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INTEGRAL THEOREMS AND A CAUCHY FORMULA IN A COMMUTATIVE THREE-DIMENSIONAL HARMONIC ALGEBRA

Summary

For monogenic functions taking values in a three-dimensional commutative harmonic algebra with the unit and two-dimensional radical, we have proved analogs of classical integral theorems of the theory of analytic functions of the complex variable: the Cauchy integral theorems for surface integral and curvilinear integral, the Morera theorem and the Cauchy integral formula.

1. Introduction

Let $\mathbb{A}_3$ be a three-dimensional commutative associative Banach algebra with the unit 1 over the field of complex numbers $\mathbb{C}$. Let $\{1, \rho_1, \rho_2\}$ be a basis of the algebra $\mathbb{A}_3$ with the multiplication table $\rho_1\rho_2 = \rho_2^2 = 0$, $\rho_1^2 = \rho_2$.

The algebra $\mathbb{A}_3$ is *harmonic* (see [1,2]) because there exist *harmonic* bases $\{e_1 = 1, e_2, e_3\}$ in $\mathbb{A}_3$ satisfying the following condition

(1) $$e_1^2 + e_2^2 + e_3^2 = 0.$$

Consider the linear envelope $E_3 := \{\zeta = x + ye_2 + ze_3 : x, y, z \in \mathbb{R}\}$ generated by the vectors $1, e_2, e_3$ over the field of real numbers $\mathbb{R}$. For a set $S \subset \mathbb{R}^3$ consider the set $S_\zeta := \{\zeta = x + ye_2 + ze_3 : (x, y, z) \in S\} \subset E_3$ congruent to S. In what follows, $\zeta = x + ye_2 + ze_3$ and $x, y, z \in \mathbb{R}$.

A continuous function $\Phi : \Omega_\zeta \to \mathbb{A}_3$ is *monogenic* in a domain $\Omega_\zeta \subset E_3$ if Φ is differentiable in the sense of Gateaux in every point of Ω_ζ, i. e. if for every $\zeta \in \Omega_\zeta$ there exists an element $\Phi'(\zeta) \in \mathbb{A}_3$ such that

$$\lim_{\varepsilon \to 0+0} \left(\Phi(\zeta + \varepsilon h) - \Phi(\zeta)\right) \varepsilon^{-1} = h\Phi'(\zeta) \quad \forall h \in E_3.$$

It follows from the equality (1) and the equality

$$\Delta_3\Phi := \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = \Phi''(\zeta)(e_1^2 + e_2^2 + e_3^2)$$

that every twice monogenic function $\Phi : \Omega_\zeta \to \mathbb{A}_3$ satisfies the three-dimensional Laplace equation $\Delta_3\Phi = 0$.

In the paper [3] for functions differentiable in the sense of Lorch in an arbitrary convex domain of commutative associative Banach algebra, some properties similar to properties of holomorphic functions of complex variable (in particular, the integral Cauchy theorem and the integral Cauchy formula, the Taylor expansion and the Morera theorem) are established. In the paper [4] the convexity of the domain is withdrawn in the mentioned results from [3].

In this paper we establish similar results for monogenic functions $\Phi : \Omega_\zeta \to \mathbb{A}_3$ given only in a domain Ω_ζ of the linear envelope E_3 instead of domain of the whole algebra $\mathbb{A}_3$. Let us note that *a priori* the differentiability of the function Φ in the sense of Gâteaux is a restriction weaker than the differentiability of this function in the sense of Lorch. Moreover, note that the integral Cauchy formula established in the papers [3, 4] is not applicable for a monogenic function $\Phi : \Omega_\zeta \to \mathbb{A}_3$ because it deals with an integration along a curve on which the function Φ is not given, generally speaking.

Note that as well as in [3,4], some hypercomplex analogues of the integral Cauchy theorem for a curvilinear integral are established in the papers [5, 6]. In the papers [5, 7–9] similar theorems are established for surface integral.

2. Cauchy integral theorem for a surface integral

A function $\Phi(\zeta)$ of the variable $\zeta \in \Omega_\zeta$ is monogenic if and only if the following Cauchy-Riemann conditions are satisfied (see Theorem 1.3 [2]):

$$\frac{\partial\Phi}{\partial y} = \frac{\partial\Phi}{\partial x}\, e_2, \qquad \frac{\partial\Phi}{\partial z} = \frac{\partial\Phi}{\partial x}\, e_3. \tag{2}$$

Along with monogenic functions, consider a function $\Psi : \Omega_\zeta \to \mathbb{A}_3$ having continuous partial derivatives of the first order in a domain Ω_ζ and satisfying the equation

$$\frac{\partial\Psi}{\partial x} + \frac{\partial\Psi}{\partial y}\, e_2 + \frac{\partial\Psi}{\partial z}\, e_3 = 0 \tag{3}$$

at every point of this domain.

In the scientific literature the different denominations are used for functions satisfying equations of the form (3). For example, in [5, 6, 10] – regular functions, and in the papers [7, 8, 11] they are called monogenic functions. As well as in the papers [9, 12, 13], we call a function *hyperholomorphic* if it satisfies the equation (3).

It is well known that in the quaternionic analysis the classes of functions determined by means conditions of the form (2) and (3) do not coincide (see [5, 14]).

Note that in the algebra $\mathbb{A}_3$ the set of monogenic functions is a subset of the set of hyperholomorphic functions because every monogenic function $\Phi : \Omega_\zeta \to \mathbb{A}_3$ satisfies the equality (3) owing to the conditions (1), (2). Yet, there exist hyperholomorphic functions which are not monogenic. For example, the function

$$\Psi(x + ye_2 + ze_3) = ze_2 - ye_3$$

satisfies the condition (3), but it does not satisfy the equalities of the form (2).

Let Ω be a bounded closed set in $\mathbb{R}^3$. For a continuous function $\Psi : \Omega_\zeta \to \mathbb{A}_3$ of the form

$$\Psi(x + ye_2 + ze_3) = \sum_{k=1}^{3} U_k(x, y, z)e_k + i\sum_{k=1}^{3} V_k(x, y, z)e_k, \tag{4}$$

where $(x, y, z) \in \Omega$, we define a volume integral by the equality

$$\int_{\Omega_\zeta} \Psi(\zeta)dxdydz := \sum_{k=1}^{3} e_k \int_{\Omega} U_k(x, y, z)dxdydz + i\sum_{k=1}^{3} e_k \int_{\Omega} V_k(x, y, z)dxdydz.$$

Let Σ be a quadrable surface in $\mathbb{R}^3$ with quadrable projections on the coordinate planes. For a continuous function $\Psi : \Sigma_\zeta \to \mathbb{A}_3$ of the form (4), where $(x, y, z) \in \Sigma$, we define a surface integral on Σ_ζ with the differential form

$$\sigma_{\alpha_1,\alpha_2,\alpha_3} := \alpha_1 dydz + \alpha_2 dzdxe_2 + \alpha_3 dxdye_3, \quad \text{where} \quad \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R},$$

by the equality

$$\begin{aligned}\int_{\Sigma_\zeta} \Psi(\zeta)\sigma_{\alpha_1,\alpha_2,\alpha_3} &:= \sum_{k=1}^{3} e_k \int_{\Sigma} \alpha_1 U_k(x, y, z)dydz + \sum_{k=1}^{3} e_2e_k \int_{\Sigma} \alpha_2 U_k(x, y, z)dzdx \\ &+ \sum_{k=1}^{3} e_3e_k \int_{\Sigma} \alpha_3 U_k(x, y, z)dxdy + i\sum_{k=1}^{3} e_k \int_{\Sigma} \alpha_1 V_k(x, y, z)dydz \\ &+ i\sum_{k=1}^{3} e_2e_k \int_{\Sigma} \alpha_2 V_k(x, y, z)dzdx + i\sum_{k=1}^{3} e_3e_k \int_{\Sigma} \alpha_3 V_k(x, y, z)dxdy.\end{aligned}$$

A connected homeomorphic image of a square in $\mathbb{R}^3$ is called *simple surface.* A surface is *locally-simple* if it is simple in a certain neighbourhood of every point.

If a simply connected domain $\Omega \subset \mathbb{R}^3$ has a closed locally-simple piecewise-smooth boundary $\partial\Omega$ and a function $\Psi : \Omega_\zeta \to \mathbb{A}_3$ is continuous together with partial derivatives of the first order up to the boundary $\partial\Omega_\zeta$, then the following analogue of the Gauss-Ostrogradski formula is true:

$$\int_{\partial\Omega_\zeta} \Psi(\zeta)\sigma = \int_{\Omega_\zeta} \left(\frac{\partial\Psi}{\partial x} + \frac{\partial\Psi}{\partial y}e_2 + \frac{\partial\Psi}{\partial z}e_3\right) dxdydz, \tag{5}$$

where $\sigma := \sigma_{1,1,1} \equiv dydz + dzdxe_2 + dxdye_3$. Now, the next theorem is a result of the formula (5) and the equality (3).

Theorem 1. *Suppose that Ω is a simply connected domain with a closed locally-simple piecewise-smooth boundary $\partial\Omega$. Suppose also that the function $\Psi : \overline{\Omega_\zeta} \to \mathbb{A}_3$ is continuous in the closure $\overline{\Omega_\zeta}$ of the domain Ω_ζ and hyperholomorphic in Ω_ζ. Then*

$$\int\limits_{\partial\Omega_\zeta} \Psi(\zeta)\sigma = 0.$$

3. Cauchy integral theorem for a curvilinear integral

Let γ be a Jordan rectifiable curve in $\mathbb{R}^3$. For a continuous function $\Psi : \gamma_\zeta \to \mathbb{A}_3$ of the form (4), where $(x, y, z) \in \gamma$, we define an integral along the curve γ_ζ by the equality

$$\int\limits_{\gamma_\zeta} \Psi(\zeta)d\zeta := \sum_{k=1}^{3} e_k \int\limits_{\gamma} U_k(x,y,z)dx + \sum_{k=1}^{3} e_2 e_k \int\limits_{\gamma} U_k(x,y,z)dy$$
$$+ \sum_{k=1}^{3} e_3 e_k \int\limits_{\gamma} U_k(x,y,z)dz + i\sum_{k=1}^{3} e_k \int\limits_{\gamma} V_k(x,y,z)dx$$
$$+ i\sum_{k=1}^{3} e_2 e_k \int\limits_{\gamma} V_k(x,y,z)dy + i\sum_{k=1}^{3} e_3 e_k \int\limits_{\gamma} V_k(x,y,z)dz,$$

where

$$d\zeta := dx + e_2 dy + e_3 dz.$$

If a function $\Phi : \Omega_\zeta \to \mathbb{A}_3$ is continuous together with partial derivatives of the first order in a domain Ω_ζ, Σ is a piecewise-smooth surface in Ω, and the edge γ of the surface Σ is a rectifiable Jordan curve, then the following analogue of the Stokes formula is true:

$$\int\limits_{\gamma_\zeta} \Phi(\zeta)d\zeta = \int\limits_{\Sigma_\zeta} \left(\frac{\partial\Phi}{\partial x}e_2 - \frac{\partial\Phi}{\partial y}\right) dxdy + \left(\frac{\partial\Phi}{\partial y}e_3 - \frac{\partial\Phi}{\partial z}e_2\right) dydz$$

$$+ \left(\frac{\partial\Phi}{\partial z} - \frac{\partial\Phi}{\partial x}e_3\right) dzdx. \tag{6}$$

Now, the next theorem is a result of the formula (6) and the equalities (2).

Theorem 2. *Suppose that $\Phi : \Omega_\zeta \to \mathbb{A}_3$ is a monogenic function in a domain Ω_ζ, Σ is a piecewise-smooth surface in Ω, and the edge γ of the surface Σ is a rectifiable Jordan curve. Then*

$$\int\limits_{\gamma_\zeta} \Phi(\zeta)d\zeta = 0. \tag{7}$$

Now, similarly to the proof of Theorem 3.2 [4] we can prove the following

Theorem 3. *Let $\Phi : \Omega_\zeta \to \mathbb{A}_3$ be a monogenic function in a domain Ω_ζ. Then for every closed Jordan rectifiable curve γ homotopic to a point in Ω, the equality* (7) *holds.*

For functions taking values in the algebra $\mathbb{A}_3$, the following Morera theorem can be established in the usual way.

Theorem 4. *If a function $\Phi : \Omega_\zeta \to \mathbb{A}_3$ is continuous in a domain Ω_ζ and satisfies the equality*

$$\int_{\partial\triangle_\zeta} \Phi(\zeta)d\zeta = 0 \tag{8}$$

for every triangle $\triangle_\zeta$ such that $\overline{\triangle_\zeta} \subset \Omega_\zeta$, then the function Φ is monogenic in the domain Ω_ζ.

4. Cauchy integral formula

In what follows, we consider a harmonic basis $\{e_1, e_2, e_3\}$ with the following decomposition with respect to the basis $\{1, \rho_1, \rho_2\}$:

$$e_1 = 1, \qquad e_2 = i + \frac{1}{2}\,i\rho_2, \qquad e_3 = -\rho_1 - \frac{\sqrt{3}}{2}\,i\rho_2.$$

It follows from Lemma 1.1 [2] that

$$\zeta^{-1} = \frac{1}{x+iy} + \frac{z}{(x+iy)^2}\,\rho_1 + \left(\frac{i}{2}\,\frac{\sqrt{3}z - y}{(x+iy)^2} + \frac{z^2}{(x+iy)^3}\right)\rho_2 \tag{9}$$

for all $\zeta = x + ye_2 + ze_3 \in E_3 \setminus \{ze_3 : z \in \mathbb{R}\}$. Thus, it is obvious that the straight line $\{ze_3 : z \in \mathbb{R}\}$ is contained in the radical of the algebra $\mathbb{A}_3$.

Using the equality (9), it is easy to calculate that

$$\int_{\tilde{\gamma}_\zeta} \tau^{-1} d\tau = 2\pi i, \tag{10}$$

where $\tilde{\gamma}_\zeta := \{\tau = x + ye_2 : x^2 + y^2 = R^2\}$.

Theorem 5. *Let Ω be a domain convex in the direction of the axis Oz and $\Phi : \Omega_\zeta \to \mathbb{A}_3$ be a monogenic function in the domain Ω_ζ. Then for every point $\zeta_0 \in \Omega_\zeta$ the following equality is true:*

$$\Phi(\zeta_0) = \frac{1}{2\pi i}\int_{\gamma_\zeta} \Phi(\zeta)\,(\zeta - \zeta_0)^{-1}\,d\zeta, \tag{11}$$

where γ_ζ is an arbitrary closed Jordan rectifiable curve in Ω_ζ, which surrounds once the straight line $\{\zeta_0 + ze_3 : z \in \mathbb{R}\}$.

Proof. We represent the integral on the right-hand side of the equality (11) as the sum of the following two integrals:

$$\int_{\gamma_\zeta} \Phi(\zeta)\,(\zeta-\zeta_0)^{-1}\,d\zeta = \int_{\gamma_\zeta} (\Phi(\zeta)-\Phi(\zeta_0))\,(\zeta-\zeta_0)^{-1}\,d\zeta +$$

$$+\Phi(\zeta_0)\int_{\gamma_\zeta} (\zeta-\zeta_0)^{-1}\,d\zeta =: I_1 + I_2.$$

Inasmuch as the domain Ω is convex in the direction of the axis Oz and the curve γ_ζ surrounds once the straight line $\{\zeta_0 + ze_3 : z \in \mathbb{R}\}$, γ is homotopic to the circle

$$K(R) := \{(x-x_0)^2 + (y-y_0)^2 = R^2, z = z_0\}, \text{ where } \zeta_0 = x_0 + y_0e_2 + z_0e_3.$$

Then using the equality (10), for $\tau = \zeta - \zeta_0$, we have $I_2 = 2\pi i\Phi(\zeta_0)$.

Let us prove that $I_1 = 0$. First, we choose on the curve γ two points A and B in which there are tangents to γ, and we choose also two points A_1, B_1 on the circle $K(\varepsilon)$ which is completely contained in the domain Ω. Let γ^1, γ^2 be connected components of the set $\gamma \setminus \{A, B\}$. By K^1 and K^2 we denote connected components of the set $K(\varepsilon) \setminus \{A_1, B_1\}$ in such a way that after a choice of smooth arcs Γ^1, Γ^2 each of the closed curves $\gamma^1 \cup \Gamma^2 \cup K^1 \cup \Gamma^1$ and $\gamma^2 \cup \Gamma^1 \cup K^2 \cup \Gamma^2$ will be homotopic to a point of the domain $\Omega \setminus \{(x_0, y_0, z) : z \in \mathbb{R}\}$.

Then it follows from Theorem 3 that

$$\int_{\gamma^1_\zeta \cup \Gamma^2_\zeta \cup K^1_\zeta \cup \Gamma^1_\zeta} (\Phi(\zeta)-\Phi(\zeta_0))\,(\zeta-\zeta_0)^{-1}\,d\zeta = 0, \tag{12}$$

$$\int_{\gamma^2_\zeta \cup \Gamma^1_\zeta \cup K^2_\zeta \cup \Gamma^2_\zeta} (\Phi(\zeta)-\Phi(\zeta_0))\,(\zeta-\zeta_0)^{-1}\,d\zeta = 0. \tag{13}$$

Inasmuch as each of the curves Γ^1_ζ, Γ^2_ζ has different orientations in the equalities (12), (13), after addition of the mentioned equalities we obtain

$$\int_{\gamma_\zeta} (\Phi(\zeta)-\Phi(\zeta_0))\,(\zeta-\zeta_0)^{-1}\,d\zeta = \int_{K_\zeta(\varepsilon)} (\Phi(\zeta)-\Phi(\zeta_0))\,(\zeta-\zeta_0)^{-1}\,d\zeta, \tag{14}$$

where the curves $K_\zeta(\varepsilon)$ and γ_ζ have the same orientation.

The integrand in the right-hand side of the equality (14) is bounded by a constant which does not depend on ε. Therefore, passing to the limit in the equality (14) as $\varepsilon \to 0$, we obtain $I_1 = 0$ and the theorem is proved.

Using the formula (11), we obtain the Taylor expansion of monogenic function in the usual way. Thus, as in the complex plane, one can give different equivalent definitions of a monogenic function $\Phi : \Omega_\zeta \to \mathbb{A}_3$, i. e. the following theorem is true:

Theorem 6. *Let Ω be a domain convex in the direction of the axis Oz. Then a function $\Phi : \Omega_\zeta \to \mathbb{A}_3$ is monogenic in the domain Ω_ζ if and only if one of the following conditions is satisfied:*

(I) *the components* $U_k : \Omega \to \mathbb{C}$, $k = \overline{1,3}$, *of the decomposition*

$$\Phi(\zeta) = \sum_{k=1}^{3} U_k(x,y,z)\, e_k\,,$$

of the function Φ *are differentiable with respect to the variables* x, y, z *in* Ω *and the conditions* (2) *are satisfied in the domain* Ω_ζ;

(II) *the function* Φ *is continuous in* Ω_ζ *and satisfies the equality* (8) *for every triangle* $\triangle_\zeta$ *such that* $\overline{\triangle_\zeta} \subset \Omega_\zeta$;

(III) *for every* $\zeta_0 \in \Omega_\zeta$ *there exists a neighbourhood in which the function* Φ *is expressed as the sum of the power series*

$$\Phi(\zeta) = \sum_{k=0}^{\infty} c_k\, (\zeta - \zeta_0)^k, \qquad c_k \in \mathbb{A}_3.$$

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TWIERDZENIA CAŁKOWE I WZÓR CAUCHY'EGO W PRZEMIENNEJ TRÓJWYMIAROWEJ ALGEBRZE HARMONICZNEJ

S t r e s z c z e n i e

Dla funkcji jednoznacznie odwracalnych o wartościach w trójwymiarowej przemiennej algebrze harmonicznej z jedynką i dwuwymiarowym pierwiastkiem, dowodzimy odpowiedników klasycznych twierdzeń całkowych teorii funkcji analitycznych jednej zmiennej zespolonej: twierdzeń całkowych Cauchy'ego dla całki powierzchniowej i krzywoliniowej twierdzenia Morery i wzoru całkowego Cauchy'ego.