An efficient method for solving equations in generalized quaternion and octonion algebras

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Abstract. Quaternions often appear in wide areas of applied science and engineering such as wireless communications systems, mechanics, etc. It is known that are two types of non-isomorphic generalized quaternion algebras, namely: the algebra of quaternions and the algebra of coquaternions. In this paper, we present the formulae to pass from a basis in the generalized quaternion algebras to a basis in the division quaternions algebra or to a basis in the coquaternions algebra and vice versa. The same result was obtained for the generalized octonion algebra. Moreover, we emphasize the applications of these results to the algebraic equations and De Moivre's formula in generalized quaternion algebras and in generalized octonion division algebras.

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0. Introduction

Let $\gamma_1, \gamma_2 \in \mathbb{R} \setminus \{0\}$, let $\mathbb{H}(\gamma_1, \gamma_2)$ be the generalized quaternion algebra with basis $\{1, e_1, e_2, e_3\}$ and $\mathbb{H}(1,1)$ be the quaternion division algebra with basis $\{1, i, j, k\}$. The multiplication table is given below:

$\cdot$	1	e_1	e_2	e_3
1	1	e_1	e_2	e_3
e_1	e_1	$-\gamma_1$	e_3	$-\gamma_1 e_2$
e_2	e_2	$-e_3$	$-\gamma_2$	$\gamma_2 e_1$
e_3	e_3	$\gamma_1 e_2$	$-\gamma_2 e_1$	$-\gamma_1 \gamma_2$

The algebra $\mathbb{H}(1,-1)$ is called *the algebra of coquaternions* [2] or also called *the algebra of para-quaternions* [8], or *the algebra of split-quaternions* [5], [16], or *the algebra of anti-quaternions*, or *the algebra of pseudo-quaternions* [18, p. 389], or *hyperbolic quaternions* [1]. We denote by $\{1, i_1, i_2, i_3\}$ the basis of coquaternion algebra.

Proposition 1. ([10], Proposition 1.1) *The quaternion algebra $\mathbb{H}(\beta_1, \beta_2)$ is isomorphic with quaternion algebra $\mathbb{H}\left(x^2\beta_1, y^2\beta_2\right)$, where $x, y \in K^*$.* $\Box$

The real octonion division algebras are a non-associative and non-commutative extension of the algebra of quaternions. Among all the real division algebras, octonion algebra forms the largest normed division algebra.

Let $\mathbb{O}(\alpha,\beta,\gamma)$ be a generalized octonion algebra over $\mathbb{R}$, with basis $\{1, f_1, ..., f_7\}$ and the multiplication given in the following table:

$\cdot$	1	f_1	f_2	f_3	f_4	f_5	f_6	f_7
1	1	f_1	f_2	f_3	f_4	f_5	f_6	f_7
f_1	f_1	$-\alpha$	f_3	$-\alpha f_2$	f_5	$-\alpha f_4$	$-f_7$	αf_6
f_2	f_2	$-f_3$	$-\beta$	βf_1	f_6	f_7	$-\beta f_4$	$-\beta f_5$
f_3	f_3	αf_2	$-\beta f_1$	$-\alpha\beta$	f_7	$-\alpha f_6$	βf_5	$-\alpha\beta f_4$
f_4	f_4	$-f_5$	$-f_6$	$-f_7$	$-\gamma$	γf_1	γf_2	γf_3
f_5	f_5	αf_4	$-f_7$	αf_6	$-\gamma f_1$	$-\alpha\gamma$	$-\gamma f_3$	$\alpha\gamma f_2$
f_6	f_6	f_7	βf_4	$-\beta f_5$	$-\gamma f_2$	γf_3	$-\beta\gamma$	$-\beta\gamma f_1$
f_7	f_7	$-\alpha f_6$	βf_5	$\alpha\beta f_4$	$-\gamma f_3$	$-\alpha\gamma f_2$	$\beta\gamma f_1$	$-\alpha\beta\gamma$

The algebra $\mathbb{O}(\alpha,\beta,\gamma)$ is non-commutative, non-associative but it is *alternative* (i.e. $x^2 y = x(xy)$ and $yx^2 = (yx)x, \forall x, y \in \mathbb{O}(\alpha,\beta,\gamma)$), *flexible* (i.e. $x(yx) = (xy)x, \forall x, y \in \mathbb{O}(\alpha,\beta,\gamma)$), *power-associative* (i.e. for each $x \in \mathbb{O}(\alpha,\beta,\gamma)$ the subalgebra generated by x is an associative algebra).

If $a \in \mathbb{O}(\alpha,\beta,\gamma)$, $a = a_0 + a_1 f_1 + a_2 f_2 + a_3 f_3 + a_4 f_4 + a_5 f_5 + a_6 f_6 + a_7 f_7$ then $\bar{a} = a_0 - a_1 f_1 - a_2 f_2 - a_3 f_3 - a_4 f_4 - a_5 f_5 - a_6 f_6 - a_7 f_7$ is called the *conjugate* of the element a. Let $A = \mathbb{O}(\alpha,\beta,\gamma)$ and $a \in A$. We have that $t(a) = a + \overline{a} \in K$ and

$$N(a) = a\overline{a} = a_0^2 + \alpha a_1^2 + \beta a_2^2 + \alpha\beta a_3^2 + \gamma a_4^2 + \alpha\gamma a_5^2 + \beta\gamma a_6^2 + \alpha\beta\gamma a_7^2 \in K.$$

These elements are called the *trace*, respectively, the *norm* of the element $a \in A$. It follows that $(a + \overline{a})\, a = a^2 + \overline{a}a = a^2 + n(a) \cdot 1$ and $a^2 - t(a)\, a + N(a) = 0, \forall a \in A$, therefore the generalized octonion algebra is *quadratic*.

The subset $A_0 = \{x \in A \mid t(x) = 0\}$ of A is a subspace of the algebra A. It is obvious that $A = K \cdot 1 \oplus A_0$, therefore each element $x \in A$ has the form $x = x_0 \cdot 1 + \overrightarrow{x}$, with $x_0 \in K$ and $\overrightarrow{x} \in A_0$. For $K = \mathbb{R}$, we call x_0 *the scalar part* and $\overrightarrow{x}$ *the vector part* for the octonion x.

If for $x \in A$, the relation $N(x) = 0$ implies $x = 0$, then the algebra A is called a *division* algebra. For other details, the reader is referred to [19].

In the papers [6], [7] are considered some algebraic equations in generalized quaternion and octonion algebras. Due to Proposition 1, in the present paper we reduced the study of an algebraic equation in an arbitrary algebra $\mathbb{H}(\gamma_1, \gamma_2)$ with $\gamma_1, \gamma_2 \in \mathbb{R} \setminus \{0\}$ to study of the corresponding algebraic equation in one of the following two algebras: division quaternion algebra or coquaternion algebra. Moreover, De Moivre's formula and Euler's formula in generalized quaternion algebras, founded in [11], was proved using this new method, for $\gamma_1, \gamma_2 > 0$. With this technique, the above mentioned results were also obtained for the octonions.

1. An isomorphism between the algebras $\mathbb{H}(\gamma_1, \gamma_2)$, with $\gamma_1, \gamma_2 > 0$, and $\mathbb{H}(1, 1)$

Everywhere in this section, we will consider $\gamma_1, \gamma_2 > 0$. An isomorphism between the algebras $\mathbb{H}(\gamma_1, \gamma_2)$ and $\mathbb{H}(1,1)$ is given by the operator A and its inverse A^{-1}, where

$$A: \quad e_1 \mapsto i\sqrt{\gamma_1}, \quad e_2 \mapsto j\sqrt{\gamma_2}, \quad e_3 \mapsto k\sqrt{\gamma_1\gamma_2}.$$

It is easy to prove the following properties for the operator $A:$

1) $A(\lambda x) = \lambda A(x), \forall \lambda \in \mathbb{R}, \forall x \in \mathbb{H}(\gamma_1, \gamma_2);$

2) $A(x+y) = A(x) + A(y), \forall x, y \in \mathbb{H}(\gamma_1, \gamma_2);$

3) $A(xy) = A(x)A(y), \forall x, y \in \mathbb{H}(\gamma_1, \gamma_2).$

From here, it results that the operators A and A^{-1} are additive and multiplicative.

Proposition 1.1. *The operators A and A^{-1} are continuous and their norms are equal with* 1.

Proof. We denote by $\|\cdot\|_{\mathbb{H}(\gamma_1,\gamma_2)}$ the Euclidian norm in $\mathbb{H}(\gamma_1, \gamma_2)$. Since the spaces $\mathbb{H}(\gamma_1, \gamma_2)$ and $\mathbb{H}(1,1)$ are normed spaces, then the continuity of A is equivalent with the boundedness of A, i.e. there is a real constant c such that for all $x \in \mathbb{H}(\gamma_1, \gamma_2)$, we have

$$\frac{\|A(x)\|_{\mathbb{H}(1,1)}}{\|x\|_{\mathbb{H}(\gamma_1,\gamma_2)}} \leq c.$$

It results that

$$\frac{\left\|x_0 + x_1 i\sqrt{\gamma_1} + x_2 j\sqrt{\gamma_2} + x_3 k\sqrt{\gamma_1\gamma_2}\right\|}{\|x_0 + x_1 e_1 + x_2 e_2 + x_3 e_3\|} =$$

$$= \frac{\sqrt{x_0^2 + x_1^2\gamma_1 + x_2^2\gamma_2 + x_3^2\gamma_3}}{\sqrt{x_0^2 + x_1^2\gamma_1 + x_2^2\gamma_2 + x_3^2\gamma_3}} = 1.$$

□

2. The algebra $\mathbb{H}(\gamma_1, \gamma_2)$, with $\gamma_1, \gamma_2 < 0$ or $\gamma_1\gamma_2 < 0$

In this situation, the algebra $\mathbb{H}(\gamma_1, \gamma_2)$ is isomorphic with $\mathbb{H}(1,-1)$. We suppose first that $\gamma_1, \gamma_2 < 0$. An isomorphism between the algebra $\mathbb{H}(\gamma_1, \gamma_2)$, where $\gamma_1, \gamma_2 < 0$, and the algebra $\mathbb{H}(1,-1)$ is given by the operator B and its inverse B^{-1}, where

$$B: \quad e_1 \mapsto i_3\sqrt{-\gamma_1}, \quad e_2 \mapsto i_2\sqrt{-\gamma_2}, \quad e_3 \mapsto i_1\sqrt{\gamma_1\gamma_2}.$$

For $\gamma_1 > 0, \gamma_2 < 0$, an isomorphism between the algebra $\mathbb{H}(\gamma_1, \gamma_2)$ and the algebra $\mathbb{H}(1, -1)$ is given by the operator C and its inverse C^{-1}, where

$$C: \quad e_1 \mapsto i_1\sqrt{\gamma_1}, \quad e_2 \mapsto i_2\sqrt{-\gamma_2}, \quad e_3 \mapsto i_3\sqrt{-\gamma_1\gamma_2}.$$

For $\gamma_1 < 0, \gamma_2 > 0$, an isomorphism between the algebra $\mathbb{H}(\gamma_1, \gamma_2)$ and the algebra $\mathbb{H}(1, -1)$ is given by the operator D and its inverse D^{-1}, where

$$D: \quad e_1 \mapsto i_3\sqrt{-\gamma_1}, \quad e_2 \mapsto i_1\sqrt{\gamma_2}, \quad e_3 \mapsto i_2\sqrt{-\gamma_1\gamma_2}.$$

The properties of the operators $B, B^{-1}, C, C^{-1}, D, D^{-1}$ are similarly with the properties of the operator A.

Since each algebra $\mathbb{H}(\gamma_1, \gamma_2)$ is isomorphic with algebra of quaternions or coquaternions, it results that the above operators provide us a simple way to generalize known results in these two algebras to generalized quaternion algebra.

3. Application to the algebraic equations

Let $x = x_0 + x_1e_1 + x_2e_3 + x_3e_3 \in \mathbb{H}(\gamma_1, \gamma_2)$ and let $f : \mathbb{H}(\gamma_1, \gamma_2) \to \mathbb{H}(\gamma_1, \gamma_2)$ be a continuous function of the form $f(x) = f_0(x_0, x_1, x_2, x_3) + f_1(x_0, x_1, x_2, x_3)e_1 + f_2(x_0, x_1, x_2, x_3)e_2 + f_3(x_0, x_1, x_2, x_3)e_3$. Let F be the one of the operators A, B, C or D, depending on the signs of γ_1 and γ_2. We define the operator $\mathfrak{F}$ which for any continuous function f, taking values in $\mathbb{H}(\gamma_1, \gamma_2)$, maps it in the continuous function $\mathfrak{F}f$, taking values in $\mathbb{H}(1, 1)$ or $\mathbb{H}(1, -1)$ by the rule:

$$\mathfrak{F}f := f_0 + f_1\,F(e_1) + f_2\,F(e_2) + f_3\,F(e_3).$$

Theorem 3.1. *Let $x^0 \in \mathbb{H}(\gamma_1, \gamma_2)$ be a root of the equation $f(x) = 0$ in $\mathbb{H}(\gamma_1, \gamma_2)$. Then $F(x^0)$ is a root of the equation $\mathfrak{F}f\big(F(x)\big) = 0$ in $\mathbb{H}(1, 1)$ or $\mathbb{H}(1, -1)$, depending on the signs of γ_1 and γ_2. The converse is also true.*

Proof. Let $\gamma_1, \gamma_2 > 0$. Applying operator A to the equality $f(x^0) = 0$ and using the continuity of A, we obtain

$$A\big(f(x^0)\big) = Af\big(A(x^0)\big) = A(0) = 0.$$

To prove the converse statement we apply the operator A^{-1} to the equality $f(x^0) = 0$. The remaining cases can be proved similarly. □

Therefore, all results from quaternionic equations and from coquaternionic equations can be generalized in $\mathbb{H}(\gamma_1, \gamma_2)$.

It is known that any polynomial of degree n with coefficients in a field K has at most n roots in K. If the coefficients are in $\mathbb{H}(1, 1)$, the situation is different. For the real division quaternion algebra over the real field, there is a kind of a fundamental theorem of algebra: *If a polynomial has only one term of the greatest degree, then it has at least one root in* $\mathbb{H}(1,1)$. ([21], Theorem 65; [4], Theorem 1).

We consider the polynomial of degree n of the form

$$f(x) = a_0 x a_1 x \ldots a_{n-1} x a_n + \varphi(x), \tag{3.1.}$$

where x, a_0, $a_1, \ldots, a_{n-1}$, $a_n \in \mathbb{H}(1,1)$, with $a_\ell \neq 0$ for $\ell \in \{0, 1, \ldots, n\}$ and $\varphi(x)$ is a sum of a finite number of monomials of the form $b_0 x b_1 x \ldots b_{t-1} x b_t$ where $t < n$. From the above, it results that the equation $f(x) = 0$ has at least one root. Applying operator A^{-1} to this last equality, the equation $\left(A^{-1} f\right)\left(A^{-1}(x)\right) = 0$, with $x = x_0 + x_1 e_1 + x_2 e_2 + x_3 e_3$, has at least one root in $\mathbb{H}(\gamma_1, \gamma_2)$. Therefore, we proved the following result:

Theorem 3.2. *In the generalized quaternion algebra* $\mathbb{H}(\gamma_1, \gamma_2)$ *with* $\gamma_1, \gamma_2 > 0$ *any polynomial of the form* (3.1) *has at least one root.* □

In the following, we consider the equation $x^2 + ax + b = 0$, where $\mathbb{H}(\gamma_1, \gamma_2)$ with $\gamma_1, \gamma_2 > 0$. We say that a root $x_0 = a_0 + a_1 i_1 + a_2 i_2 + a_3 i_3 \in \mathbb{H}(\gamma_1, \gamma_2)$, $\gamma_1, \gamma_2 > 0$ is a *ellipsoidal root* if every element $x_1 \in \mathbb{H}(\gamma_1, \gamma_2)$ of the form $x_1 = a_0 + b_1 i_1 + b_2 i_2 + b_3 i_3$ such that $\gamma_1 a_1^2 + \gamma_2 a_2^2 + \gamma_1 \gamma_2 a_3^2 = \gamma_1 b_1^2 + \gamma_2 b_2^2 + \gamma_1 \gamma_2 b_3^2$ is also a root of this equation.

Using Theorem 3 from [12] and Theorem 3.1, we just proved the following theorem:

Theorem 3.3. *In the generalized quaternion algebra* $\mathbb{H}(\gamma_1, \gamma_2)$ *with* $\gamma_1, \gamma_2 > 0$ *the equation* $x^2 + ax + b = 0$ *has ellipsoidal root if and only if* a *and* b *are real numbers and* $a^2 - 4b < 0$. □

More results on the structure of roots of the quadratic quaternionic equations can be found in see [22], [13], [14], [15].

In the following, we apply the above results to the coquaternion algebra. We will consider one of the three cases: $\gamma_1, \gamma_2 < 0$; $\gamma_1 < 0, \gamma_2 > 0$; $\gamma_1 > 0, \gamma_2 < 0$. We say that a root $x_0 = a_0 + a_1 i_1 + a_2 i_2 + a_3 i_3 \in \mathbb{H}(\gamma_1, \gamma_2)$ is a *hyperboloidal root* if every element $x_1 \in \mathbb{H}(\gamma_1, \gamma_2)$ of the form $x_1 = a_0 + b_1 i_1 + b_2 i_2 + b_3 i_3$ such that $\gamma_1 a_1^2 + \gamma_2 a_2^2 + \gamma_1 \gamma_2 a_3^2 = \gamma_1 b_1^2 + \gamma_2 b_2^2 + \gamma_1 \gamma_2 b_3^2$ is also a root of this equation. Using Theorem 2.5 of [17] and the above Theorem 3.1, we proved:

Theorem 3.4. *Suppose that coefficients* r_k, $k = 0, 1, \ldots, n$ *of the polynomial equation* $r_n x^n + r_{n-1} x^{n-1} + \ldots + r_0 = 0, \quad x \in \mathbb{H}(\gamma_1, \gamma_2)$ *are real numbers. Therefore all its non-real roots are hyperboloidal.* □

Solutions of linear equations in coquaternionic algebra can be found in [9], [5] and solutions of linear equations in quaternion algebra can be found, for example, in [23], [20].

Let $x = x_0 + x_1 i + x_2 j + x_3 k \in \mathbb{H}(1,1), \ x_0, x_1, x_2, x_3 \in \mathbb{R}$. We denoted by

$$\widetilde{x} = A^{-1}(x) = x_0 + x_1 \frac{e_1}{\sqrt{\gamma_1}} + x_2 e_2 \frac{e_2}{\sqrt{\gamma_2}} + x_3 e_3 \frac{e_1}{\sqrt{\gamma_1 \gamma_2}}.$$

Example 3.5. Consider the general polynomial equation in $\mathbb{H}(1,1)$:

$$\sum_{p=1}^{n}\left(\sum_{l=1}^{m_p} a_{p,l,1}xa_{p,l,2}x...a_{p,l,p}xa_{p,l,p+1}\right)+c=0.$$

This equation is "equivalent" to the following equation in $\mathbb{H}(\gamma_1,\gamma_2)$, with $\gamma_1,\gamma_2>0$:

$$\sum_{p=1}^{n}\left(\sum_{l=1}^{m_p} \widetilde{a}_{p,l,1}\widetilde{x}\,\widetilde{a}_{p,l,2}\,\widetilde{x}\ldots\widetilde{a}_{p,l,p}\,\widetilde{x}\,\widetilde{a}_{p,l,p+1}\right)+\widetilde{c}=0.$$

Therefore, the algebraic equations $f(x)=0$, with f continuous, in usual quaternions $\mathbb{H}(1,1)$ can be reduced to the similar equations in all algebras $\mathbb{H}(\gamma_1,\gamma_2)$ for $\gamma_1,\gamma_2>0$ and vice versa. A similar result can be found for different signs of γ_1 and γ_2.

4. De Moivre's formula

In the following, we will use some ideas and notations from [3]. Let $q=q_0+q_1e_1+q_2e_2+q_3e_3\in\mathbb{H}(\gamma_1,\gamma_2)$, $\gamma_1,\gamma_2>0$, $q_0,q_1,q_2,q_3\in\mathbb{R}$ and

$$|q|=\sqrt{q_0^2+\gamma_1q_1^2+\gamma_2q_2^2+\gamma_1\gamma_2q_3^2}.$$

Consider the sets

$$\mathcal{S}_G^3=\{q\in\mathbb{H}(\gamma_1,\gamma_2),\gamma_1,\gamma_2>0:\ |q|=1\},$$
$$\mathcal{S}_G^2=\{q\in\mathbb{H}\left(\gamma_1,\gamma_2\right),\gamma_1,\gamma_2>0:\ q_0=0,|q|=1\}.$$

Any $q\in\mathcal{S}_G^3$ can be expressed as $q=\cos\theta+\varepsilon\sin\theta$, where

$$\cos\theta=q_0,\ \ \varepsilon=\frac{q_1e_1+q_2e_2+q_3e_3}{\sqrt{\gamma_1q_1^2+\gamma_2q_2^2+\gamma_1\gamma_2q_3^2}}.$$

Using Proposition 2 from [3] and applying the operator A^{-1} we will find De Moivre's formula for $\mathbb{H}\left(\gamma_1,\gamma_2\right),\gamma_1,\gamma_2>0$.

Theorem 4.1. *Let* $q=\cos\theta+\varepsilon\sin\theta\in\mathcal{S}_G^3$, $\theta\in\mathbb{R}$. *Then* $q^n=\cos n\theta+\varepsilon\sin n\theta$ *for every integer* n.

Theorem 4.1 is the same with Theorem 7 from the paper [11], obtained with another proof.□

Using Corollary 3 from [3] and Theorem 3.1, we obtain the next statement.

Proposition 4.2. *i) In $\mathbb{H}(\gamma_1, \gamma_2)$, $\gamma_1, \gamma_2 > 0$ the equation $x^n = 1$ whit n integer and $n \geq 3$ has infinity of roots, namely*

$$q = \cos \frac{2\pi}{n} + \varepsilon \sin \frac{2\pi}{n} \in \mathcal{S}_G^3, \ \varepsilon \in \mathcal{S}_G^2.$$

ii) In $\mathbb{H}(\gamma_1, \gamma_2)$, $\gamma_1, \gamma_2 > 0$ the equation $x^n = a$, $n \in \mathbb{N}$, $a \in \mathbb{R}$ has infinity of roots, namely $\sqrt[n]{a}\, q$, where $q = \cos \frac{2\pi}{n} + \varepsilon \sin \frac{2\pi}{n} \in \mathcal{S}_G^3$, with $\varepsilon \in \mathcal{S}_G^2$. If n is even it is necessary that $a > 0$.$\Box$

5. De Moivre's formula and Euler's formula for octonions

In the following, we will generalize in a natural way De Moivre formula and Euler's formula for the division octonion algebra $\mathbb{O}(1,1,1)$. For this, we will use some ideas and notations from [3]. We consider the sets

$$\mathcal{S}^3 = \{a \in \mathbb{O}(1,1,1) : \ N(a) = 1\},$$

$$\mathcal{S}_G^3 = \{a \in \mathbb{O}(\alpha, \beta, \gamma) : \ N(a) = 1\},$$

$$\mathcal{S}^2 = \{a \in \mathbb{O}(1,1,1) : \ t(a) = 0, N(a) = 1\}.$$

$$\mathcal{S}_G^2 = \{a \in \mathbb{O}(1,1,1) : \ t(a) = 0, N(a) = 1\}.$$

We remark that for all elements $a \in \mathcal{S}^2$, we have $a^2 = -1$. Let $a \in \mathcal{S}^3, a = a_0 + a_1 f_1 + a_2 f_2 + a_3 f_3 + a_4 f_4 + a_5 f_5 + a_6 f_6 + a_7 f_7$. This element can be write under the form

$$a = \cos \lambda + w \sin \lambda,$$

where $\cos \lambda = a_0$ and

$$w = \frac{a_1 f_1 + a_2 f_2 + a_3 f_3 + a_4 f_4 + a_5 f_5 + a_6 f_6 + a_7 f_7}{\sqrt{a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2 + a_7^2}} =$$

$$= \frac{a_1 f_1 + a_2 f_2 + a_3 f_3 + a_4 f_4 + a_5 f_5 + a_6 f_6 + a_7 f_7}{\sqrt{1 - a_0^2}}.$$

Since $w^2 = -1$, we obtain the following Euler's formula:

$$\begin{aligned} e^{\lambda w} &= \sum_{i=1}^{\infty} \frac{(\lambda w)^n}{n!} = \sum_{i=1}^{\infty} \frac{(-1)^n \lambda^{2n}}{(2n)!} + w \sum_{i=1}^{\infty} \frac{(-1)^{n-1} \lambda^{2n-1}}{(2n-1)!} = \\ &= \cos \lambda + w \sin \lambda. \end{aligned}$$

Proposition 5.1. *The cosinus function is constant for all elements in $\mathcal{S}^2$.*

Proof. Indeed, $\cos w = \sum\limits_{i=1}^{\infty} \frac{(-1)^n \lambda^{2n}}{(2n)!} = \cos i.\Box$

Proposition 5.2. For $w \in \mathcal{S}^2$, we have $(\cos \lambda_1 + w \sin \lambda_1)(\cos \lambda_2 + w \sin \lambda_2) = \cos(\lambda_1 + \lambda_2) + w \sin(\lambda_1 + \lambda_2)$.

Proof. By straightforward calculations $\Box$

Proposition 5.3. (De Moivre formula for octonions) *With the above notations, we have that*

$$a^n = e^{n\lambda w} = (\cos \lambda + w \sin \lambda)^n = \cos n\lambda + w \sin n\lambda,$$

where $a \in \mathcal{S}^3$, $n \in \mathbb{Z}$ *and* $\lambda \in \mathbb{R}$.

Proof. For $n > 0$, by induction. We obtain

$$\begin{aligned} a^{n+1} &= (\cos \lambda + w \sin \lambda)^{n+1} = \\ &= (\cos \lambda + w \sin \lambda)^n (\cos \lambda + w \sin \lambda) = \\ &= (\cos n\lambda + w \sin n\lambda)(\cos \lambda + w \sin \lambda) = \\ &= \cos(n+1)\lambda + w \sin(n+1)\lambda. \end{aligned}$$

Since $a^{-1} = \cos \lambda - w \sin \lambda = \cos(-\lambda) + w \sin(-\lambda)$, it results the asked formula for all $n \in \mathbb{Z}.\Box$

Remark 5.4. It is known that any polynomial of degree n with coefficients in a field K has at most n roots in K. If the coefficients are in $\mathbb{O}(1,1,1)$ there is a kind of a fundamental theorem of algebra: If a polynomial has only one term of the greatest degree, then it has at least one root in $\mathbb{O}(1,1,1)$ (see [21], Theorem 65).

Theorem 5.5. *Equation* $x^n = a$, *where* $a \in \mathbb{O}(1,1,1) \setminus \mathbb{R}$, *has* n *roots.*

Proof. The octonion a can be written under the form $a = \sqrt{N(a)} \frac{a}{\sqrt{N(a)}}$. The octonion $b = \frac{a}{\sqrt{N(a)}}$ is in $\mathcal{S}^3$, then we can find the elements $w \in \mathcal{S}^2$ and $\lambda \in \mathbb{R}$ such that $b = \cos \lambda + w \sin \lambda$. From Proposition 5.3, we have that the solutions of the above equation are $x_r = \sqrt[n]{Q}\left(\cos \frac{\lambda + 2r\pi}{n} + w \sin \frac{\lambda + 2r\pi}{n}\right)$, where $Q = \sqrt{N(a)}$ and $r \in \{0, 1, ..., n-1\}.\Box$

Corollary 5.6. *If* $a \in \mathbb{R}$, *therefore the equation* $x^n = a$ *has an infinity of roots.*

Proof. Indeed, if $a \in \mathbb{R}$, we can write $a = a \cdot 1 = a(\cos 2\pi + w \sin 2\pi)$, where $w \in \mathcal{S}^2$ is an arbitrary element. $\Box$

Remark 5.7. The rotation of the octonion $x \in \mathbb{O}(1,1,1)$ on the angle λ around the unit vector $w \in \mathcal{S}^2$ is defined by the formula

$$x^r = \overline{u} x u,$$

where $u \in \mathcal{S}^3, u = \cos \frac{\lambda}{2} + w \sin \frac{\lambda}{2}$ and $\overline{u} = \cos \frac{\lambda}{2} - w \sin \frac{\lambda}{2}$.

Using the form $x = x_0 \cdot 1 + \overrightarrow{x}$, $y = y_0 \cdot 1 + \overrightarrow{y}$ for the octonions $x, y \in \mathbb{O}\,(1,1,1)$, we obtain the following expression for the product of two octonions:

$$\begin{aligned} xy &= (x_0 \cdot 1 + \overrightarrow{x})\,(y_0 \cdot 1 + \overrightarrow{y}) = \\ &= x_0 y_0 \cdot 1 + x_0 \overrightarrow{y} + y_0 \overrightarrow{x} + < \overrightarrow{x}, \overrightarrow{y} > + \overrightarrow{x} \times \overrightarrow{y}, \end{aligned}$$

where $< \overrightarrow{x}, \overrightarrow{y} >$ is the inner product of two octonionic-vector and $\overrightarrow{x} \times \overrightarrow{y}$ is the cross product. From here, we obtain that $x^r = \overline{u} x u =$
$= \left(\cos \frac{\lambda}{2} - w \sin \frac{\lambda}{2}\right)(x_0 \cdot 1 + \overrightarrow{x})\left(\cos \frac{\lambda}{2} + w \sin \frac{\lambda}{2}\right) =$
$= x_0 + \overrightarrow{x} \cos \lambda - w < w, \overrightarrow{x} > (1 - \cos \lambda) - (w \times \overrightarrow{x}) \sin \lambda$. It results that that rotation does not transform the octonion-scalar part, but the octonion-vector part $\overrightarrow{x}$ is rotated on the angle λ around w.

6. An isomorphism between the algebras $\mathbb{O}(\alpha, \beta, \gamma)$ and some its applications

In the following, we will consider the generalized real octonion algebra $\mathbb{O}(\alpha, \beta, \gamma)$ and the algebras $\mathbb{O}(1,1,1)$ and $\mathbb{O}(1,1,-1)$. Let $\{1, f_1, \ldots, f_7\}$ be a basis in $\mathbb{O}(\alpha, \beta, \gamma)$, and $\{1, \widetilde{f}_1, \ldots, \widetilde{f}_7\}$ be the canonical basis in $\mathbb{O}(1,1,1)$, and $\{1, \widehat{f}_1, \ldots, \widehat{f}_7\}$ be the canonical basis in $\mathbb{O}(1,1,-1)$.

We prove that the algebra $\mathbb{O}(\alpha, \beta, \gamma)$ with $\alpha, \beta, \gamma \in \mathbb{R} \setminus \{0\}$ is isomorphic with algebra $\mathbb{O}(1,1,1)$ or $\mathbb{O}(1,1,-1)$ and indicate the formulae to pass from one basis to another basis. Thus, if $\alpha, \beta, \gamma > 0$ then the real octonion algebra $\mathbb{O}(\alpha, \beta, \gamma)$ is isomorphic with algebra $\mathbb{O}(1,1,1)$ and this isomorphism is given by the relations:

$$\begin{aligned} A_1: \quad & f_1 \mapsto \widetilde{f}_1 \sqrt{\alpha}, \quad f_2 \mapsto \widetilde{f}_2 \sqrt{\beta}, \quad f_3 \mapsto \widetilde{f}_3 \sqrt{\alpha\beta}, \\ & f_4 \mapsto \widetilde{f}_4 \sqrt{\gamma}, \quad f_5 \mapsto \widetilde{f}_5 \sqrt{\alpha\gamma}, \quad f_6 \mapsto \widetilde{f}_6 \sqrt{\beta\gamma}, \quad f_7 \mapsto \widetilde{f}_7 \sqrt{\alpha\beta\gamma}\,. \end{aligned}$$

If $\alpha, \beta > 0, \gamma < 0$, then the real octonion algebra $\mathbb{O}(\alpha, \beta, \gamma)$ is isomorphic with algebra $\mathbb{O}(1,1,-1)$ and this isomorphism is given by the relations:

$$\begin{aligned} A_2: \quad & f_1 \mapsto \widehat{f}_1 \sqrt{\alpha}, \quad f_2 \mapsto \widehat{f}_2 \sqrt{\beta}, \quad f_3 \mapsto \widehat{f}_3 \sqrt{\alpha\beta}, \\ & f_4 \mapsto \widehat{f}_4 \sqrt{-\gamma}, \quad f_5 \mapsto \widehat{f}_5 \sqrt{-\alpha\gamma}, \quad f_6 \mapsto \widehat{f}_6 \sqrt{-\beta\gamma}, \quad f_7 \mapsto \widehat{f}_7 \sqrt{-\alpha\beta\gamma}\,. \end{aligned}$$

If $\alpha, \gamma > 0, \beta < 0$, then the real octonion algebra $\mathbb{O}(\alpha, \beta, \gamma)$ is isomorphic with algebra $\mathbb{O}(1,1,-1)$ and this isomorphism is given by the relations:

$$\begin{aligned} A_3: \quad & f_1 \mapsto \widehat{f}_1 \sqrt{\alpha}, \quad f_2 \mapsto \widehat{f}_4 \sqrt{-\beta}, \quad f_3 \mapsto \widehat{f}_5 \sqrt{-\alpha\beta}, \\ & f_4 \mapsto \widehat{f}_2 \sqrt{\gamma}, \quad f_5 \mapsto \widehat{f}_3 \sqrt{\alpha\gamma}, \quad f_6 \mapsto \widehat{f}_6 \sqrt{-\beta\gamma}, \quad f_7 \mapsto \widehat{f}_7 \sqrt{-\alpha\beta\gamma}\,. \end{aligned}$$

If $\alpha > 0, \beta, \gamma < 0$, then the real octonion algebra $\mathbb{O}(\alpha, \beta, \gamma)$ is isomorphic with algebra $\mathbb{O}(1, 1, -1)$ and this isomorphism is given by the relations:

$$
\begin{aligned}
A_4: \quad & f_1 \mapsto \widehat{f_1}\sqrt{\alpha}, \quad f_2 \mapsto \widehat{f_4}\sqrt{-\beta}, \quad f_3 \mapsto \widehat{f_5}\sqrt{-\alpha\beta}, \\
& f_4 \mapsto \widehat{f_6}\sqrt{-\gamma}, \quad f_5 \mapsto \widehat{f_7}\sqrt{-\alpha\gamma}, \quad f_6 \mapsto \widehat{f_2}\sqrt{\beta\gamma}, \quad f_7 \mapsto \widehat{f_3}\sqrt{\alpha\beta\gamma}\,.
\end{aligned}
$$

If $\alpha < 0, \beta, \gamma > 0$, then the real octonion algebra $\mathbb{O}(\alpha, \beta, \gamma)$ is isomorphic with algebra $\mathbb{O}(1, 1, -1)$ and this isomorphism is given by the relations:

$$
\begin{aligned}
A_5: \quad & f_1 \mapsto \widehat{f_4}\sqrt{-\alpha}, \quad f_2 \mapsto \widehat{f_1}\sqrt{\beta}, \quad f_3 \mapsto \widehat{f_5}\sqrt{-\alpha\beta}, \\
& f_4 \mapsto \widehat{f_2}\sqrt{\gamma}, \quad f_5 \mapsto \widehat{f_6}\sqrt{-\alpha\gamma}, \quad f_6 \mapsto \widehat{f_3}\sqrt{\beta\gamma}, \quad f_7 \mapsto \widehat{f_7}\sqrt{-\alpha\beta\gamma}\,.
\end{aligned}
$$

If $\alpha, \gamma < 0, \beta > 0$, then real octonion algebra $\mathbb{O}(\alpha, \beta, \gamma)$ is isomorphic with algebra $\mathbb{O}(1, 1, -1)$ and this isomorphism is given by the relations:

$$
\begin{aligned}
A_6: \quad & f_1 \mapsto \widehat{f_4}\sqrt{-\alpha}, \quad f_2 \mapsto \widehat{f_1}\sqrt{\beta}, \quad f_3 \mapsto \widehat{f_5}\sqrt{-\alpha\beta}, \\
& f_4 \mapsto \widehat{f_6}\sqrt{-\gamma}, \quad f_5 \mapsto \widehat{f_2}\sqrt{\alpha\gamma}, \quad f_6 \mapsto \widehat{f_7}\sqrt{-\beta\gamma}, \quad f_7 \mapsto \widehat{f_3}\sqrt{\alpha\beta\gamma}\,.
\end{aligned}
$$

If $\alpha, \beta < 0, \gamma > 0$, then the real octonion algebra $\mathbb{O}(\alpha, \beta, \gamma)$ is isomorphic with algebra $\mathbb{O}(1, 1, -1)$ and this isomorphism is given by the relations:

$$
\begin{aligned}
A_7: \quad & f_1 \mapsto \widehat{f_4}\sqrt{-\alpha}, \quad f_2 \mapsto \widehat{f_5}\sqrt{-\beta}, \quad f_3 \mapsto \widehat{f_1}\sqrt{\alpha\beta}, \\
& f_4 \mapsto \widehat{f_2}\sqrt{\gamma}, \quad f_5 \mapsto \widehat{f_6}\sqrt{-\alpha\gamma}, \quad f_6 \mapsto \widehat{f_7}\sqrt{-\beta\gamma}, \quad f_7 \mapsto \widehat{f_3}\sqrt{\alpha\beta\gamma}\,.
\end{aligned}
$$

If $\alpha, \beta, \gamma < 0$, then the real octonion algebra $\mathbb{O}(\alpha, \beta, \gamma)$ is isomorphic with algebra $\mathbb{O}(1, 1, -1)$ and this isomorphism is given by the relations:

$$
\begin{aligned}
A_8: \quad & f_1 \mapsto \widehat{f_4}\sqrt{-\alpha}, \quad f_2 \mapsto \widehat{f_5}\sqrt{-\beta}, \quad f_3 \mapsto \widehat{f_1}\sqrt{\alpha\beta}, \\
& f_4 \mapsto \widehat{f_6}\sqrt{-\gamma}, \quad f_5 \mapsto \widehat{f_2}\sqrt{\alpha\gamma}, \quad f_6 \mapsto \widehat{f_3}\sqrt{\beta\gamma}, \quad f_7 \mapsto \widehat{f_7}\sqrt{-\alpha\beta\gamma}\,.
\end{aligned}
$$

It is easy to prove that the operators $A_k,\ k = \overline{1,8}$ is additive and multiplicative. The following statement can be proved completely analogous as Proposition 1.1.

Proposition 6.1. *The operators $A_k,\ k = \overline{1,8}$ are continuous and have norm* 1.□

Let $x = x_0 + \sum\limits_{k=1}^{7} x_k f_k \in \mathbb{O}(\alpha, \beta, \gamma)$ and let $g : \mathbb{O}(\alpha, \beta, \gamma) \to \mathbb{O}(\alpha, \beta, \gamma)$ be a continuous function of the form $g(x) = g_0(x_0, \ldots, x_7) + \sum\limits_{k=1}^{7} g_k(x_0, \ldots, x_7) f_k$. Let L be one of the operators $A_k,\ k = \overline{1,8}$, depending on the signs of α, β and γ. We define the operator $\mathfrak{L}$ by the rule:

$$\mathfrak{L}g := f_0 + \sum_{k=1}^{7} g_k\, L(f_k).$$

The operator $\mathfrak{L}$ for any continuous function g, taking values in $\mathbb{O}(\alpha,\beta,\gamma)$, maps it in the continuous function $\mathfrak{L}g$, taking values in $\mathbb{O}(1,1,1)$ or $\mathbb{O}(1,1,-1)$,

The following statement can be analogously proved as Theorem 3.1.

Theorem 6.2. *Let $x^0 \in \mathbb{O}(\alpha,\beta,\gamma)$, be a root of the equation $g(x)=0$ in $\mathbb{O}(\alpha,\beta,\gamma)$. Then $L(x^0)$ is a root of the equation $\mathfrak{L}g\big(L(x)\big)=0$ in $\mathbb{O}(1,1,1)$ or $\mathbb{O}(1,1,-1)$, depending on the signs of α,β, and γ. The converse is also true.* $\square$

Thus, the study of algebraic equations in an arbitrary algebra $\mathbb{O}(\alpha,\beta,\gamma)$ with $\alpha,\beta,\gamma \in \mathbb{R}\backslash\{0\}$ was reduced to study of the corresponding algebraic equation in one of the following two algebras: division octonion algebra $\mathbb{O}(1,1,1)$ or algebra $\mathbb{O}(1,1,-1)$.

Using the above notations, we can prove the following theorem.

Theorem 6.3. *Equation $x^n=a$, where $a \in \mathbb{O}(\alpha,\beta,\gamma)\setminus\mathbb{R}, \alpha,\beta,\gamma>0$, has n roots.*

Proof. The octonion $b=\frac{a}{\sqrt{N(a)}}$ is in $\mathcal{S}_G^3$, then there are $w \in \mathcal{S}_G^2$, $w = A_1^{-1}(\widetilde{w}), \widetilde{w} \in \mathcal{S}^2$ and $\lambda \in \mathbb{R}$ such that $b=\cos\lambda+\widetilde{w}\sin\lambda$. From Proposition 5.3, we have that the solutions of the above equation are $x_r = A_1^{-1}(\widetilde{x}_r) = \sqrt[2n]{N(a)}\left(\cos\frac{\lambda+2r\pi}{n}+\widetilde{w}\sin\frac{\lambda+2r\pi}{n}\right)$, where $r\in\{0,1,\ldots,n-1\}$ and $\widetilde{x}_r$ is a solution of the equation $\widetilde{x}^n=\widetilde{a}$ in $\mathbb{O}(1,1,1)$.$\square$

Remark 6.4. Using the operator A_1^{-1}, the rotation of the octonion $x \in \mathbb{O}(\alpha,\beta,\gamma)$ on the angle λ around the unit vector $w \in \mathcal{S}_G^2$ is defined by the formula

$$x^r=\overline{u}xu,$$

where $u \in \mathcal{S}_G^3, w \in \mathcal{S}_G^2,\ u=\cos\frac{\lambda}{2}+w\sin\frac{\lambda}{2}$ and $\overline{u}=\cos\frac{\lambda}{2}-w\sin\frac{\lambda}{2}$.

By straightforward calculations, it results that rotation does not transform the octonion-scalar part, but the octonion-vector part $\overrightarrow{x}$ is rotated on the angle λ around w.

Example 6.5. 1) Let $a \in \mathcal{S}^3$,
$a=\frac{\sqrt{2}}{2}+\frac{1}{\sqrt{14}}\widetilde{f}_1+\frac{1}{\sqrt{14}}\widetilde{f}_2+\frac{1}{\sqrt{14}}\widetilde{f}_3+\frac{1}{\sqrt{14}}\widetilde{f}_4+\frac{1}{\sqrt{14}}\widetilde{f}_5+\frac{1}{\sqrt{14}}\widetilde{f}_6+\frac{1}{\sqrt{14}}\widetilde{f}_7$, we have $\cos\lambda=\frac{\sqrt{2}}{2}, \sin\lambda=\frac{\sqrt{2}}{2}$. It results that $a=\cos\frac{\pi}{4}+v\sin\frac{\pi}{4}$, where $v=\frac{1}{\sqrt{7}}(\widetilde{f}_1+\widetilde{f}_2+\widetilde{f}_3+\widetilde{f}_4+\widetilde{f}_5+\widetilde{f}_6+\widetilde{f}_7)$. The vector a corresponds to the rotation of the space $\mathbb{R}^8$on the angle $\frac{\pi}{2}$ around the vector $v=\left(\frac{1}{\sqrt{7}},\frac{1}{\sqrt{7}},\ldots,\frac{1}{\sqrt{7}}\right)\in\mathbb{R}^7$ written in the canonical basis.

2) In the algebra $\mathbb{O}(2,4,7)$, for the above element $a \in \mathcal{S}^3$, we have
$b{=}A^{-1}(a){=}\frac{\sqrt{2}}{2}{+}\frac{1}{2\sqrt{7}}f_1{+}\frac{1}{2\sqrt{14}}f_2{+}\frac{1}{7\sqrt{2}}f_3{+}\frac{1}{4\sqrt{7}}f_4{+}\frac{1}{14}f_5{+}\frac{1}{14\sqrt{2}}f_6{+}\frac{1}{28}f_7 \in \mathcal{S}_G^3$ and corresponds to the rotation of the space $\mathbb{R}^8$on the angle $\frac{\pi}{2}$ around the vector $v=\left(\frac{1}{2\sqrt{7}},\frac{1}{2\sqrt{14}},\frac{1}{7\sqrt{2}},\frac{1}{4\sqrt{7}},\frac{1}{14},\frac{1}{14\sqrt{2}},\frac{1}{28}\right)\in\mathbb{R}^7$written in the basis $\{f_1,\ldots,f_7\}$.

Case when $\alpha=\beta=1,\gamma=-1$

In this case, the octonion algebra $\mathbb{O}(1,1,-1)$ is not a division algebra (is a split algebra). The norm of an octonion $a\in\mathbb{O}(1,1,-1)$, $a=a_0+a_1f_1+a_2f_2+a_3f_3+a_4f_4+a_5f_5+a_6f_6+a_7f_7$, in this situation, can be positive, zero or negative. In the following, we used definitions and propositions obtained for the split quaternions as in [16] to generalized them to similar results for the split octonions. A split octonion is called *spacelike, timelike or lightlike* if $N(a)<0, N(a)>0$ or $N(a)=0$. If $N(a)=1$, then a is called *the unit split octonion*.

Spacelike octonions

Let $a\in\mathbb{O}(1,1,-1)$ such that $N(a)=-1$, be a spacelike octonion. For the octonion $w=\frac{a_1f_1+a_2f_2+a_3f_3+a_4f_4+a_5f_5+a_6f_6+a_7f_7}{\sqrt{1+a_0^2}}$, we have $N(w)=-1$ and $t(w)=0$, therefore $w^2=1$. Denoting $\sinh\lambda=a_0$ and $\cosh\lambda=\sqrt{1+a_0^2},\lambda\in\mathbb{R}$, it results:

$$a=e^{\lambda w}=\sinh\lambda+w\cosh\lambda.$$

If $a\in\mathbb{O}(1,1,-1)$ with $N(a)<0$, we have $a=\sqrt{|N(a)|}(\sinh\lambda+w\cosh\lambda)$.

Proposition 6.6. *We have that* $a^n=(\sqrt{|N(a)|})^n(\sinh\lambda+w\cosh\lambda)$ *for* n *odd and* $a^n=(\sqrt{|N(a)|})^n(\cosh\lambda+w\sinh\lambda)$ *for* n *even.* $\Box$

Timelike octonions

Let $a\in\mathbb{O}(1,1,-1)$ such that $N(a)=1$, be a timelike octonion. If $1-a_0^2>0$, for the octonion $w=\frac{a_1f_1+a_2f_2+a_3f_3+a_4f_4+a_5f_5+a_6f_6+a_7f_7}{\sqrt{1-a_0^2}}$, we have $N(w)=1$ and $t(w)=0$, therefore $w^2=-1$ Denoting $\cos\lambda=a_0$ and $\sin\lambda=\sqrt{1-a_0^2},\lambda\in\mathbb{R}$, it results:

Proposition 6.7. *With the above notations, we have the Euler's formula:*

$$a=e^{\lambda w}=\cos\lambda+w\sin\lambda.$$

Proof. Indeed, $e^{\lambda w}=\sum\limits_{i=1}^{\infty}\frac{(\lambda w)^n}{n!}=\sum\limits_{i=1}^{\infty}\frac{(-1)^n\lambda^{2n}}{(2n)!}+w\sum\limits_{i=1}^{\infty}\frac{(-1)^{n-1}\lambda^{2n-1}}{(2n-1)!}=\cos\lambda+w\sin\lambda.\Box$

If $a\in\mathbb{O}(1,1,-1)$ with $N(a)>0$, it results $a=\sqrt{N(a)}\left(\cos\lambda+w\sin\lambda\right).$

Proposition 6.8. *We have that* $a^n=(\sqrt{N(a)})^n(\cos n\lambda+w\sin n\lambda).\Box$

Proposition 6.9.

1)If $a\in\mathbb{O}(1,1,-1)$, *it results* $a^n=(\sqrt{N(a)})^n(\cos n\lambda+w\sin n\lambda)$.

2) The equation $x^n=a$ *has* n *roots:* $\sqrt[2n]{|N(a)|}\left(\cosh\frac{\lambda}{n}+w\sinh\frac{\lambda}{n}\right).\Box$

If $1-a_0^2<0$, we have $w=\frac{a_1f_1+a_2f_2+a_3f_3+a_4f_4+a_5f_5+a_6f_6+a_7f_7}{\sqrt{a_0^2-1}}$, with $N(w)=-1$ and $t(w)=0$, therefore $w^2=1$. Denoting $\cosh\lambda=a_0$ and $\sinh\lambda=\sqrt{a_0^2-1},\lambda\in\mathbb{R}$, it results:

Proposition 6.10. *With the above notations, we have Euler's formula:*

$$a = e^{\lambda w} = \cosh \lambda + w \sinh \lambda.$$

Proof. Indeed, $e^{\lambda w} = \sum\limits_{n=0}^{\infty} \frac{(\lambda w)^n}{n!} = \sum\limits_{n=0}^{\infty} \frac{\lambda^{2n}}{(2n)!} + w \sum\limits_{n=0}^{\infty} \frac{\lambda^{2n+1}}{(2n+1)!} = \cosh \lambda + w \sinh \lambda. \square$

If $a \in a \in \mathbb{O}(1,1,-1)$ with $N(a) < 0$, it results $a = \sqrt{|N(a)|}(\cosh \lambda + w \sinh \lambda)$.

Proposition 6.11.

1) If $a \in \mathbb{O}(1,1,-1)$, *then* $a^n = (\sqrt{|N(a)|})^n (\cosh n\lambda + w \sinh n\lambda)$.

2) The equation $x^n = a$ *has only one root:* $\sqrt[2n]{|N(a)|} \left(\cosh \frac{\lambda}{n} + w \sinh \frac{\lambda}{n}\right). \square$

Remark 6. 12. Using the above technique, De Moivre's formula and Euler's formula can be easy proved for the octonion algebra $\mathbb{O}(\alpha, \beta, \gamma)$, with $\alpha, \beta, \gamma \in \mathbb{R} \setminus \{0\}$ such that $\mathbb{O}(\alpha, \beta, \gamma)$ is split. Thus, the study of algebraic equations in an arbitrary algebra $\mathbb{O}(\alpha, \beta, \gamma)$ with $\alpha, \beta, \gamma \in \mathbb{R} \setminus \{0\}$ was reduced to study of the corresponding algebraic equation in one of the following two algebras: division octonion algebra $\mathbb{O}(1,1,1)$ or algebra $\mathbb{O}(1,1,-1)$.

Conclusion. In this paper, we used isomorphism between the real quaternion algebras $\mathbb{H}(\gamma_1, \gamma_2)$ and $\mathbb{H}(1,1)$ or $\mathbb{H}(1,-1)$ to reduced the study of some algebraic equations in an arbitrary algebra $\mathbb{H}(\gamma_1, \gamma_2)$ with $\gamma_1, \gamma_2 \in \mathbb{R} \setminus \{0\}$ to study of the corresponding algebraic equation in one of the following two algebras: division quaternion algebra or coquaternion algebra. The same result was obtained for the generalized octonion algebra $\mathbb{O}(\alpha, \beta, \gamma)$. De Moivre's formula in generalized quaternion algebras and generalized octonion division algebras was proved using this new method.

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