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Monogenic functions of double variable

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Dedicated to memory of Professor Promarz M. Tamrazov

We establish a constructive description of twice-monogenic functions of double variable by means twice-differentiable functions of real variable.

Встановлено конструктивний опис двічі моногенних функцій подвійної змінної за допомогою двічі диференційовних функцій дійсної змінної.

1. Introduction. An effectiveness of analytic function methods applicable for researching plane potential fields inspires developing similar methods for other models of mathematical physics. In this paper we develop such methods for the wave equation

$$\frac{\partial^2 F}{\partial x^2} - \frac{\partial^2 F}{\partial y^2} = 0. \tag{1}$$

Let $\mathbb{P} := \{x + jy : j^2 := 1, x, y \in \mathbb{R}\}$ be the algebra of double numbers over the field of real numbers $\mathbb{R}$ (see, e. g., [1, p. 52]). In the algebra $\mathbb{P}$ there exists a basis $\{I_1, I_2\}$ such that $I_1^2 = I_1$, $I_2^2 = I_2$, $I_1 I_2 = 0$ and $I_1 + I_1 = 1$. In this case,

$$1 = I_1 + I_2, \quad j = I_1 - I_2 \tag{2}$$

and obviously, $z = x+jy = (x+y)I_1+(x-y)I_2$. Algebraic operations with double numbers are defined by the usual way, and the division is defined for all elements of $\mathbb{P}$ except the set of zero divisors $\{x + jy : y = \pm x\}$.

In many papers (see, e. g., [1 — 7]) differentiable functions in $\mathbb{P}$ are studied, and their physical applications are considered. In this paper, in contrast to previous papers, we consider the differentiable functions in the sense of Gâteaux that is more weak assumption *a priori*.

2. Monogenic functions of double variable. We associate the set $D_z := \{z = x + jy : (x, y) \in D\}$ in the plane $\mathbb{P}$ with a set D of the two-dimensional real space $\mathbb{R}^2$.

We say that a continuous function $\Phi : D_z \to \mathbb{P}$ is *monogenic* in a domain D_z if Φ is differentiable in the sense of Gateaux in every point of D_z, i. e. if for every $z \in D_z$ there exists an element $\Phi'(z) \in \mathbb{P}$ such that

$$\lim_{\varepsilon \to 0+0} (\Phi(z + \varepsilon h) - \Phi(z))\, \varepsilon^{-1} = h\Phi'(z) \quad \forall h \in \mathbb{P}. \tag{3}$$

Theorem 1. *Let $u(x, y)$, $v(x, y)$ be differentiable functions in a domain $D \subset \mathbb{R}^2$. A function $\Phi : D_z \to \mathbb{P}$ of the form*

$$\Phi(z) = u(x, y) + jv(x, y) \tag{4}$$

is monogenic in a domain D_z if and only if the following conditions are fulfilled in D:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}. \tag{5}$$

The conditions (5) are analogous to the Cauchy – Riemann conditions.

It is easy see that all elementary functions introduced in the paper [5, p. 64] are monogenic.

Let Γ be a Jordan rectifiable curve in the plane $\mathbb{R}^2$. For a function $\Phi : \Gamma_z \to \mathbb{P}$ of the form (4) we define an integral along the curve Γ_z by the equality

$$\int_{\Gamma_z} \Phi(z)dz := \int_{\Gamma} udx + vdy + j \int_{\Gamma} vdx + udy. \tag{6}$$

The following analogue of the Cauchy theorem is proved in a such way as in the complex analysis (see, e. g., [8, p. 88]).

Theorem 2. *Suppose that a domain D is bounded by a closed Jordan rectifiable curve Γ, and the functions $u(x, y), v(x, y)$ are continuously differentiable in D. Suppose also that a function $\Phi : D_z \to \mathbb{P}$ of the form (4) is monogenic in the domain D_z and continuous in the closure $\overline{D}_z$. Then*

$$\int_{\Gamma_z} \Phi(z)dz = 0.$$

It is easy to prove the following analogue of Morera theorem for functions taking values in the algebra $\mathbb{P}$.

Theorem 3. *If a function $\Phi : D_z \to \mathbb{P}$ is continuous in a simply connected domain D_z and satisfies the equality*

$$\int\limits_{T_z} \Phi(z)dz = 0 \tag{7}$$

for every triangle $T_z \subset D_z$, then Φ is monogenic in the domain D_z.

3. Relation between twice-monogenic functions and the wave equation. Twice continuously differentiable solutions of the equation (1) are called *wave functions.* Denote by $C^2(D)$ the set of all twice continuously differentiable functions in a domain D. We say that $\Phi : D_z \to \mathbb{P}$ is a *twice-monogenic* function if the Gateaux derivative Φ' is continuous and differentiable in the sense of Gateaux in the domain D_z.

The next theorem follows from the conditions (5).

Theorem 4. *Let a function of the form* (4) *be twice-monogenic in a domain D_z, and $u, v \in C^2(D)$. Then u and v are wave functions in D.*

Two wave functions $u(x, y), v(x, y)$ is called *conjugate* if they are related by the conditions (5).

Theorem 5. *Let $u(x, y)$ be a wave function in a simple connected domain D. Then there exist one* (*accurate to a real constant*) *wave function $v(x, y)$ conjugate to $u(x, y)$ in the domain D.*

Proof. Consider the integral

$$v_0(x, y) = \int\limits_{z_0}^{z} \frac{\partial u}{\partial y}\, dx + \frac{\partial u}{\partial x}\, dy\,, \tag{8}$$

where $z_0 := z_0 + jy_0$ is a fixed point and $z = x + jy$ is an arbitrary point in D_z.

Since $u(x, y)$ is a wave function, then $\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)$. Therefore, the integral (8) does not depend on the way of integration and is a function of the point z only. Taking it into account, in the following equalities the integral is taken from z to $z + h$ along the segment on which $dy = 0$:

$$\frac{\partial v_0}{\partial x} = h^{-1} \lim_{h\to 0} [v_0(x + h, y) - v_0(x, y)] = h^{-1} \lim_{h\to 0} \int\limits_{z}^{z+h} \frac{\partial u}{\partial y} dx = \frac{\partial u}{\partial y}\,.$$

The equality $\frac{\partial v_0}{\partial y} = \frac{\partial u}{\partial x}$ can be proved similarly.

Thus, $v_0(x, y)$ is a wave function conjugate to $u(x, y)$, and $v(x, y) = v_0(x, y) + C$ with a real constant C.

4. Constructive description of twice-monogenic functions. Now we construct a representation of any twice-monogenic function using two differentiable functions of a real variable.

Note that the sets $\mathcal{I}_1 := \{\lambda_1 I_1 : \lambda_1 \in \mathbb{R}\}, \mathcal{I}_2 := \{\lambda_2 I_2 : \lambda_2 \in \mathbb{R}\}$ are maximal ideals in the algebra $\mathbb{P}$. Consider the linear functionals f_1 and f_2 defined on $\mathbb{P}$, whose kernel is the ideals $\mathcal{I}_1$ and $\mathcal{I}_2$, respectively:

$$f_1(I_1) = 0, \quad f_1(I_2) = 1,$$

$$f_2(I_1) = 1, \quad f_2(I_2) = 0.$$

Therefore, $f_1(z) = x - y$, $f_2(z) = x + y$. It is obvious that $f_1(\alpha) = f_2(\alpha) = \alpha$ for all $\alpha \in \mathbb{R}$. Note that the functionals f_1, f_2 are continuous and multiplicative.

A domain D is called *convex in the direction of the straight line* L, if it contains each segment that connects two points of D and is parallel to the straight line L.

In the case where a segment l is parallel to a straight line L in $\mathbb{R}^2$, we shall say that the segment l_z is parallel to the straight line L_z in $\mathbb{P}$.

Denote by L^1 and L^2 the straight lines $y = x$ and $y = -x$, respectively. For $z_1, z_2 \in \mathbb{P}$ and $z_1 \neq z_2$, denote by $[z_1 z_2]$ the segment connecting the points z_1 and z_2.

Lemma. 1) *Let a domain $D \subset \mathbb{R}^2$ be convex in the direction of the straight line L^1 and a function of the form* (4) *be twice-monogenic in D_z, and $u, v \in C^2(D)$. If the points $z_1, z_2 \in D_z$ are such that the segment $[z_1 z_2]$ is parallel to L^1_z, then $\Phi(z_2) - \Phi(z_1) \in \mathcal{I}_1$.*

2) *Let a domain $D \subset \mathbb{R}^2$ be convex in the direction of the straight line L^2 and a function of the form* (4) *be twice-monogenic in D_z, and $u, v \in C^2(D)$. If the points $z_1, z_2 \in D_z$ are such that the segment $[z_1 z_2]$ is parallel to L^2_z, then $\Phi(z_2) - \Phi(z_1) \in \mathcal{I}_2$.*

Proof. Consider the case 1) of Lemma. In this case there exists a real number λ such that $z_2 = z_1 + 2\lambda I_1$ and $[z_1 z_2]$ is completely contained in D. Since $\{I_1, I_2\}$ is a basis in $\mathbb{P}$, the following decomposition is true:

$$\Phi(z_1) - \Phi(z_2) = \alpha I_1 + \beta I_2 \,,$$

where $\alpha, \beta \in \mathbb{R}$.

To complete the proof, it is sufficient to show that $\beta = 0$. Using the equalities (2), we have:

$$\Phi(z_1) - \Phi(z_2) = \Phi(z_1) - \Phi(z_1 + 2\lambda I_1) = u(x_1, y_1) + jv(x_1, y_1) -$$

$$-u(x_1+\lambda, y_1+\lambda) - jv(x_1+\lambda, y_1+\lambda) = \alpha I_1 + \beta I_2 = \frac{1}{2}(\alpha+\beta) + \frac{1}{2}j(\alpha-\beta),$$

whence we obtain the system of equations

$$\begin{cases} u(x_1, y_1) - u(x_1 + \lambda, y_1 + \lambda) = \dfrac{1}{2}\,(\alpha + \beta), \\ v(x_1, y_1) - v(x_1 + \lambda, y_1 + \lambda) = \dfrac{1}{2}\,(\alpha - \beta). \end{cases} \tag{9}$$

Since the integral (8) does not depend on a way of integration but depend on the endpoint only, and since $u(x, y)$, $v(x, y)$ are conjugate wave functions, we obtain

$$v(x_1, y_1) - v(x_1 + \lambda, y_1 + \lambda) = -\int\limits_{z_1}^{z_1+2\lambda I_1} \frac{\partial u}{\partial y}\,dx + \frac{\partial u}{\partial x}\,dy\,.$$

Since $dx = dy$ along the segment $[z_1 z_2]$, then

$$v(x_1, y_1) - v(x_1 + \lambda, y_1 + \lambda) = -\int\limits_{z_1}^{z_1+2\lambda I_1} \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial x}dx =$$

$$= -\int\limits_{z_1}^{z_1+2\lambda I_1} du = -u(x_1 + \lambda, y_1 + \lambda) + u(x_1, y_1). \tag{10}$$

It follows from the equality (10) and the system of equation (9) that

$$\alpha + \beta = \alpha - \beta,$$

whence $\beta = 0$. The statement 1) of Lemma is proved. The statement 2) is similarly proved.

Let a domain $D \subset \mathbb{R}^2$ be convex in the directions of the straight lines L_1 and L_2. Then $\Delta_1 := f_1(D_z)$, $\Delta_2 := f_2(D_z)$ are intervals on the real axis. Consider the linear operators A_1 and A_2 that assign the functions

$F_1 : \Delta_1 \to \mathbb{R}$ and $F_2 : \Delta_2 \to \mathbb{R}$, respectively, to every twice-monogenic function $\Phi : D_z \to \mathbb{P}$ by the formulas $F_1(t_1) := f_1(\Phi(z))$ and $F_2(t_2) := := f_2(\Phi(z))$, where $t_1 := f_1(z) = x - y$ and $t_2 := f_2(z) = x + y$.

It follows from Lemma that the values $F_1(t_1)$, $F_2(t_2)$ do not depend on a choice of a point z for which $f_1(z) = t_1$ or $f_2(z) = t_2$.

Theorem 6. *Let a domain $D \subset \mathbb{R}^2$ be convex in the directions of the straight lines L_1 and L_2. Then every twice-monogenic in D_z function of the form* (4) *with $u, v \in C^2(D)$ can be represented in the form*

$$\Phi(z) = F_1(t_1)I_2 + F_2(t_2)I_1, \tag{11}$$

where $F_1(t_1)$ and $F_2(t_2)$ are certain twice-differentiable functions on the intervals Δ_1 and Δ_2, respectively.

Proof. Let a function Φ have the form

$$\Phi(z) = U(x,y)I_1 + V(x,y)I_2. \tag{12}$$

Acting by the linear functionals f_1, f_2 on the equality (12), we obtain

$$f_1(\Phi(z)) = F_1(t_1) = V(x,y),$$

$$f_2(\Phi(z)) = F_2(t_2) = U(x,y).$$

From these equalities and the equality (12) we obtain the representation (11). It remains to prove the twice-differentiability of functions $F_1(t_1)$ and $F_2(t_2)$.

From the representation (11) we obtain the equalities

$$F_1(t_1) = u(x,y) - v(x,y), \quad F_2(t_2) = u(x,y) + v(x,y). \tag{13}$$

Since the functions u, v are twice-differentiable in D, the functions $F_1(t_1)$, $F_2(t_2)$ are also twice-differentiable on the intervals Δ_1, Δ_2, respectively, due to the equalities (13). Theorem is proved.

Passing in the equality (11) to the basis $\{1, j\}$, we obtain the wave functions in the domain D:

$$u(x,y) = \frac{1}{2}\big(F_1(t_1) + F_2(t_2)\big), \quad v(x,y) = \frac{1}{2}\big(F_2(t_2) - F_1(t_1)\big),$$

that coincides with the well-known general solution of the wave equation (see, e. g., [9, p. 51]).

Note that the equality (11) can be rewritten as

$$\Phi(z) = A_1(\Phi(z))I_2 + A_2(\Phi(z))I_1.$$

Let $\Pi_z := \{z \in \mathbb{P} : f_1(z) = \Delta_1\} \cap \{z \in \mathbb{P} : f_2(z) = \Delta_2\}$. The next theorem follows directly from the equality (11), where the right-hand part is a monogenic function in the rectangular domain Π_z.

Theorem 7. *Let a domain $D \subset \mathbb{R}^2$ be convex in the directions of the straight lines L_1 and L_2. Then every twice-monogenic in D_z function of the form* (4) *with $u, v \in C^2(D)$ can be extended to a function monogenic in the domain Π_z.*

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