Contemporary Mathematics
Volume **553**, 2011

Commutative Algebras of Monogenic Functions associated with Classic Equations of Mathematical Physics

S.A. Plaksa, S.V. Gryshchuk, and V.S. Shpakivskyi

Abstract. The idea of an algebraic-analytic approach to equations of mathematical physics means finding commutative Banach algebras such that monogenic functions defined on them form an algebra and have components satisfying previously given equations with partial derivatives. We obtain constructive descriptions of monogenic functions taking values in commutative algebras associated with the two-dimensional biharmonic equation and the three-dimensional Laplace equation by means of analytic functions of the complex variable.

1. Introduction

1.1. Algebras associated with the Laplace equation. An important achievement of mathematics is the description of plane potential fields by means of analytic functions of complex variable.

A potential $u(x,y)$ and a flow function $v(x,y)$ of plane stationary potential solenoid field satisfy the Cauchy–Riemann conditions

$$\frac{\partial u(x,y)}{\partial x} = \frac{\partial v(x,y)}{\partial y}\,, \qquad \frac{\partial u(x,y)}{\partial y} = -\frac{\partial v(x,y)}{\partial x}\,,$$

and they form the complex potential $F(x+iy) = u(x,y)+iv(x,y)$ being an analytic function of complex variable $x+iy$. In turn, every analytic function $F(x+iy)$ satisfies the two-dimensional Laplace equation

$$\Delta_2 F := \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} \equiv F''(x+iy)\;(1^2+i^2) = 0$$

due to the equality $1^2+i^2=0$ for unit 1 and the imaginary unit i of the algebra of complex numbers.

The effectiveness of analytic function methods in the complex plane for researching plane potential fields inspires mathematicians to develop analogous methods for spatial fields.

2010 *Mathematics Subject Classification.* Primary 30G35; Secondary 35J05, 31A30.

Key words and phrases. Laplace equation, biharmonic equation, harmonic commutative Banach algebra, biharmonic algebra, monogenic function, Cauchy–Riemann conditions.

Apparently, W. Hamilton (1843) made the first attempts to construct an algebra associated with the three-dimensional Laplace equation

$$\Delta_3 u(x,y,z) := \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)u(x,y,z)=0 \tag{1.1}$$

in the sense that components of hypercomlex functions satisfy Eq. (1.1). However, the Hamilton quaternions form a noncommutative algebra, and after constructing the quaternion algebra he did not study the problem about constructing any other algebra (see [**3**]).

Let $\mathbb{A}$ be a commutative associative Banach algebra of rank n ($3 \le n \le \infty$) over either the field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$. Let $\{e_1,e_2,e_3\}$ be a part of the basis of $\mathbb{A}$, and $E_3 := \{\zeta := xe_1+ye_2+ze_3 \ : \ x,y,z \in \mathbb{R}\}$ be the linear envelope generated by the vectors e_1,e_2,e_3.

A function $\Phi : Q_\zeta \to \mathbb{A}$ is *analytic* in the domain $Q_\zeta \subset E_3$ if in a certain neighborhood of every point $\zeta_0 \in Q_\zeta$ it can be represented in the form of the sum of convergent power series with coefficients belonging to the algebra $\mathbb{A}$.

It is obvious that if the basic elements e_1,e_2,e_3 satisfy the condition

$$e_1^2+e_2^2+e_3^2=0\,, \tag{1.2}$$

then every analytic function $\Phi : Q_\zeta \to \mathbb{A}$ satisfies Eq. (1.1), because

$$\Delta_3\Phi(\zeta) \equiv \Phi''(\zeta)\,\left(e_1^2+e_2^2+e_3^2\right)=0\,, \qquad \zeta = xe_1+ye_2+ze_3\,. \tag{1.3}$$

We say that an algebra $\mathbb{A}$ is *harmonic* (see [**2, 8, 9**]) if in $\mathbb{A}$ there exists a triad of linearly independent vectors $\{e_1,e_2,e_3\}$ satisfying the equality (1.2) provided that $e_k^2 \ne 0$ for $k=1,2,3$. We say also that such a triad $\{e_1,e_2,e_3\}$ is *harmonic*.

P.W. Ketchum [**2**] considered the C. Segre algebra of quaternions [**10**] in its relations with the three-dimensional Laplace equation. Indeed, in the Segre algebra of quaternions there is unit 1, and the multiplication table for the basis $\{1,i,j,k\}$ is of the following form: $i^2=j^2=-1,\quad k^2=1,\quad i\,j=k,\ \ i\,k=-j,\ \ j\,k=-i$. Therefore, there are harmonic triads, in particular: $e_1=\sqrt{2}, e_2=i, e_3=j$.

K.S. Kunz [**5**] developed a method for a formal construction of solutions of Eq. (1.1) using power series in any commutative associative algebra over the field $\mathbb{C}$.

I.P. Mel'nichenko [**6**] noticed that doubly differentiable functions in the sense of Gateaux form the largest class of functions $\Phi(\zeta)$ satisfying identically the equality (1.3). He suggested an algebraic-analytic approach to equations of mathematical physics which means finding a commutative Banach algebra such that differentiable in the sense of Gateaux functions with values in this algebra have components satisfying the given equation with partial derivatives.

We say that a continuous function $\Phi : \Omega_\zeta \to \mathbb{A}$ is *monogenic* in a domain $\Omega_\zeta \subset E_3$ if Φ is differentiable in the sense of Gateaux in every point of Ω_ζ, i.e., if for every $\zeta \in \Omega_\zeta$ there exists an element $\Phi'(\zeta) \in \mathbb{A}$ such that

$$\lim_{\varepsilon\to 0+0}\left(\Phi(\zeta+\varepsilon h)-\Phi(\zeta)\right)\varepsilon^{-1} = h\Phi'(\zeta) \quad \forall h \in E_3.$$

Because monogenic functions take values in a commutative Banach algebra form a functional algebra, note that a relation between these functions and solutions of given equation with partial derivatives is important for constructing the mentioned solutions.

It is quite natural that for Eq. (1.1) in such a way a quantity of fulfilled operations will be minimal in an algebra of third rank. In the paper [**6**], I.P. Mel'nichenko established that there does not exist a harmonic algebra of third rank with unit over the field $\mathbb{R}$, but he constructed a three-dimensional harmonic algebra over the field $\mathbb{C}$. At the same time, for commutative associative algebras of third rank over the field $\mathbb{C}$ in the papers [**8, 9**], I. P. Mel'nichenko developed a method for extracting all harmonic bases. In addition, in [**9**] monogenic functions with values in three-dimensional harmonic algebras are explicitly constructed in the form of principal extensions of analytic functions of complex variable.

1.2. An algebra associated with the biharmonic equation. Such an algebra is constructed in the paper [**4**].

We say that an associative commutative two-dimensional algebra $\mathbb{B}$ with unit 1 over the field $\mathbb{C}$ is *biharmonic* if in $\mathbb{B}$ there exists a *biharmonic* basis $\{e_1, e_2\}$ satisfying the conditions

$$(e_1^2 + e_2^2)^2 = 0, \qquad e_1^2 + e_2^2 \neq 0\,. \tag{1.4}$$

V. F. Kovalev and I. P. Mel'nichenko [**4**] found a multiplication table for a biharmonic basis $\{e_1, e_2\}$:

$$e_1 = 1, \qquad e_2^2 = e_1 + 2ie_2. \tag{1.5}$$

In the paper [**7**], I. P. Mel'nichenko proved that there exists the unique biharmonic algebra $\mathbb{B}$ with a non-biharmonic basis $\{1, \rho\}$, for which $\rho^2 = 0$. Moreover, he constructed all biharmonic bases in the form:

$$e_1 = \alpha_1 + \alpha_2\,\rho\,, \qquad e_2 = \pm\, i \left(\alpha_1 + \left(\alpha_2 - \frac{1}{2\alpha_1}\right)\rho\right), \tag{1.6}$$

where complex numbers $\alpha_1 \neq 0$, α_2 can be chosen arbitrarily. In particular, for the basis (1.5) in the equalities (1.6) we choose $\alpha_1 = 1$, $\alpha_2 = 0$ and $+$ of the double sign:

$$e_1 = 1, \qquad e_2 = i - \frac{i}{2}\,\rho, \tag{1.7}$$

Note that every analytic function $\Phi(\zeta)$ of the variable $\zeta = xe_1 + ye_2$ satisfies the two-dimensional biharmonic equation

$$(\Delta_2)^2 U(x,y) := \left(\frac{\partial^4}{\partial x^4} + 2\,\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}\right) U(x,y) = 0 \tag{1.8}$$

due to the relations (1.4) and $(\Delta_2)^2\Phi = \Phi^{(4)}(\zeta)\,(e_1^2 + e_2^2)^2$.

1.3. Below we obtain a constructive description of monogenic functions taking values in a harmonic algebra constructed in [**8, 9**] by means of analytic functions of the complex variable. We also prove similar results for monogenic functions taking values in the biharmonic algebra. In addition, we prove that every biharmonic function in a bounded simply connected domain is the first component of a monogenic function defined in the corresponding domain of the biharmonic plane.

2. Monogenic functions in a three-dimensional harmonic algebra with the two-dimensional radical

2.1. A harmonic algebra $\mathbb{A}_3$. Let $\mathbb{A}_3$ be a three-dimensional commutative associative Banach algebra with unit 1 over the field of complex numbers $\mathbb{C}$. Let $\{1, \rho_1, \rho_2\}$ be a basis of the algebra $\mathbb{A}_3$ with the multiplication table

$$\rho_1\rho_2 = \rho_2^2 = 0, \quad \rho_1^2 = \rho_2.$$

The algebra $\mathbb{A}_3$ is harmonic. All harmonic bases in $\mathbb{A}_3$ are described in Theorem 1.6 [**9**], namely, the basis $\{e_1, e_2, e_3\}$ is harmonic if decompositions of its elements with respect to the basis $\{1, \rho_1, \rho_2\}$ are of the form

$$\begin{aligned} e_1 &= 1, \\ e_2 &= n_1 + n_2\rho_1 + n_3\rho_2, \\ e_3 &= m_1 + m_2\rho_1 + m_3\rho_2, \end{aligned} \tag{2.1}$$

where n_k and m_k for $k = 1, 2, 3$ are complex numbers satisfying the system of equations

$$\begin{aligned} &1 + n_1^2 + m_1^2 &&= 0, \\ &n_1n_2 + m_1m_2 &&= 0, \\ &xn_2^2 + m_2^2 + 2(n_1n_3 + m_1m_3) &&= 0 \end{aligned} \tag{2.2}$$

and the inequality $n_2m_3 - n_3m_2 \neq 0$, and moreover, at least one of the numbers in each of the pairs (n_1, n_2) and (m_1, m_2) is not equal to zero. Any harmonic basis in $\mathbb{A}_3$ can be obtained as a result of multiplication of elements of harmonic basis (2.1) by any invertible element of the algebra $\mathbb{A}_3$.

For example, if $n_1 = i$, $n_2 = i/2$, $n_3 = m_1 = 0$, $m_2 = -1$, $m_3 = -\sqrt{3}\, i/2$, then we have a harmonic basis $\{e_1, e_2, e_3\}$ with the following decomposition with respect to the basis $\{1, \rho_1, \rho_2\}$:

$$e_1 = 1, \qquad e_2 = i + \frac{1}{2}\, i\rho_2, \qquad e_3 = -\rho_1 - \frac{\sqrt{3}}{2}\, i\rho_2. \tag{2.3}$$

The algebra $\mathbb{A}_3$ has the unique maximum ideal $\mathcal{I} := \{\lambda_1\rho_1 + \lambda_2\rho_2 : \lambda_1, \lambda_2 \in \mathbb{C}\}$ which is also the radical of $\mathbb{A}_3$.

Consider the linear functional $f : \mathbb{A}_3 \to \mathbb{C}$ such that the maximum ideal $\mathcal{I}$ is its kernel and $f(1) = 1$. It is well-known [**1**] that f is also a multiplicative functional, i.e., the equality $f(ab) = f(a)f(b)$ is fulfilled for all $a, b \in \mathbb{A}_3$.

2.2. A constructive description of monogenic functions taking values in the algebra $\mathbb{A}_3$. Let $\{e_1, e_2, e_3\}$ be a harmonic basis of the form (2.1) and $\zeta = x + ye_2 + ze_3$, where $x, y, z \in \mathbb{R}$.

It follows from the equality

$$\begin{aligned} (t-\zeta)^{-1} = {} & \frac{1}{t - x - n_1y - m_1z} + \frac{n_2y + m_2z}{(t - x - n_1y - m_1z)^2}\,\rho_1 \\ & + \left(\frac{n_3y + m_3z}{(t - x - n_1y - m_1z)^2} + \frac{(n_2y + m_2z)^2}{(t - x - n_1y - m_1z)^3}\right)\rho_2 \end{aligned}$$

$$\forall\, t \in \mathbb{C} : t \neq x + n_1y + m_1z$$

(see. [**9**, p. 30]) that the element $\zeta = x + ye_2 + ze_3 \in E_3$ is noninvertible in $\mathbb{A}_3$ if and only if the point (x, y, z) belongs to the following straight line in $\mathbb{R}^3$:

$$L : \quad \begin{cases} x + y\mathrm{Re}\, n_1 + z\mathrm{Re}\, m_1 = & 0, \\ \quad y\mathrm{Im}\, n_1 + z\mathrm{Im}\, m_1 = & 0\,. \end{cases}$$

We say that the domain $\Omega \subset \mathbb{R}^3$ is *convex in the direction of the straight line* L if Ω contains every segment parallel to L and connecting two points $(x_1, y_1, z_1), (x_2, y_2, z_2) \in \Omega$.

Associate with a set $Q \subset \mathbb{R}^3$ the set $Q_\zeta := \{\zeta = xe_1 + ye_2 + ze_3 : (x, y, z) \in Q\}$ in E_3.

To obtain a constructive description of monogenic functions given in the domain Ω_ζ and taking values in the algebra $\mathbb{A}_3$, consider an auxiliary statement.

LEMMA 2.1. *Let a domain $\Omega \subset \mathbb{R}^3$ be convex in the direction of the straight line L and $\Phi : \Omega_\zeta \to \mathbb{A}_3$ be a monogenic function in the domain Ω_ζ. If $\zeta_1, \zeta_2 \in \Omega_\zeta$ and $\zeta_2 - \zeta_1 \in L_\zeta$, then*

$$\Phi(\zeta_1) - \Phi(\zeta_2) \in \mathcal{I}. \tag{2.4}$$

PROOF. Let the segment connecting the points $(x_1, y_1, z_1), (x_2, y_2, z_2) \in \Omega$ be parallel to the straight line L.

Let us construct in Ω two surfaces Q and Σ satisfying the following conditions:

- Q and Σ have the same edge;
- the surface Q contains the point (x_1, y_1, z_1) and the surface Σ contains the point (x_2, y_2, z_2);
- restrictions of the functional f onto the sets Q_ζ and Σ_ζ are one-to-one mappings of these sets onto the same domain G of the complex plane;
- for every $\zeta_0 \in Q_\zeta$ (and $\zeta_0 \in \Sigma_\zeta$) the equality

$$\lim_{\varepsilon \to 0+0} \left(\Phi(\zeta_0 + \varepsilon(\zeta - \zeta_0)) - \Phi(\zeta_0)\right) \varepsilon^{-1} = \Phi'(\zeta_0)(\zeta - \zeta_0) \tag{2.5}$$

  is fulfilled for all $\zeta \in Q_\zeta$ for which $\zeta_0 + \varepsilon(\zeta - \zeta_0) \in Q_\zeta$ for all $\varepsilon \in (0, 1)$ (or for all $\zeta \in \Sigma_\zeta$ for which $\zeta_0 + \varepsilon(\zeta - \zeta_0) \in \Sigma_\zeta$ for all $\varepsilon \in (0, 1)$, respectively).

As the surface Q, we can take an equilateral triangle having the center (x_1, y_1, z_1) and apexes A_1, A_2, A_3, and, in addition, the plane of this triangle is perpendicular to the straight line L.

To construct the surface Σ, first consider a triangle with the center (x_2, y_2, z_2) and apexes A_1', A_2', A_3' such that the segments $A_1'A_2'$, $A_2'A_3'$, $A_1'A_3'$ are parallel to the segments A_1A_2, A_2A_3, A_1A_3, respectively, and, in addition, the length of $A_1'A_2'$ is less than the length of A_1A_2. Inasmuch as the domain Ω is convex in the direction of the straight line L, the prism with vertexes $A_1', A_2', A_3', A_1'', A_2'', A_3''$ is completely contained in Ω, where the points A_1'', A_2'', A_3'' are located in the plane of triangle $A_1A_2A_3$ and the edges $A_m'A_m''$ are parallel to L for $m = \overline{1,3}$.

Further, set a triangle with apexes B_1, B_2, B_3 such that the point B_m is located on the segment $A_m'A_m''$ for $m = \overline{1,3}$ and the truncated pyramid with vertexes $A_1, A_2, A_3, B_1, B_2, B_3$ and lateral edges A_mB_m, $m = \overline{1,3}$, is completely contained in the domain Ω.

Finally, in the plane of triangle $A_1'A_2'A_3'$, set a triangle T with apexes C_1, C_2, C_3 such that the segments C_1C_2, C_2C_3, C_1C_3 are parallel to the segments $A_1'A_2'$, $A_2'A_3'$, $A_1'A_3'$, respectively, and, in addition, the length of C_1C_2 is less than the length of $A_1'A_2'$. It is evident that the truncated pyramid with vertexes $B_1, B_2, B_3, C_1, C_2, C_3$ and lateral edges B_mC_m, $m = \overline{1,3}$, is completely contained in the domain Ω.

Now for the surface Σ, denote the surface formed by the triangle T and the lateral surfaces of mentioned truncated pyramids $A_1A_2A_3B_1B_2B_3$ and $B_1B_2B_3C_1C_2C_3$.

For each $\xi \in G$, define two complex-valued functions H_1 and H_2 so that

$$H_1(\xi) := f(\Phi(\zeta)), \text{ where } \xi = f(\zeta) \text{ and } \zeta \in Q_\zeta,$$

$$H_2(\xi) := f(\Phi(\zeta)), \text{ where } \xi = f(\zeta) \text{ and } \zeta \in \Sigma_\zeta.$$

Inasmuch as f is a linear continuous multiplicative functional, from the equality (2.5) it follows that

$$\lim_{\varepsilon\to 0+0}\left(f(\Phi(\zeta_0+\varepsilon(\zeta-\zeta_0)))-f(\Phi(\xi))\right)\varepsilon^{-1}=f(\Phi'(\zeta_0))(f(\zeta)-f(\zeta_0)).$$

Thus, there exist all directional derivatives of the functions H_1, H_2 in the point $f(\zeta_0)\in G$, and, moreover, these derivatives are equal for each of the functions H_1, H_2. Therefore, by Theorem 21 [**12**], the functions H_1, H_2 are analytic in the domain G, i.e., they are holomorphic in the case where $\xi=\tau+i\eta$, and they are antiholomorphic in the case where $\xi=\tau-i\eta$, $\tau,\eta\in\mathbb{R}$.

Inasmuch as $H_1(\xi)\equiv H_2(\xi)$ on the boundary of domain G, this identity is fulfilled everywhere in G. Therefore, the equalities

$$f(\Phi(\zeta_2)-\Phi(\zeta_1))=f(\Phi(\zeta_2))-f(\Phi(\zeta_1))=0$$

are fulfilled for $\zeta_1:=x_1+y_1e_2+z_1e_3$ and $\zeta_2:=x_2+y_2e_2+z_2e_3$. Thus, $\Phi(\zeta_2)-\Phi(\zeta_1)$ belongs to the kernel $\mathcal{I}$ of functional f. $\square$

Let $D:=f(\Omega_\zeta)$ and A be the linear operator which assigns the function $F:D\to\mathbb{C}$ to every monogenic function $\Phi:\Omega_\zeta\to\mathbb{A}_3$ by the formula $F(\xi):=f(\Phi(\zeta))$, where $\zeta=xe_1+ye_2+ze_3$ and $\xi:=f(\zeta)=x+n_1y+m_1z$. It follows from Lemma 2.1 that the value $F(\xi)$ does not depend on a choice of a point ζ, for which $f(\zeta)=\xi$.

Now the following theorem can easily be proved in the same way as Theorem 2.4 [**9**].

THEOREM 2.2. *If a domain $\Omega\subset\mathbb{R}^3$ is convex in the direction of the straight line L, then every monogenic function $\Phi:\Omega_\zeta\to\mathbb{A}_3$ can be expressed in the form*

$$\Phi(\zeta)=\frac{1}{2\pi i}\int_{\Gamma_\zeta}(A\Phi)(t)(t-\zeta)^{-1}dt+\Phi_0(\zeta)\qquad\forall\,\zeta\in\Omega_\zeta\,, \tag{2.6}$$

where Γ_ζ is an arbitrary closed Jordan rectifiable curve in D that embraces the point $f(\zeta)$, and $\Phi_0:\Omega_\zeta\to\mathcal{I}$ is a monogenic function taking values in the radical $\mathcal{I}$.

Note that the complex number $\xi=f(\zeta)$ is the spectrum of $\zeta\in\mathbb{A}_3$, and the integral in the equality (2.6) is the principal extension of analytic function $F(\xi)=(A\Phi)(\xi)$ of the complex variable ξ into the domain Ω_ζ.

It follows from Theorem 2.2 that the algebra of functions monogenic in Ω_ζ is decomposed into the direct sum of the algebra of principal extensions of analytic functions of the complex variable and the algebra of monogenic functions in Ω_ζ taking values in the radical $\mathcal{I}$.

In Theorem 1.7 [**9**], the principal extension of analytic function $F:D\to\mathbb{C}$ into the domain $\Pi_\zeta:=\{\zeta\in E_3:f(\zeta)\in D\}$ was explicitly constructed in the form

$$\begin{aligned}\frac{1}{2\pi i}\int_{\Gamma_\zeta}F(t)(t-\zeta)^{-1}dt&=F(x+n_1y+m_1z)\\&+(n_2y+m_2z)F'(x+n_1y+m_1z)\rho_1\\+\Big((n_3y+m_3z)F'(x+n_1y+m_1z)&+\frac{(n_2y+m_2z)^2}{2}F''(x+n_1y+m_1z)\Big)\rho_2\\&\forall\zeta=xe_1+ye_2+ze_3\in\Pi_\zeta.\end{aligned} \tag{2.7}$$

It is evident that the domain $\Pi \subset \mathbb{R}^3$ congruent to Π_ζ is an infinite cylinder, and its generatrix is parallel to L.

In the following theorem, we describe all monogenic functions given in the domain Ω_ζ and taking values in the radical $\mathcal{I}$.

THEOREM 2.3. *If a domain $\Omega \subset \mathbb{R}^3$ is convex in the direction of the straight line L, then every monogenic function $\Phi_0 : \Omega_\zeta \to \mathcal{I}$ can be expressed in the form*

$$\Phi_0(\zeta) = F_1(\xi)\,\rho_1 + (F_2(\xi) + (n_2 y + m_2 z)F_1'(\xi))\,\rho_2 \qquad (2.8)$$

$$\forall \zeta = xe_1 + ye_2 + ze_3 \in \Omega_\zeta\,,$$

where F_1, F_2 are complex-valued analytic functions in the domain D and $\xi = x + n_1 y + m_1 z$.

PROOF. A function $\Phi_0(\zeta)$ of the variable $\zeta = x + ye_2 + ze_3$, where $x, y, z \in \mathbb{R}$, is monogenic in Ω_ζ if and only if the following Cauchy–Riemann conditions are satisfied (see [**9**, Theorem 1.3]):

$$\frac{\partial \Phi_0}{\partial y} = \frac{\partial \Phi_0}{\partial x}\, e_2, \qquad \frac{\partial \Phi_0}{\partial z} = \frac{\partial \Phi_0}{\partial x}\, e_3. \qquad (2.9)$$

The function Φ_0 is of the form

$$\Phi_0(\zeta) = V_1(x,y,z)\rho_1 + V_2(x,y,z)\rho_2\,, \qquad (2.10)$$

where $V_k : \Omega \to \mathbb{C}$ for $k = 1, 2$.

Substituting the expressions (2.1), (2.10) into the equalities (2.9) and taking into account the uniqueness of decomposition of element of $\mathbb{A}_3$ with respect to the basis $\{1, \rho_1, \rho_2\}$, we get the following system for the determination of functions V_1, V_2:

$$\begin{aligned} &\frac{\partial V_1}{\partial y} = n_1 \frac{\partial V_1}{\partial x}\,,\\ &\frac{\partial V_2}{\partial y} = n_2 \frac{\partial V_1}{\partial x} + n_1 \frac{\partial V_2}{\partial x}\,,\\ &\frac{\partial V_1}{\partial z} = m_1 \frac{\partial V_1}{\partial x}\,,\\ &\frac{\partial V_2}{\partial z} = m_2 \frac{\partial V_1}{\partial x} + m_1 \frac{\partial V_2}{\partial x}\,. \end{aligned} \qquad (2.11)$$

Inasmuch as

$$\xi = (x + y\mathrm{Re}\, n_1 + z\mathrm{Re}\, m_1) + i(y\mathrm{Im}\, n_1 + z\mathrm{Im}\, m_1) =: \tau + i\eta\,, \qquad (2.12)$$

from the first and the third equations of the system (2.11), we get

$$\frac{\partial V_1}{\partial \eta}\,\mathrm{Im}\, n_1 = i\,\frac{\partial V_1}{\partial \tau}\,\mathrm{Im}\, n_1\,, \qquad \frac{\partial V_1}{\partial \eta}\,\mathrm{Im}\, m_1 = i\,\frac{\partial V_1}{\partial \tau}\,\mathrm{Im}\, m_1\,. \qquad (2.13)$$

It follows from the first equation of the system (2.2) that at least one of the numbers $\mathrm{Im}\, n_1$, $\mathrm{Im}\, m_1$ is not equal to zero. Therefore, from (2.13) we get the equality

$$\frac{\partial V_1}{\partial \eta} = i\,\frac{\partial V_1}{\partial \tau}\,. \qquad (2.14)$$

Let us prove that $V_1(x_1, y_1, z_1) = V_1(x_2, y_2, z_2)$ for the points (x_1, y_1, z_1), $(x_2, y_2, z_2) \in \Omega$ such that the segment connecting these points is parallel to the straight line L. Consider two surfaces Q, Σ in Ω and the domain G in $\mathbb{C}$ that are defined in the proof of Lemma 2.1. For each $\xi \in G$, define two complex-valued functions H_1 and H_2 so that

$$H_1(\xi) := V_1(x, y, z) \text{ for } (x, y, z) \in Q,$$

$$H_2(\xi) := V_1(x, y, z) \text{ for } (x, y, z) \in \Sigma,$$

where the correspondence between the points (x, y, z) and $\xi \in G$ is determined by the relation (2.12). The functions H_1, H_2 are analytic in the domain G due to the equality (2.14) and Theorem 6 [**11**]. Further, the identity $H_1(\xi) \equiv H_2(\xi)$ in G can be proved in the same way as in the proof of Lemma 2.1.

Thus, the function V_1 of the form $V_1(x, y, z) := F_1(\xi)$, where $F_1(\xi)$ is an arbitrary function analytic in D, is the general solution of the system consisting of the first and the third equations of the system (2.11).

Now from the second and the fourth equations of the system (2.11), we get the following system for the determination of function $V_2(x, y, z)$:

$$\begin{aligned} \frac{\partial V_2}{\partial y} - n_1 \frac{\partial V_2}{\partial x} &= n_2 \frac{\partial F_1}{\partial x}\,, \\ \frac{\partial V_2}{\partial z} - m_1 \frac{\partial V_2}{\partial x} &= m_2 \frac{\partial F_1}{\partial x}\,. \end{aligned} \tag{2.15}$$

The function $(n_2 y + m_2 z)F_1'(\xi)$ is a particular solution of this system and, therefore, the general solution of the system (2.15) is represented in the form

$$V_2(x, y, z) = F_2(\xi) + (n_2 y + m_2 z)F_1'(\xi),$$

where F_2 is an arbitrary function analytic in the domain D. $\square$

It follows from the equalities (2.6), (2.8) that in the case where a domain $\Omega \subset \mathbb{R}^3$ is convex in the direction of the straight line L, any monogenic function $\Phi : \Omega_\zeta \to \mathbb{A}_3$ can be constructed by means of three complex analytic in D functions F, F_1, F_2 in the form:

$$\begin{aligned} \Phi(\zeta) = \frac{1}{2\pi i} \int\limits_{\Gamma_\zeta} F(t)(t-\zeta)^{-1} dt + \rho_1 F_1(x + n_1 y + m_1 z) \\ + \rho_2 \Big(F_2(x + n_1 y + m_1 z) + (n_2 y + m_2 z) F_1'(x + n_1 y + m_1 z) \Big) \\ \forall \zeta = x e_1 + y e_2 + z e_3 \in \Omega_\zeta \end{aligned} \tag{2.16}$$

and in this case the equality (2.7) is applicable.

It is evident that the following statement follows from the equality (2.16).

Theorem 2.4. *If a domain $\Omega \subset \mathbb{R}^3$ is convex in the direction of the straight line L, then every monogenic function $\Phi : \Omega_\zeta \to \mathbb{A}_3$ can be continued to a function monogenic in the domain Π_ζ.*

Note that the condition of convexity of Ω in the direction of the line L is essential for the veracity of Lemma 2.1 and consequently for that of Theorems 2.2 – 2.4.

EXAMPLE 2.5. Let us construct a domain Ω, which is not convex in the direction of the straight line L, and an example of monogenic function $\Phi : \Omega_\zeta \to \mathbb{A}_3$ for which the relation (2.4) is not fulfilled for a certain $\zeta_1, \zeta_2 \in \Omega_\zeta$ such that $\zeta_2 - \zeta_1 \in L_\zeta$.

Consider a harmonic basis (2.3). In this case, the straight line L coincides with the axis Oz. Consider the domain Ω_ζ which is the union of sets

$$\Omega_\zeta^{(1)} := \{x + ye_2 + ze_3 \in E_3 : |x+iy| < 2,\ 0 < z < 2,\ -\pi/4 < \arg(x+iy) < 3\pi/2\},$$

$$\Omega_\zeta^{(2)} := \{x + ye_2 + ze_3 \in E_3 : |x+iy| < 2,\ 2 \le z \le 4,\ \pi/2 < \arg(x+iy) < 3\pi/2\},$$

$$\Omega_\zeta^{(3)} := \{x + ye_2 + ze_3 \in E_3 : |x+iy| < 2,\ 4 < z < 6,\ \pi/2 < \arg(x+iy) < 9\pi/4\}.$$

It is evident that the domain $\Omega \subset \mathbb{R}^3$ congruent to Ω_ζ is not convex in the direction of the axis Oz.

In the domain $\{\xi \in \mathbb{C} : |\xi| < 2, -\pi/4 < \arg\xi < 3\pi/2\}$ of the complex plane, consider a holomorphic branch $H_1(\xi)$ of the analytic function $\operatorname{Ln}\xi$ for which $H_1(1) = 0$. In the domain $\{\xi \in \mathbb{C} : |\xi| < 2,\ \pi/2 < \arg\xi < 9\pi/4\}$, consider also a holomorphic branch $H_2(\xi)$ of the function $\operatorname{Ln}\xi$ for which $H_2(1) = 2\pi i$.

Further, consider the principal extension Φ_1 of the function H_1 into the set $\Omega_\zeta^{(1)} \cup \Omega_\zeta^{(2)}$ and the principal extension Φ_2 of function H_2 into the set $\Omega_\zeta^{(2)} \cup \Omega_\zeta^{(3)}$ constructed using the formula (2.7):

$$\Phi_1(\zeta) = H_1(x+iy) - \frac{2z - iy}{2(x+iy)}\,\rho_1 - \left(\frac{\sqrt{3}iz}{2(x+iy)} + \frac{(2z-iy)^2}{8(x+iy)^2}\right)\rho_2\,,$$

$$\Phi_2(\zeta) = H_2(x+iy) - \frac{2z - iy}{2(x+iy)}\,\rho_1 - \left(\frac{\sqrt{3}iz}{2(x+iy)} + \frac{(2z-iy)^2}{8(x+iy)^2}\right)\rho_2\,,$$

where $\zeta = x + ye_2 + ze_3$.

Now the function

$$\Phi(\zeta) := \begin{cases} \Phi_1(\zeta) & \zeta \in \Omega_\zeta^{(1)} \cup \Omega_\zeta^{(2)}, \\ \Phi_2(\zeta) & \zeta \in \Omega_\zeta^{(3)} \end{cases}$$

is monogenic in the domain Ω_ζ, because $\Phi_1(\zeta) \equiv \Phi_2(\zeta)$ everywhere in $\Omega_\zeta^{(2)}$. At the same time, for the points $\zeta_1 = 1 + e_3$ and $\zeta_2 = 1 + 5\,e_3$ we have $\zeta_2 - \zeta_1 \in L_\zeta$ but

$$\Phi(\zeta_2) - \Phi(\zeta_1) = 2\pi i - 4\,\rho_1 - (12 + 2\sqrt{3}\,i)\rho_2 \ \notin \mathcal{I}\,,$$

i.e., the relation (2.4) is not fulfilled.

The following statement is true for monogenic functions in an arbitrary domain Ω_ζ.

THEOREM 2.6. *For every monogenic function $\Phi : \Omega_\zeta \to \mathbb{A}_3$ in an arbitrary domain Ω_ζ, the Gateaux n-th derivatives $\Phi^{(n)}$ are monogenic in Ω_ζ for any n.*

PROOF. Consider an arbitrary point $(x_0, y_0, z_0) \in \Omega$ and a ball $\mho \subset \Omega$ with the center in the point (x_0, y_0, z_0). Inasmuch as $\mho$ is a convex set, in the neighbourhood $\mho_\zeta$ of the point $\zeta_0 = x_0 + y_0e_2 + z_0e_3$ we have the equality (2.6), where the integral has the Gateaux n-th derivatives in $\mho_\zeta$ for any n. Furthermore, the function Φ_0 is represented in $\mho_\zeta$ in the form (2.8) and is infinitely differentiable with respect to the variables x, y, z. Therefore, the Gateaux derivative Φ_0' satisfies the conditions of the form (2.9) in $\mho_\zeta$, i.e., Φ_0' is a monogenic function. In the same way, it can be proved that the Gateaux n-th derivatives $\Phi_0^{(n)}$ are monogenic functions in $\mho_\zeta$ for any n. □

For monogenic functions $\Phi : \Omega_\zeta \to \mathbb{A}_3$, analogs of the Cauchy integral theorem and the Cauchy integral formula can be proved. It yields the Taylor expansion of monogenic functions in the usual way. An analog of Morera theorem can be also established.

Thus, as in the complex plane, one can give different equivalent definitions of monogenic functions $\Phi : \Omega_\zeta \to \mathbb{A}_3$.

3. Monogenic functions in the biharmonic algebra

3.1. A constructive description of monogenic functions given in a biharmonic plane. The algebra $\mathbb{B}$ has the unique maximum ideal $\mathcal{I} := \{\lambda\rho : \lambda \in \mathbb{C}\}$ which is also the radical of $\mathbb{B}$. In what follows, $f : \mathbb{B} \to \mathbb{C}$ is the linear functional such that the maximum ideal $\mathcal{I}$ is its kernel and $f(1) = 1$.

Consider a *biharmonic plane* $\mu := \{\zeta = x\,e_1 + y\,e_2 : x, y \in \mathbb{R}\}$ which is a linear envelope generated by the elements e_1, e_2 of a biharmonic basis (1.6). In what follows, $\zeta = x\,e_1 + y\,e_2$ and $x, y \in \mathbb{R}$.

Let G_ζ be a domain in the biharmonic plane μ. Inasmuch as divisors of zero don't belong to the plane μ, the Gateaux derivative of function $\Phi : G_\zeta \to \mathbb{B}$ coincides with the derivative

$$\Phi'(\zeta) := \lim_{h\to 0,\, h\in\mu} \big(\Phi(\zeta + h) - \Phi(\zeta)\big)\, h^{-1} .$$

Therefore, we define *monogenic* functions as functions $\Phi : G_\zeta \to \mathbb{B}$ for which the derivative $\Phi'(\zeta)$ exists in every point $\zeta \in G_\zeta$.

Let $D := f(G_\zeta)$ and A be the linear operator which assigns the function $F : D \to \mathbb{C}$ to every function $\Phi : G_\zeta \to \mathbb{B}$ by the formula $F(\xi) := f(\Phi(\zeta))$, where $\xi := f(\zeta) = \alpha_1(x \pm iy)$.

It is evident that if Φ is a monogenic function in the domain G_ζ, then F is an analytic function in the domain D, i.e., F is either holomorphic in the case where $\xi = \alpha_1(x + iy)$ or antiholomorphic in the case where $\xi = \alpha_1(x - iy)$.

The following theorem can be proved similarly to Theorem 2.4 [**9**].

Theorem 3.1. *Every monogenic function $\Phi : G_\zeta \to \mathbb{B}$ can be expressed in the form*

$$\Phi(\zeta) = \frac{1}{2\pi i} \int\limits_{\Gamma_\zeta} (A\Phi)(t)(t - \zeta)^{-1}\, dt + \Phi_0(\zeta) \qquad \forall\, \zeta \in G_\zeta, \tag{3.1}$$

where Γ_ζ is an arbitrary closed rectifiable curve in D that embraces the point $f(\zeta)$, and $\Phi_0 : G_\zeta \to \mathcal{I}$ is a monogenic function taking values in the radical $\mathcal{I}$.

Note that the complex number $\xi = f(\zeta)$ is the spectrum of $\zeta \in \mathbb{B}$, and the integral in the equality (3.1) is the principal extension of analytic function $F(\xi) = (A\Phi)(\xi)$ of the complex variable ξ into the domain G_ζ.

It follows from Theorem 3.1 that the algebra of monogenic functions in G_ζ is decomposed into the direct sum of the algebra of principal extensions of analytic functions of the complex variable and the algebra of monogenic functions in G_ζ taking values in the radical $\mathcal{I}$.

It is established in the paper [**4**] that a function $\Phi(\zeta)$ is monogenic in a domain of a biharmonic plane generated by the biharmonic basis (1.7) if and only if the following Cauchy–Riemann condition

$$\frac{\partial\Phi(\zeta)}{\partial y} = \frac{\partial\Phi(\zeta)}{\partial x}\, e_2. \tag{3.2}$$

is satisfied.

It can similarly be proved that a function $\Phi : G_\zeta \to \mathbb{B}$ is monogenic in a domain G_ζ of an arbitrary biharmonic plane μ if and only if the following equality is fulfilled

$$\frac{\partial\Phi(\zeta)}{\partial y}\, e_1 = \frac{\partial\Phi(\zeta)}{\partial x}\, e_2 \qquad \forall\, \zeta = xe_1 + ye_2 \in G_\zeta. \tag{3.3}$$

In the following theorem, we describe all monogenic functions given in the domain G_ζ and taking values in the radical $\mathcal{I}$.

THEOREM 3.2. *Every monogenic function $\Phi_0 : G_\zeta \to \mathcal{I}$ can be expressed in the form*

$$\Phi_0(\zeta) = F_0(\xi)\rho \qquad \forall\, \zeta \in G_\zeta\,, \tag{3.4}$$

where $F_0 : D \to \mathbb{C}$ is an analytic function and $\xi = f(\zeta)$.

PROOF. Substituting the function (3.4) in the equality (3.3) in place of Φ, we get

$$\frac{\partial F_0(\xi)}{\partial y}\, \rho e_1 = \frac{\partial F_0(\xi)}{\partial x}\, \rho e_2 \quad \forall\, \xi \in D. \tag{3.5}$$

Using the equality $e_1^{-1} = \frac{1}{\alpha_1}\left(1 - \frac{\alpha_2}{\alpha_1}\rho\right)$ and the relations (1.6), we obtain the equality $\rho e_2 e_1^{-1} = \pm i\rho$. Then, as a result of multiplication of the equality (3.5) by e_1^{-1}, we get

$$\frac{\partial F_0(\xi)}{\partial y}\, \rho = \pm i\, \frac{\partial F_0(\xi)}{\partial x}\, \rho \qquad \forall\, \xi \in D.$$

From this, taking into account the uniqueness of decomposition of element of $\mathbb{B}$ with respect to the basis $\{1, \rho\}$, we obtain the equality

$$\frac{\partial F_0(\xi)}{\partial y} = \pm i\, \frac{\partial F_0(\xi)}{\partial x} \quad \forall\, \xi \in D.$$

Thus, the function F_0 is either holomorphic in D in the case where $\xi = \alpha_1(x+iy)$ or antiholomorphic in D in the case where $\xi = \alpha_1(x - iy)$, i.e., F_0 is analytic in the domain D. $\square$

It follows from equalities (3.1), (3.4) that any monogenic function $\Phi : G_\zeta \to \mathbb{B}$ can be constructed by means of two complex analytic functions F, F_0 in D in the form:

$$\Phi(\zeta) = \frac{1}{2\pi i} \int\limits_{\Gamma_\zeta} F(t)(t-\zeta)^{-1} dt + F_0(f(\zeta))\rho \qquad \forall \zeta \in G_\zeta. \tag{3.6}$$

Moreover, using the expression

$$(t-\zeta)^{-1} = \frac{1}{t-\xi} - \frac{1}{2\alpha_1}\frac{2\alpha_2\xi \pm iy}{(t-\xi)^2}\, \rho$$
$$\forall\, \zeta = xe_1 + ye_2 \in G_\zeta \qquad \forall\, t \in \mathbb{C} : t \neq \xi = \alpha_1(x \pm iy),$$

the principal extension of analytic function F in D into G_ζ can explicitly be constructed in the form

$$(3.7)\quad \frac{1}{2\pi i}\int\limits_{\Gamma_\zeta} F(t)(t-\zeta)^{-1}dt = F(\xi) - \frac{F'(\xi)}{\alpha_1}\left(\alpha_2\xi \pm \frac{iy}{2}\right)\rho, \qquad \xi = f(\zeta) \in D,$$

$$\forall\, \zeta = xe_1 + ye_2 \in G_\zeta.$$

Note that in a particular case, in the paper [**4**], principal extensions of analytic functions of a complex variable were explicitly constructed into the biharmonic plane generated by the biharmonic basis (1.7).

The following theorem can be proved similarly to Theorem 2.6.

THEOREM 3.3. *Every monogenic function $\Phi : G_\zeta \to \mathbb{B}$ has derivatives of all orders in the domain G_ζ.*

Monogenic functions $\Phi : G_\zeta \to \mathbb{B}$ have properties similar to properties of analytic functions of complex variable, namely, the Cauchy integral theorem and the Cauchy integral formula and the Taylor expansion and the Morera theorem are true in the biharmonic plane μ.

3.2. A representation of biharmonic function in the form of the first component of monogenic function. In what follows, the basic elements e_1, e_2 are defined by the equalities (1.7).

$U : G \to \mathbb{R}$ is called a *biharmonic* function in a domain $G \subset \mathbb{R}^2$ if it satisfies the equation (1.8) in G.

We shall prove that every biharmonic function $U_1(x, y)$ in a bounded simply connected domain $G \subset \mathbb{R}^2$ is the first component of some monogenic function

$$(3.8)\quad \Phi(\zeta) = U_1(x,y)\, e_1 + U_2(x,y)\, ie_1 + U_3(x,y)\, e_2 + U_4(x,y)\, ie_2, \quad \zeta = xe_1 + ye_2,$$

in the corresponding domain $G_\zeta := \{\zeta = xe_1 + ye_2 : (x, y) \in G\}$ of biharmonic plane μ, where $U_k : G \to \mathbb{R}$ for $k = \overline{1,4}$.

First, consider the following auxiliary statements.

LEMMA 3.4. *Every monogenic function (3.8) with $U_1 \equiv 0$ is of the form*

$$\begin{aligned} \Phi(\zeta) = {} & i(-ax^2 + kx - ay^2 - by + n) + e_2(2ay^2 + 2by + c) \\ & + ie_2(-2axy - bx + ky + m) \qquad \forall\, \zeta = xe_1 + ye_2, \end{aligned} \tag{3.9}$$

where a, b, c, k, m, n are arbitrary real constants.

To prove Lemma 3.4, taking into account the identity $U_1 \equiv 0$, one should integrate the Cauchy–Riemann condition (3.2) rewritten in expanded form:

$$\begin{aligned} 0 &= \frac{\partial U_3(x,y)}{\partial x}, \\ \frac{\partial U_2(x,y)}{\partial y} &= \frac{\partial U_4(x,y)}{\partial x}, \\ \frac{\partial U_3(x,y)}{\partial y} &= -2\frac{\partial U_4(x,y)}{\partial x}, \\ \frac{\partial U_4(x,y)}{\partial y} &= \frac{\partial U_2(x,y)}{\partial x} + 2\frac{\partial U_3(x,y)}{\partial x}. \end{aligned}$$

LEMMA 3.5. *If F is a holomorphic function in a bounded simply connected domain $D \subset \mathbb{C}$, then the functions*

$$\begin{aligned}
\Phi_1(\zeta) &= u(x,y) + iv(x,y) - e_2 v(x,y) + ie_2 u(x,y),\\
\Phi_2(\zeta) &= yu(x,y) + iyv(x,y) + e_2\left(\mathcal{U}(x,y) - yv(x,y)\right)\\
&\qquad + ie_2\left(\mathcal{V}(x,y) + yu(x,y)\right),\\
\Phi_3(\zeta) &= xu(x,y) + ixv(x,y) + e_2\left(\mathcal{V}(x,y) - xv(x,y)\right)\\
&\qquad + ie_2\left(xu(x,y) - \mathcal{U}(x,y)\right) \quad \forall\, \zeta = xe_1 + ye_2 \in G_\zeta
\end{aligned}$$

are monogenic in the domain $G_\zeta \equiv \{\zeta = xe_1 + ye_2 : x + iy \in D\}$ of the biharmonic plane μ, where

$$u(x,y) := \operatorname{Re} F(\xi), \quad v(x,y) := \operatorname{Im} F(\xi),$$

$$\mathcal{U}(x,y) := \operatorname{Re} \mathcal{F}(\xi), \quad \mathcal{V}(x,y) := \operatorname{Im} \mathcal{F}(\xi) \quad \forall\, \xi = x + iy \in D$$

and $\mathcal{F}$ is a primitive function for the function F.

To prove Lemma 3.5, it is easy to show that the functions Φ_1, Φ_2, Φ_3 satisfy the conditions of the form (3.2).

It is well-known that every biharmonic function $U_1(x,y)$ in the domain G is expressed by the Goursat formula

$$U_1(x,y) = \operatorname{Re}\left(\varphi(\xi) + \bar{\xi}\psi(\xi)\right), \qquad \xi = x + iy, \tag{3.10}$$

where φ, ψ are holomorphic functions in the domain $D \equiv \{x + iy : (x,y) \in G\}$, $\bar{\xi} := x - iy$.

THEOREM 3.6. *Every biharmonic function $U_1(x,y)$ in a bounded simply connected domain $G \subset \mathbb{R}^2$ is the first component in the decomposition (3.8) of the function*

$$\Phi(\zeta) = \varphi(\xi) + \bar{\xi}\psi(\xi) + ie_2\left(\varphi(\xi) + \bar{\xi}\psi(\xi) - 2\mathcal{F}(\xi)\right), \tag{3.11}$$

$$\zeta = xe_1 + ye_2, \qquad \xi = x + iy,$$

monogenic in the corresponding domain G_ζ of biharmonic plane μ, where φ, ψ are the same functions as in the equality (3.10) and $\mathcal{F}$ is a primitive function for the function ψ. Moreover, all monogenic functions in G_ζ for which the first component in the decomposition (3.8) is the given function U_1 are expressed as the sum of the functions (3.9) and (3.11).

PROOF. Introducing the functions $u_1(x,y) := \operatorname{Re}\varphi(z)$, $u_2(x,y) := \operatorname{Re}\psi(z)$, $v_2(x,y) := \operatorname{Im}\psi(z)$, we rewrite the equality (3.10) in the form

$$U_1(x,y) = u_1(x,y) + xu_2(x,y) + yv_2(x,y). \tag{3.12}$$

Now it follows from the equality (3.12) and Lemma 3.5 that the function (3.11) is monogenic in the domain G_ζ and the first component in the decomposition (3.8) is the given function U_1. Finally, it evidently follows from Lemma 3.4 that all monogenic functions in G_ζ for which the first component in the decomposition (3.8) is the given function U_1 are expressed as the sum of functions (3.9) and (3.11). □

References

[1] E. Hille and R. S. Phillips, *Functional Analysis and Semi-Groups*, Amer. Math. Soc., Providence, R.I., 1957.
[2] P. W. Ketchum, *Analytic functions of hypercomplex variables*, Trans. Amer. Math. Soc. **30** (1928), 641–667.
[3] F. Klein, *Vorlesungen über die Entwicklung der Mathematik im 19 Jahrhundert*, vol. 1, Springer Verlag, Berlin, 1926.
[4] V. F. Kovalev and I. P. Mel'nichenko, *Biharmonic functions on biharmonic plane*, Dop. AN Ukr. Ser. A., no. 8 (1981), 25–27. [in Russian]
[5] K. S. Kunz, *Application of an algebraic technique to the solution of Laplace's equation in three dimensions*, SIAM J. Appl. Math. **21** (1971), 425–441.
[6] I. P. Mel'nichenko, *The representation of harmonic mappings by monogenic functions*, Ukr. Math. J. **27** (1975), 499–505.
[7] I. P. Mel'nichenko, *Biharmonic bases in algebras of the second rank*, Ukr. Math. J. **38** (1986), 224–226.
[8] I. P. Mel'nichenko, *Algebras of functionally invariant solutions of the three-dimensional Laplace equation*, Ukr. Math. J. **55** (2003), 1551–1557.
[9] I. P. Mel'nichenko and S. A. Plaksa, *Commutative algebras and spatial potential fields*, Kiev: Inst. Math. NAS Ukraine, 2008. [in Russian]
[10] C. Segre, *The real representations of complex elements and extentions to bicomlex systems*, Math. Ann. **40** (1892), 413–467.
[11] G. P. Tolstov, *On the curvilinear and iterated integral*, Trudy Mat. Inst. Steklov., Acad. Sci. USSR **35** (1950), 3–101. [in Russian]
[12] Ju. Ju. Trohimchuk, *Continuous Mappings and Conditions of Monogenity*, Israel Program for Scientific Translations, Jerusalem; Daniel Davey & Co., Inc., New York, 1964.

Institute of Mathematics of the National Academy of Sciences of Ukraine, 3 Tereshchenkivska Street, Kiev, UKRAINE

E-mail address: plaksa@imath.kiev.ua

Institute of Mathematics of the National Academy of Sciences of Ukraine, 3 Tereshchenkivska Street, Kiev, UKRAINE

Current address: Department of Pure and Applied Mathematics, University of Padova, via Trieste, 63-35121 Padova, Italy

E-mail address: gr_sergey_v@mail.ru

Institute of Mathematics of the National Academy of Sciences of Ukraine, 3 Tereshchenkivska Street, Kiev, UKRAINE

E-mail address: shpakivskyi@mail.ru