zeta \to \mathbb{A}_3$ is called *monogenic* in a domain $\Omega_\zeta \subset E_3$ if Φ is Gâteaux differentiable at every point of this domain, i.e., if, for every $\zeta \in \Omega_\zeta$, there exists an element $\Phi'(\zeta)$ of the algebra $\mathbb{A}_3$ such that

$$\lim_{\varepsilon \to 0+0} \big(\Phi(\zeta + \varepsilon h)) - \Phi(\zeta) \big) \varepsilon^{-1} = h \Phi'(\zeta) \qquad \forall h \in E_3 .$$

The element $\Phi'(\zeta)$ is called the *Gâteaux derivative* of the function Φ at the point ζ.

Necessary and sufficient conditions for the monogeneity of a function Φ (Cauchy–Riemann conditions) were established in Theorem 1.3 in [3]. We write these conditions here in a compact form:

$$\frac{\partial \Phi}{\partial y} = \frac{\partial \Phi}{\partial x} e_2 , \qquad \frac{\partial \Phi}{\partial z} = \frac{\partial \Phi}{\partial x} e_3 . \tag{5}$$

The monogeneity of a complex-valued function $F(\xi)$ of a complex variable ξ is understood as its holomorphy in the case where $\xi = \tau + i\eta$ or antiholomorphy in the case where $\xi = \tau - i\eta$, $\tau, \eta \in \mathbb{R}$.

Let f be a linear continuous functional defined on $\mathbb{A}_3$ whose kernel is the maximal ideal

$$\mathcal{I} := \{ \lambda_1 \rho_1 + \lambda_2 \rho_2 : \lambda_1, \lambda_2 \in \mathbb{C} \}$$

and let $f(1) = 1$. It is known [6, p. 147] that f is also a multiplicative functional, i.e., the equality $f(ab) = f(a)f(b)$ holds for all $a, b \in \mathbb{A}_3$.

It follows from the decomposition of the resolvent (see [3, p. 30])

$$\begin{aligned}(t - \zeta)^{-1} = {} & \frac{1}{t - x - n_1 y - m_1 z} + \frac{n_2 y + m_2 z}{(t - x - n_1 y - m_1 z)^2} \rho_1 \\ & + \left(\frac{n_3 y + m_3 z}{(t - x - n_1 y - m_1 z)^2} + \frac{(n_2 y + m_2 z)^2}{(t - x - n_1 y - m_1 z)^3} \right) \rho_2 \\ & \forall t \in \mathbb{C} : \; t \neq x + n_1 y + m_1 z \end{aligned}$$

that the points $(x, y, z) \in \mathbb{R}^3$ corresponding to the noninvertible elements $\zeta = x e_1 + y e_2 + z e_3$ of the algebra $\mathbb{A}_3$ form the straight line

$$L : \quad \begin{cases} x + y \operatorname{Re} n_1 + z \operatorname{Re} m_1 = 0, \\ y \operatorname{Im} n_1 + z \operatorname{Im} m_1 = 0 \end{cases}$$

in the three-dimensional space $\mathbb{R}^3$.

A domain $\Omega \subset \mathbb{R}^3$ is called *convex in the direction of the straight line* L if it contains each segment that connects two points of it and is parallel to the straight line L.

Lemma 1. *Let a domain $\Omega \subset \mathbb{R}^3$ be convex in the direction of the straight line L and let Φ: $\Omega_\zeta \to \mathbb{A}_3$ be a monogenic function in the domain Ω_ζ. If points $\zeta_1, \zeta_2 \in \Omega_\zeta$ are such that $\zeta_2 - \zeta_1 \in L_\zeta$, then*

$$\Phi(\zeta_1) - \Phi(\zeta_2) \in \mathcal{I}. \tag{6}$$

Proof. Let (x_1, y_1, z_1) and (x_2, y_2, z_2) be points of the domain Ω such that the segment that connects them is parallel to the straight line L.

In the domain Ω, we construct two surfaces with common edge, namely a surface Q that contains the point (x_1, y_1, z_1) and a surface Σ that contains the point (x_2, y_2, z_2), such that the restrictions of the functional f to the corresponding subsets Q_ζ and Σ_ζ of the domain Ω_ζ are bijections of these subsets to the same domain G of the complex plane, and, moreover, at every point $\zeta_0 \in Q_\zeta$ (or $\zeta_0 \in \Sigma_\zeta$), one has

$$\lim_{\varepsilon \to 0+0} \left(\Phi(\zeta_0 + \varepsilon(\zeta - \zeta_0)) - \Phi(\zeta_0) \right) \varepsilon^{-1} = \Phi'(\zeta_0)(\zeta - \zeta_0) \tag{7}$$

for all $\zeta \in Q_\zeta$ such that $\zeta_0 + \varepsilon(\zeta - \zeta_0) \in Q_\zeta$ for any $\varepsilon \in (0,1)$ $\big($or, respectively, for all $\zeta \in \Sigma_\zeta$ such that $\zeta_0 + \varepsilon(\zeta - \zeta_0) \in \Sigma_\zeta$ for any $\varepsilon \in (0,1)\big)$.

As the surface Q in the domain Ω, we take a fixed equilateral triangle with vertices A_1, A_2, and A_3 centered at the point (x_1, y_1, z_1) the plane of which is perpendicular to the straight line L. We now continue the construction of the surface Σ.

Consider the triangle with vertices A_1', A_2', and A_3' centered at the point (x_2, y_2, z_2), lying in the domain Ω, and such that its sides $A_1'A_2'$, $A_2'A_3'$, and $A_1'A_3'$ are parallel to the segments A_1A_2, A_2A_3, and A_1A_3, respectively, and have smaller lengths than the sides of the triangle $A_1A_2A_3$. Since the domain Ω is convex in the direction of the straight line L, we conclude that the prism with vertices $A_1', A_2', A_3', A_1'', A_2''$, and A_3'' such that the points A_1'', A_2'', and A_3'' lie in the plane of the triangle $A_1A_2A_3$ and its edges $A_m'A_m''$, $m = \overline{1, 3}$, are parallel to the straight line L is completely contained in Ω.

We now fix a triangle with vertices B_1, B_2, and B_3 such that the point B_m lies on the segment $A_m'A_m''$ for $m = \overline{1, 3}$ and the truncated pyramid with vertices A_1, A_2, A_3, B_1, B_2, and B_3 and lateral edges A_mB_m, $m = \overline{1, 3}$, is completely contained in the domain Ω.

Finally, in the plane of the triangle $A_1'A_2'A_3'$, we fix a triangle T with vertices C_1, C_2, and C_3 such that its sides C_1C_2, C_2C_3, and C_1C_3 are parallel to the segments $A_1'A_2'$, $A_2'A_3'$, and $A_1'A_3'$, respectively, and have smaller lengths than the sides of the triangle $A_1'A_2'A_3'$. By construction, the truncated pyramid with vertices B_1, B_2, B_3, C_1, C_2, and C_3 and lateral edges B_mC_m, $m = \overline{1, 3}$, is completely contained in the domain Ω.

Let Σ denote the surface formed by the triangle T and the lateral surfaces of the truncated pyramids $A_1A_2A_3B_1B_2B_3$ and $B_1B_2B_3C_1C_2C_3$.

Since the surfaces Q and Σ have a common edge, the sets Q_ζ and Σ_ζ are mapped by the functional f onto the same domain G of the complex plane. In the domain G, we define two complex-valued functions H_1 and H_2 such that, for every $\xi \in G$, one has

$$H_1(\xi) = f(\Phi(\zeta)), \quad \text{where} \quad \xi = f(\zeta) \quad \text{and} \quad \zeta \in Q_\zeta,$$

$$H_2(\xi) = f(\Phi(\zeta)), \quad \text{where} \quad \xi = f(\zeta) \quad \text{and} \quad \zeta \in \Sigma_\zeta.$$

Let us show that H_1 and H_2 are functions of the complex variable ξ monogenic in G. Note that, acting by the functional f on equality (7) and using the linearity, continuity, and multiplicativity of the functional, we get

$$\lim_{\varepsilon \to 0+0} \big(f(\Phi(\zeta_0 + \varepsilon(\zeta - \zeta_0))) - f(\Phi(\xi)) \big) \varepsilon^{-1} = f(\Phi'(\zeta_0))(f(\zeta) - f(\zeta_0)).$$

This implies that the functions H_1 and H_2 have derivatives at the point $f(\zeta_0) \in G$ in all directions, and, furthermore, these derivatives are equal for each of the functions H_1 and H_2. Therefore, according to Theorem 21 in [7], the functions H_1 and H_2 are monogenic in the domain G.

According to the definition of the functions H_1 and H_2, we have $H_1(\xi) \equiv H_2(\xi)$ on the boundary of the domain G. By virtue of the monogeneity of the functions H_1 and H_2 in the domain G, the identity $H_1(\xi) \equiv H_2(\xi)$ holds everywhere in G. Consequently, for $\zeta_1 := x_1e_1 + y_1e_2 + z_1e_3$ and $\zeta_2 := x_2e_1 + y_2e_2 + z_2e_3$, one has

$$f(\Phi(\zeta_2) - \Phi(\zeta_1)) = f(\Phi(\zeta_2)) - f(\Phi(\zeta_1)) = 0,$$

i.e., $\Phi(\zeta_2) - \Phi(\zeta_1)$ belongs to the kernel $\mathcal{I}$ of the functional f.

The lemma is proved.

Note that the condition of the convexity of the domain Ω in the direction of the straight line L in Lemma 1 is essential. In what follows, we construct an example of a domain Ω that is not convex in the direction of the straight line L and a monogenic function $\Phi : \Omega_\zeta \to \mathbb{A}_3$ for which relation (6) does not hold for some $\zeta_1, \zeta_2 \in \Omega_\zeta$ such that $\zeta_2 - \zeta_1 \in L_\zeta$.

Let D denote the domain in $\mathbb{C}$ onto which the domain Ω_ζ is mapped by the functional f. Consider the linear operator A that associates every monogenic function $\Phi : \Omega_\zeta \to \mathbb{A}_3$ with a function $F : D \to \mathbb{C}$ according to the relation $F(\xi) := f(\Phi(\zeta))$, where $\zeta = xe_1 + ye_2 + ze_3$ and $\xi := f(\zeta) = x + n_1 y + m_1 z$. It follows from Lemma 1 that the value of $F(\xi)$ is independent of the choice of a point ζ for which $f(\zeta) = \xi$.

By analogy with Theorem 2.4 in [3], we prove the following statement:

Theorem 1. *Let a domain Ω be convex in the direction of the straight line L. Then every function $\Phi : \Omega_\zeta \to \mathbb{A}_3$ monogenic in the domain Ω_ζ can be represented in the form*

$$\Phi(\zeta) = \frac{1}{2\pi i} \int_{\Gamma_\zeta} (A\Phi)(t)(t - \zeta)^{-1} dt + \Phi_0(\zeta) \qquad \forall \zeta \in \Omega_\zeta, \tag{8}$$

where Γ_ζ is a closed Jordan rectifiable curve that lies in the domain D and encloses the point $f(\zeta)$, and $\Phi_0 : \Omega_\zeta \to \mathcal{I}$ is a certain function monogenic in the domain Ω_ζ and taking values in the ideal $\mathcal{I}$.

Note that the complex number $\xi = f(\zeta)$ is the spectrum of the element ζ of the algebra $\mathbb{A}_3$, and the integral in equality (8) is the principal extension of the monogenic function $F(\xi) = (A\Phi)(\xi)$ of the complex variable ξ to the domain Ω_ζ.

It follows from Theorem 1 that the algebra of functions monogenic in the domain Ω_ζ can be decomposed into the direct sum of the algebra of principal extensions of monogenic functions of a complex variable in Ω_ζ and the algebra of functions monogenic in Ω_ζ and taking values in the ideal $\mathcal{I}$.

The principal extension of a function of a complex variable $F : D \to \mathbb{C}$ to the domain $\Pi_\zeta := \{\zeta \in E_3 : f(\zeta) \in D\}$ was constructed in explicit form in Theorem 1.7 in [3]; the decomposition of this extension in the basis $\{1, \rho_1, \rho_2\}$ has the form

$$\frac{1}{2\pi i}\int_{\Gamma_\zeta} F(t)(t-\zeta)^{-1}dt = F(x+n_1y+m_1z) + (n_2y+m_2z)\,F'(x+n_1y+m_1z)\rho_1$$

$$+\left((n_3y+m_3z)\,F'(x+n_1y+m_1z) + \frac{(n_2y+m_2z)^2}{2}F''(x+n_1y+m_1z)\right)\rho_2 \qquad (9)$$

$$\forall \zeta = xe_1 + ye_2 + ze_3 \in \Pi_\zeta .$$

It is obvious that a domain Π of the space $\mathbb{R}^3$ congruent to the domain Π_ζ is an infinite cylinder whose generatrices are parallel to the straight line L.

In the theorem below, all monogenic functions defined in the domain Ω_ζ and taking values in the ideal $\mathcal{I}$ are described with the use of monogenic functions of the corresponding complex variable.

Theorem 2. *Let a domain Ω be convex in the direction of the straight line L. Then every monogenic function $\Phi_0 : \Omega_\zeta \to \mathcal{I}$ that takes values in the maximal ideal $\mathcal{I}$ can be represented in the form*

$$\Phi_0(\zeta) = F_1(\xi)\rho_1 + \big(F_2(\xi) + (n_2y+m_2z)\,F_1'(\xi)\big)\rho_2 \qquad (10)$$

$$\forall \zeta = xe_1 + ye_2 + ze_3 \in \Omega_\zeta ,$$

where F_1 and F_2 are arbitrary functions monogenic in the domain D and $\xi = x + n_1y + m_1z$.

Proof. Since Φ_0 takes values in the maximal ideal, we have

$$\Phi_0(\zeta) = V_1(x,y,z)\,\rho_1 + V_2(x,y,z)\,\rho_2 , \qquad (11)$$

where $V_k : \Omega \to \mathbb{C}$ for $k = 1, 2$. The function $\Phi_0(\zeta)$ satisfies the conditions of monogeneity (5) for $\Phi = \Phi_0$. Substituting relations (3) and (11) into these conditions and taking into account the uniqueness of the decomposition of elements of the algebra $\mathbb{A}_3$ in the basis $\{1, \rho_1, \rho_2\}$, we obtain the following system of equations for the determination of the functions V_1 and V_2:

$$\begin{aligned}
\frac{\partial V_1}{\partial y} &= n_1 \frac{\partial V_1}{\partial x}, \\
\frac{\partial V_2}{\partial y} &= n_2 \frac{\partial V_1}{\partial x} + n_1 \frac{\partial V_2}{\partial x}, \\
\frac{\partial V_1}{\partial z} &= m_1 \frac{\partial V_1}{\partial x}, \\
\frac{\partial V_2}{\partial z} &= m_2 \frac{\partial V_1}{\partial x} + m_1 \frac{\partial V_2}{\partial x}.
\end{aligned} \tag{12}$$

Using the first and the third equation of system (12), we determine the function V_1. For this purpose, we first separate the real and the imaginary part of the expression

$$\xi = (x + y \operatorname{Re} n_1 + z \operatorname{Re} m_1) + i (y \operatorname{Im} n_1 + z \operatorname{Im} m_1) =: \tau + i\eta \tag{13}$$

and note that the indicated equations yield

$$\frac{\partial V_1}{\partial \eta} \operatorname{Im} n_1 = i \frac{\partial V_1}{\partial \tau} \operatorname{Im} n_1, \qquad \frac{\partial V_1}{\partial \eta} \operatorname{Im} m_1 = i \frac{\partial V_1}{\partial \tau} \operatorname{Im} m_1. \tag{14}$$

It follows from the first equation of system (14) that at least one of the numbers $\operatorname{Im} n_1$ and $\operatorname{Im} m_1$ is not equal to zero. Using (14), we get

$$\frac{\partial V_1}{\partial \eta} = i \frac{\partial V_1}{\partial \tau}. \tag{15}$$

We prove that $V_1(x_1, y_1, z_1) = V_1(x_2, y_2, z_2)$ for points $(x_1, y_1, z_1), (x_2, y_2, z_2) \in \Omega$ such that the segment that connects these points is parallel to the straight line L. To this end, we consider the domain G in $\mathbb{C}$ and the surfaces Q and Σ in Ω defined in the proof of Lemma 1 and introduce two complex-valued functions H_1 and H_2 in G as follows:

$$\begin{aligned}
H_1(\xi) &= V_1(x, y, z) \quad \text{for} \quad (x, y, z) \in Q, \\
H_2(\xi) &= V_1(x, y, z) \quad \text{for} \quad (x, y, z) \in \Sigma,
\end{aligned}$$

where the correspondence between the points (x, y, z) and $\xi \in G$ is described by relation (13).

By virtue of equality (15) and Theorem 6 in [8], the functions H_1 and H_2 are monogenic in the domain G. Further, the identity $H_1(\xi) \equiv H_2(\xi)$ in G is proved in the same way as in the proof of Lemma 1. Therefore, the equality $V_1(x_1, y_1, z_1) = V_1(x_2, y_2, z_2)$ is proved.

Thus, a function V_1 of the form $V_1(x, y, z) := F_1(\xi)$, where $F_1(\xi)$ is an arbitrary function monogenic in the domain D, is a general solution of the system

$$\begin{aligned} \frac{\partial V_1}{\partial y} &= n_1 \frac{\partial V_1}{\partial x}, \\ \frac{\partial V_1}{\partial z} &= m_1 \frac{\partial V_1}{\partial x}, \end{aligned} \tag{16}$$

which consists of the first and the third equation of system (12).

Using the second and the fourth equation of system (12), we obtain the following system of equations for the determination of the function $V_2(x, y, z)$:

$$\begin{aligned} \frac{\partial V_2}{\partial y} - n_1 \frac{\partial V_2}{\partial x} &= n_2 \frac{\partial F_1}{\partial x}, \\ \frac{\partial V_2}{\partial z} - m_1 \frac{\partial V_2}{\partial x} &= m_2 \frac{\partial F_1}{\partial x}. \end{aligned} \tag{17}$$

A particular solution of this system is the function

$$v_2(x, y, z) := (n_2 y + m_2 z) F_1'(\xi).$$

Indeed, substituting v_2 into the first equation of system (17), we obtain

$$n_2 F_1'(\xi) + n_2 y \frac{\partial F_1'(\xi)}{\partial y} + m_2 z \frac{\partial F_1'(\xi)}{\partial y} = n_2 \frac{\partial F_1(\xi)}{\partial x} + n_1 n_2 y \frac{\partial F_1'(\xi)}{\partial x} + n_1 m_2 z \frac{\partial F_1'(\xi)}{\partial x},$$

which is true by virtue of the identities

$$F_1'(\xi) \equiv \frac{\partial F_1(\xi)}{\partial x}, \qquad \frac{\partial F_1(\xi)}{\partial y} \equiv n_1 \frac{\partial F_1(\xi)}{\partial x}.$$

By analogy, we establish that the function v_2 satisfies the second equation of system (17).

Therefore, the general solution of system (17) is represented as the sum of its particular solution and the general solution of the corresponding homogenous system [analogous to system (16)]

$$V_2(x, y, z) = F_2(\xi) + (n_2 y + m_2 z) F_1'(\xi),$$

where F_2 is an arbitrary function monogenic in the domain D.

The theorem is proved.

By virtue of (8) and (10), in the case where the domain Ω is convex in the direction of the straight line L all monogenic functions $\Phi : \Omega_\zeta \to \mathbb{A}_3$ can be constructed with the use of arbitrary three complex-valued monogenic functions $F(\xi)$, $F_1(\xi)$, and $F_2(\xi)$ of a complex variable $\xi \in D$ as follows:

$$\Phi(\zeta) = \frac{1}{2\pi i}\int_{\Gamma_\zeta} F(t)(t-\zeta)^{-1}dt + \rho_1 F_1(x+n_1y+m_1z)$$

$$+ \rho_2\big(F_2(x+n_1y+m_1z) + (n_2y+m_2z)F_1'(x+n_1y+m_1z)\big) \tag{18}$$

$$\forall \zeta = xe_1+ye_2+ze_3 \in \Omega_\zeta;$$

moreover, the decomposition (9) of the principal extension of the function F in the basis $\{1,\rho_1,\rho_2\}$ is also true.

Theorem 3. *Let a domain* Ω *be convex in the direction of the straight line* L *and let a function* $\Phi:\Omega_\zeta \to \mathbb{A}_3$ *be monogenic in the domain* Ω_ζ. *Then* Φ *can be extended to a function monogenic in the domain* Π_ζ.

The statement of the theorem follows directly from equality (18), whose right-hand side is a monogenic function in the domain Π_ζ.

Let us construct an example of a domain Ω that is not convex in the direction of the straight line L and a monogenic function $\Phi:\Omega_\zeta\to\mathbb{A}_3$ for which relation (6) does not hold for some $\zeta_1, \zeta_2 \in \Omega_\zeta$ such that $\zeta_2-\zeta_1\in L_\zeta$.

Consider the harmonic basis

$$e_1 = 1,$$

$$e_2 = i + \frac{1}{2}i\rho_2, \tag{19}$$

$$e_3 = -\rho_1 - \frac{\sqrt{3}}{2}i\rho_2$$

[i.e., we have $n_1 = i$, $n_2 = i/2$, $n_3 = m_1 = 0$, $m_2 = -1$, and $m_3 = -\sqrt{3}\,i/2$ in decompositions (3)]; furthermore, the straight line L coincides with the axis Oz.

Consider the domain Ω_ζ that is the union of the following three sets:

$$\Omega_\zeta^{(1)} := \{xe_1+ye_2+ze_3\in E_3 : |x+iy|<2,\ 0<z<2,\ -\pi/4<\arg(x+iy)<3\pi/2\},$$

$$\Omega_\zeta^{(2)} := \{xe_1+ye_2+ze_3\in E_3 : |x+iy|<2,\ 2\le z\le 4,\ \pi/2<\arg(x+iy)<3\pi/2\},$$

$$\Omega_\zeta^{(3)} := \{xe_1+ye_2+ze_3\in E_3 : |x+iy|<2,\ 4<z<6,\ \pi/2<\arg(x+iy)<9\pi/4\}.$$

It is clear that a domain Ω of the space $\mathbb{R}^3$ that is congruent to it is not convex in the direction of the straight line L.

In the domain $\{\xi \in \mathbb{C} : |\xi| < 2,\ -\pi/4 < \arg\xi < 3\pi/2\}$ of the complex plane, we consider the holomorphic branch $H_1(\xi) := \ln|\xi| + i\arg\xi$ of the analytic function $\operatorname{Ln}\xi$ for which $H_1(1) = 0$; in the domain $\{\xi \in \mathbb{C} : |\xi| < 2,\ \pi/2 < \arg\xi < 9\pi/4\}$, we consider the holomorphic branch $H_2(\xi) := \ln|\xi| + i\arg\xi$ of the function $\operatorname{Ln}\xi$ for which $H_2(1) = 2\pi i$.

We construct the principal extension Φ_1 of the function H_1 to the set $\Omega_\zeta^{(1)} \cup \Omega_\zeta^{(2)}$ and the principal extension Φ_2 of the function H_2 to the set $\Omega_\zeta^{(2)} \cup \Omega_\zeta^{(3)}$ by using relations of the form (9):

$$\Phi_1(\zeta) = H_1(x+iy) - \frac{2z-iy}{2(x+iy)}\rho_1 - \left(\frac{\sqrt{3}iz}{2(x+iy)} + \frac{(2z-iy)^2}{8(x+iy)^2}\right)\rho_2,$$

$$\Phi_2(\zeta) = H_2(x+iy) - \frac{2z-iy}{2(x+iy)}\rho_1 - \left(\frac{\sqrt{3}iz}{2(x+iy)} + \frac{(2z-iy)^2}{8(x+iy)^2}\right)\rho_2,$$

where $\zeta = xe_1 + ye_2 + ze_3$.

Since $\Phi_1(\zeta) \equiv \Phi_2(\zeta)$ on the set $\Omega_\zeta^{(2)}$, the function

$$\Phi(\zeta) = \begin{cases} \Phi_1(\zeta) & \text{for } \zeta \in \Omega_\zeta^{(1)} \cup \Omega_\zeta^{(2)}, \\ \Phi_2(\zeta) & \text{for } \zeta \in \Omega_\zeta^{(3)} \end{cases}$$

is monogenic in the domain Ω_ζ. Moreover, for $\zeta_1 = e_1 + e_3$ and $\zeta_2 = e_1 + 5e_3$, we have $\zeta_2 - \zeta_1 \in L_\zeta$, but

$$\Phi(\zeta_2) - \Phi(\zeta_1) = 2\pi i - 4\rho_1 - (12 + 2\sqrt{3}i)\rho_2 \notin \mathcal{I},$$

i.e., relation (6) is not true.

The statement below is true for monogenic functions in an arbitrary domain Ω_ζ.

Theorem 4. *Let a function $\Phi : \Omega_\zeta \to \mathbb{A}_3$ be monogenic in the domain Ω_ζ. Then the Gâteaux derivatives of all orders of the function Φ are monogenic functions in the domain Ω_ζ.*

Proof. Since a ball $\mho$ centered at an arbitrary point $(x_0, y_0, z_0) \in \Omega$ and completely contained in the domain Ω is a convex domain in the direction of the straight line L, we conclude that equality (8) is true in the neighborhood $\mho_\zeta$ of the point $\zeta_0 = x_0e_1 + y_0e_2 + z_0e_3$, and the integral in this equality has Gâteaux derivatives of all orders in $\mho_\zeta$. Furthermore, the function Φ_0 admits representation (10) in $\mho_\zeta$, by virtue of which the function Φ_0 is infinitely differentiable with respect to the variables x, y, and z. Therefore, the Gâteaux derivative Φ_0' satisfies conditions of the form (5) in $\mho_\zeta$, i.e., it is a monogenic function. By analogy, we establish that the Gâteaux derivatives of all orders of the function Φ_0 are monogenic functions in $\mho_\zeta$.

The theorem is proved.

By virtue of Theorem 4, every function $\Phi : \Omega_\zeta \to \mathbb{A}_3$ monogenic in the domain Ω_ζ is a monogenic potential in this domain.

2. On Isomorphism of Algebras of Monogenic Functions in Different Harmonic Bases

Let $\mathcal{M}(E_3, \Omega_\zeta)$ denote the algebra of functions monogenic in the domain $\Omega_\zeta \subset E_3$ and taking values in the algebra $\mathbb{A}_3$.

Parallel with a harmonic basis $\{e_1, e_2, e_3\}$, we consider another harmonic basis $\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3\}$. Let

$$\tilde{E}_3 := \left\{ \tilde{\zeta} = \tilde{x}\tilde{e}_1 + \tilde{y}\tilde{e}_2 + \tilde{z}\tilde{e}_3 : \tilde{x}, \tilde{y}, \tilde{z} \in \mathbb{R} \right\}$$

denote the linear span generated by the vectors $\tilde{e}_1$, $\tilde{e}_2$, and $\tilde{e}_3$ and let $\tilde{\Omega}_{\tilde{\zeta}}$ denote a domain in $\tilde{E}_3$.

We now indicate the correspondence between the domains Ω_ζ and $\tilde{\Omega}_{\tilde{\zeta}}$ in the case of transition from the basis $\{e_1, e_2, e_3\}$ to the basis $\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3\}$ for which the algebras of monogenic functions $\mathcal{M}(E_3, \Omega_\zeta)$ and $\mathcal{M}(\tilde{E}_3, \tilde{\Omega}_{\tilde{\zeta}})$ are isomorphic.

Consider several auxiliary statements.

Lemma 2. *Suppose that the harmonic bases* $\{e_1, e_2, e_3\}$ *and* $\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3\}$ *are related to one another as follows:*

$$\tilde{e}_1 = e_1 = 1,$$

$$\tilde{e}_2 = \alpha_1 e_1 + \alpha_2 e_2 + r_{21}\rho_1 + r_{22}\rho_2, \tag{20}$$

$$\tilde{e}_3 = \beta_1 e_1 + \beta_2 e_2 + e_3 + r_{31}\rho_1 + r_{32}\rho_2,$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$, $\alpha_2 \neq 0$, *and* $r_{21}, r_{22}, r_{31}, r_{32} \in \mathbb{C}$. *If a function* $\Phi : \Omega_\zeta \to \mathbb{A}_3$ *is monogenic in the domain* Ω_ζ, *then the function*

$$\tilde{\Phi}(\tilde{\zeta}) = \Phi(\zeta) + \Phi'(\zeta)\big((r_{21}\tilde{y} + r_{31}\tilde{z})\rho_1 + (r_{22}\tilde{y} + r_{32}\tilde{z})\rho_2\big) + \frac{1}{2}\Phi''(\zeta)(r_{21}\tilde{y} + r_{31}\tilde{z})^2\rho_2 \tag{21}$$

is monogenic in the domain $\tilde{\Omega}_{\tilde{\zeta}}$ *such that the coordinates of the corresponding points* $\tilde{\zeta} = \tilde{x}\tilde{e}_1 + \tilde{y}\tilde{e}_2 + \tilde{z}\tilde{e}_3 \in \tilde{\Omega}_{\tilde{\zeta}}$ *and* $\zeta = xe_1 + ye_2 + ze_3 \in \Omega_\zeta$ *are connected by the following relations:*

$$x = \tilde{x} + \alpha_1\tilde{y} + \beta_1\tilde{z},$$

$$y = \alpha_2\tilde{y} + \beta_2\tilde{z}, \tag{22}$$

$$z = \tilde{z}.$$

Proof. Let us show that function (21) satisfies the necessary and sufficient conditions of monogeneity:

$$\frac{\partial\tilde{\Phi}}{\partial\tilde{y}} = \frac{\partial\tilde{\Phi}}{\partial\tilde{x}}\tilde{e}_2, \qquad \frac{\partial\tilde{\Phi}}{\partial\tilde{z}} = \frac{\partial\tilde{\Phi}}{\partial\tilde{x}}\tilde{e}_3. \tag{23}$$

Relations (22) yield the operator equalities

$$\frac{\partial}{\partial\tilde{x}} = \frac{\partial}{\partial x},$$

$$\frac{\partial}{\partial\tilde{y}} = \alpha_1\frac{\partial}{\partial x} + \alpha_2\frac{\partial}{\partial y},$$

$$\frac{\partial}{\partial\tilde{z}} = \beta_1\frac{\partial}{\partial x} + \beta_2\frac{\partial}{\partial y} + \frac{\partial}{\partial z}.$$

Taking these equalities into account, we obtain the following expressions for the partial derivatives of function (21):

$$\frac{\partial\tilde{\Phi}}{\partial\tilde{x}} = \frac{\partial\Phi}{\partial x} + \frac{\partial\Phi'}{\partial x}\big((r_{21}\tilde{y} + r_{31}\tilde{z})\rho_1 + (r_{22}\tilde{y} + r_{32}\tilde{z})\rho_2\big) + \frac{1}{2}\frac{\partial\Phi''}{\partial x}(r_{21}\tilde{y} + r_{31}\tilde{z})^2\rho_2,$$

$$\frac{\partial\tilde{\Phi}}{\partial\tilde{y}} = \alpha_1\frac{\partial\Phi}{\partial x} + \alpha_2\frac{\partial\Phi}{\partial y} + \left(\alpha_1\frac{\partial\Phi'}{\partial x} + \alpha_2\frac{\partial\Phi'}{\partial y}\right)\big((r_{21}\tilde{y} + r_{31}\tilde{z})\rho_1$$

$$+ (r_{22}\tilde{y} + r_{32}\tilde{z})\rho_2\big) + \frac{\partial\Phi}{\partial x}(r_{21}\rho_1 + r_{22}\rho_2)$$

$$+ \frac{1}{2}\left(\alpha_1\frac{\partial\Phi''}{\partial x} + \alpha_2\frac{\partial\Phi''}{\partial y}\right)(r_{21}\tilde{y} + r_{31}\tilde{z})^2\rho_2 + \frac{\partial^2\Phi}{\partial x^2}(r_{21}\tilde{y} + r_{31}\tilde{z})r_{21}\rho_2,$$

$$\frac{\partial\tilde{\Phi}}{\partial\tilde{z}} = \beta_1\frac{\partial\Phi}{\partial x} + \beta_2\frac{\partial\Phi}{\partial y} + \frac{\partial\Phi}{\partial z} + \frac{\partial\Phi}{\partial x}(r_{31}\rho_1 + r_{32}\rho_2)$$

$$+ \left(\beta_1\frac{\partial\Phi'}{\partial x} + \beta_2\frac{\partial\Phi'}{\partial y} + \frac{\partial\Phi'}{\partial z}\right)\big((r_{21}\tilde{y} + r_{31}\tilde{z})\rho_1 + (r_{22}\tilde{y} + r_{32}\tilde{z})\rho_2\big)$$

$$+ \frac{1}{2}\left(\beta_1\frac{\partial\Phi''}{\partial x} + \beta_2\frac{\partial\Phi''}{\partial y} + \frac{\partial\Phi''}{\partial z}\right)(r_{21}\tilde{y} + r_{31}\tilde{z})^2\rho_2 + \frac{\partial^2\Phi}{\partial x^2}(r_{21}\tilde{y} + r_{31}\tilde{z})r_{31}\rho_2.$$

Substituting the obtained expressions for the partial derivatives of function (21) and expressions (20) for the elements $\tilde{e}_2$ and $\tilde{e}_3$ in equalities (23) and taking into account the rules of multiplication (2) and conditions (5), we establish that conditions (23) are satisfied.

The lemma is proved.

Lemma 3. *Suppose that harmonic bases* $\{e_1, e_2, e_3\}$ *and* $\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3\}$ *are connected by relations (20) and a function* $\tilde{\Phi} : \tilde{\Omega}_{\tilde{\zeta}} \to \mathbb{A}_3$ *is monogenic in the domain* $\tilde{\Omega}_{\tilde{\zeta}}$. *Then there exists a unique function* $\Phi(\zeta)$ *monogenic in the domain* Ω_ζ *that satisfies equality (21), where the coordinates of the corresponding points* $\tilde{\zeta} = \tilde{x}\tilde{e}_1 + \tilde{y}\tilde{e}_2 + \tilde{z}\tilde{e}_3 \in \tilde{\Omega}_{\tilde{\zeta}}$ *and* $\zeta = xe_1 + ye_2 + ze_3 \in \Omega_\zeta$ *are connected by relations (22).*

Proof. Consider the function

$$\Phi(\zeta) = \tilde{\Phi}(\tilde{\zeta}) - \tilde{\Phi}'(\tilde{\zeta})\big((r_{21}\tilde{y} + r_{31}\tilde{z})\rho_1 + (r_{22}\tilde{y} + r_{32}\tilde{z})\rho_2\big) + \frac{1}{2}\tilde{\Phi}''(\tilde{\zeta})(r_{21}\tilde{y} + r_{31}\tilde{z})^2\rho_2 . \tag{24}$$

The monogeneity of this function is proved by analogy with the proof of the monogeneity of function (21).

Let us show that function (24) satisfies relation (21). To this end, we multiply both sides of equality (24) by ρ_2. This yields

$$\rho_2\tilde{\Phi}(\tilde{\zeta}) = \rho_2\Phi(\zeta),$$

whence

$$\rho_2\tilde{\Phi}'(\tilde{\zeta}) = \rho_2\Phi'(\zeta), \qquad \rho_2\tilde{\Phi}''(\tilde{\zeta}) = \rho_2\Phi''(\zeta). \tag{25}$$

By analogy, multiplying both sides of equality (24) by ρ_1, we get

$$\rho_1\Phi(\zeta) = \rho_1\tilde{\Phi}(\tilde{\zeta}) - \rho_2\tilde{\Phi}'(\tilde{\zeta})(r_{21}\tilde{y} + r_{31}\tilde{z}) = \rho_1\tilde{\Phi}(\tilde{\zeta}) - \rho_2\Phi'(\zeta)(r_{21}\tilde{y} + r_{31}\tilde{z}),$$

which yields

$$\rho_1\tilde{\Phi}'(\tilde{\zeta}) = \rho_1\Phi'(\zeta) + \rho_2\Phi''(\zeta)(r_{21}\tilde{y} + r_{31}\tilde{z}). \tag{26}$$

Substituting (25) and (26) in equality (24), we obtain relation (21).

We now prove the uniqueness of a monogenic function $\Phi : \Omega_\zeta \to \mathbb{A}_3$ that satisfies equality (21). For this purpose, it suffices to show that the function $\tilde{\Phi} \equiv 0$ in $\tilde{\Omega}_{\tilde{\zeta}}$ is associated only with the function $\Phi \equiv 0$ in Ω_ζ. Indeed, for $\tilde{\Phi} \equiv 0$, equality (21) takes the form

$$\Phi(\zeta) + \Phi'(\zeta)(r_{21}\tilde{y} + r_{31}\tilde{z})\rho_1 + \Phi'(\zeta)(r_{22}\tilde{y} + r_{32}\tilde{z})\rho_2 + \frac{1}{2}\Phi''(\zeta)(r_{21}\tilde{y} + r_{31}\tilde{z})^2\rho_2 \equiv 0. \tag{27}$$

Multiplying both sides of identity (27) by ρ_2 and taking into account the rules of multiplication (2), we obtain $\Phi(\zeta)\rho_2 \equiv 0$. This identity yields

$$\Phi'(\zeta)\rho_2 \equiv 0, \qquad \Phi''(\zeta)\rho_2 \equiv 0. \tag{28}$$

Similarly, multiplying both sides of identity (27) by ρ_1 we get

$$\Phi(\zeta)\,\rho_1 + \Phi'(\zeta)\,(r_{21}\tilde{y} + r_{31}\tilde{z})\,\rho_2 \equiv 0.$$

With regard for the first relation in (28), this relation yields the identity $\Phi(\zeta)\,\rho_1 \equiv 0$. Therefore,

$$\Phi'(\zeta)\,\rho_1 \equiv 0. \tag{29}$$

Finally, using relations (27)–(29), we obtain the identity $\Phi \equiv 0$.

The lemma is proved.

Now let $\{e_1, e_2, e_3\}$ be the harmonic basis whose elements are defined by equalities (19) and let $\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3\}$ be an arbitrary harmonic basis in $\mathbb{A}_3$.

The elements of the basis $\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3\}$ can be represented in the form

$$\tilde{e}_1 = a\tilde{e}_1^{(1)}, \qquad \tilde{e}_2 = a\tilde{e}_2^{(1)}, \qquad \tilde{e}_3 = a\tilde{e}_3^{(1)}, \tag{30}$$

where a is an invertible element of the algebra $\mathbb{A}_3$, and the elements of the basis $\{\tilde{e}_1^{(1)}, \tilde{e}_2^{(1)}, \tilde{e}_3^{(1)}\}$ admit decompositions of the form (3) in the basis $\{1, \rho_1, \rho_2\}$, in which, by virtue of the equality $1 + n_1^2 + m_1^2 = 0$, it may be assumed without loss of generality that $\operatorname{Im} n_1 \neq 0$. Then the elements of the basis $\{\tilde{e}_1^{(1)}, \tilde{e}_2^{(1)}, \tilde{e}_3^{(1)}\}$ can also be represented in the form

$$\tilde{e}_1^{(1)} = e_1,$$

$$\tilde{e}_2^{(1)} = \alpha_1 e_1 + \alpha_2 e_2 + r_{21}\rho_1 + r_{22}\rho_2,$$

$$\tilde{e}_3^{(1)} = \beta_1 e_1 + \beta_2 e_2 + e_3 + r_{31}\rho_1 + r_{32}\rho_2.$$

Here and in what follows,

$$\alpha_1 := \operatorname{Re} n_1, \qquad \alpha_2 := \operatorname{Im} n_1, \qquad \beta_1 := \operatorname{Re} m_1, \qquad \beta_2 := \operatorname{Im} m_1,$$

$$r_{21} := n_2, \qquad r_{22} := n_3 - \frac{1}{2}\, i \operatorname{Im} n_1,$$

$$r_{31} := m_2 + 1, \qquad r_{32} := m_3 + \frac{\sqrt{3}}{2}\, i - \frac{1}{2}\, i \operatorname{Im} m_1.$$

Theorem 5. *Let* $\{e_1, e_2, e_3\}$ *be the harmonic basis whose elements are defined by equalities (19) and let* $\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3\}$ *be an arbitrary harmonic basis in* $\mathbb{A}_3$ *whose elements are represented in the form (30).*

Let Ω_ζ be an arbitrary domain in E_3 and let $\tilde{\Omega}_{\tilde{\zeta}}$ be a domain in $\tilde{E}_3$ such that the coordinates of the corresponding points $\tilde{\zeta} = \tilde{x}\tilde{e}_1 + \tilde{y}\tilde{e}_2 + \tilde{z}\tilde{e}_3 \in \tilde{\Omega}_{\tilde{\zeta}}$ and $\zeta = xe_1 + ye_2 + ze_3 \in \Omega_\zeta$ are connected by relations (22). Then the algebras $\mathcal{M}(E_3, \Omega_\zeta)$ and $\mathcal{M}(\tilde{E}_3, \tilde{\Omega}_{\tilde{\zeta}})$ are isomorphic, and, moreover, the correspondence between the functions $\Phi \in \mathcal{M}(E_3, \Omega_\zeta)$ and $\tilde{\Phi} \in \mathcal{M}(\tilde{E}_3, \tilde{\Omega}_{\tilde{\zeta}})$ is given by equality (21).

Proof. We define a domain $\tilde{\Omega}^{(1)}_{\tilde{\zeta}^{(1)}}$ in

$$\tilde{E}_3^{(1)} := \left\{ \tilde{\zeta}^{(1)} = \tilde{x}\tilde{e}_1^{(1)} + \tilde{y}\tilde{e}_2^{(1)} + \tilde{z}\tilde{e}_3^{(1)} : \ \tilde{x}, \tilde{y}, \tilde{z} \in \mathbb{R} \right\}$$

so that the coordinates of its points $\tilde{\zeta}^{(1)}$ are connected with the coordinates of the corresponding points $\zeta \in \Omega_\zeta$ by relations (22). We associate every function $\Phi \in \mathcal{M}(E_3, \Omega_\zeta)$ with a function $\tilde{\Phi}^{(1)} \in \mathcal{M}(\tilde{E}_3^{(1)}, \tilde{\Omega}^{(1)}_{\tilde{\zeta}^{(1)}})$ by a relation of the form (21). By virtue of Lemmas 2 and 3, this correspondence between the algebras $\mathcal{M}(E_3, \Omega_\zeta)$ and $\mathcal{M}(\tilde{E}_3^{(1)}, \tilde{\Omega}^{(1)}_{\tilde{\zeta}^{(1)}})$ is bijective. Furthermore, it follows from the equality

$$\tilde{\Phi}_1^{(1)}(\tilde{\zeta}^{(1)})\,\tilde{\Phi}_2^{(1)}(\tilde{\zeta}^{(1)}) = \Phi_1(\zeta)\,\Phi_2(\zeta) + \big(\Phi_1(\zeta)\,\Phi_2'(\zeta) + \Phi_2(\zeta)\,\Phi_1'(\zeta)\big)\big((r_{21}\tilde{y} + r_{31}\tilde{z})\rho_1 + (r_{22}\tilde{y} + r_{32}\tilde{z})\rho_2\big)$$

$$+ \frac{1}{2}\big(\Phi_1''(\zeta)\,\Phi_2(\zeta) + 2\Phi_1'(\zeta)\,\Phi_2'(\zeta) + \Phi_1(\zeta)\,\Phi_2''(\zeta)\big)(r_{21}\tilde{y} + r_{31}\tilde{z})^2\rho_2$$

that the product of functions $\tilde{\Phi}_1^{(1)}, \tilde{\Phi}_2^{(1)} \in \mathcal{M}(\tilde{E}_3^{(1)}, \tilde{\Omega}^{(1)}_{\tilde{\zeta}^{(1)}})$ corresponds to the product of functions $\Phi_1, \Phi_2 \in \mathcal{M}(E_3, \Omega_\zeta)$, i.e., the algebras $\mathcal{M}(E_3, \Omega_\zeta)$ and $\mathcal{M}(\tilde{E}_3^{(1)}, \tilde{\Omega}^{(1)}_{\tilde{\zeta}^{(1)}})$ are isomorphic.

Finally, an isomorphism between the algebras $\mathcal{M}(\tilde{E}_3^{(1)}, \tilde{\Omega}^{(1)}_{\tilde{\zeta}^{(1)}})$ and $\mathcal{M}(\tilde{E}_3, \tilde{\Omega}_{\tilde{\zeta}})$ is established by using the equality

$$\tilde{\Phi}(\tilde{\zeta}) := \tilde{\Phi}^{(1)}(\tilde{\zeta}^{(1)}),$$

where $\tilde{\zeta} = \tilde{x}\tilde{e}_1 + \tilde{y}\tilde{e}_2 + \tilde{z}\tilde{e}_3 \in \tilde{\Omega}_{\tilde{\zeta}}$ and $\tilde{\zeta}^{(1)} = \tilde{x}\tilde{e}_1^{(1)} + \tilde{y}\tilde{e}_2^{(1)} + \tilde{z}\tilde{e}_3^{(1)} \in \tilde{\Omega}^{(1)}_{\tilde{\zeta}^{(1)}}$. The monogeneity of the function $\tilde{\Phi}$ in the domain $\tilde{\Omega}_{\tilde{\zeta}}$ is an obvious corollary of monogeneity conditions of the form (5) for the function $\tilde{\Phi}^{(1)}$ and the invertibility of an element $a \in \mathbb{A}_3$.

The theorem is proved.

In view of Theorem 5, it is obvious that, in the subsequent investigation, it suffices to consider monogenic functions $\Phi \in \mathcal{M}(E_3, \Omega_\zeta)$, where the linear span E_3 is generated by the harmonic basis whose elements are defined by equalities (19).

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