PL ISSN 0459-6854

BULLETIN

DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ

2009 Vol. LIX

Recherches sur les déformations Vol. LVIII

pp. 45–58

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SOLUTION OF GENERAL QUADRATIC QUATERNIONIC EQUATIONS

Summary

We investigate a certain wide class of quadratic quaternionic equations. In article [3] the equations of form $\sum_{\ell=0}^{n} a_\ell x^\ell = 0$ are investigated and there it is proved that roots can be isolated points or constitute spheres. In our article we show, that the roots of the general quadratic quaternionic equations can constitute not only the points and spheres but also: a circle, a hyperbola, a cylindrical surface, two planes (which do not coincide with coordinate planes), and other surfaces. We describe the examples of equations, whose roots form the above mentioned structures.

1. Introduction

In paper [4] solutions of the general linear equations are specified, that is of ones of the form

$$\sum_{\ell=1}^{n} a_\ell x b_\ell = c. \tag{1}$$

The set of the solutions of such equation can be empty set, a straight line, a plane, or a three-dimensional space. As it was already mentioned, there [3] any equation of the form is investigated

$$\sum_{\ell=0}^{n} a_\ell x^\ell = 0 \quad \text{or} \quad \sum_{\ell=0}^{n} x^\ell a_\ell = 0. \tag{2}$$

The set of the roots of (2) consists of isolated points and (or) spheres, and the centers of the spheres lie on the real axis.

Because of non-commutativity of quaternions, equation of the form (2) is not a general equation of the degree n. The general form of the quaternionic polynomial equation is

$$\sum_{p=1}^{n}\left(\sum_{\ell=1}^{m_p} a_{p,\ell,1}xa_{p,\ell,2}x\dots a_{p,\ell,p}xa_{p,\ell,p+1}\right)+c=0.$$

In [2] it is solved any equation of the following form

$$x^2+xa+bx+c=0.$$

Roots of the last equation can be isolated points or constitute a sphere with the center in any point.

Moreover in paper [1] any equation of a kind

$$x^2+\sum_{\ell=1}^{n} a_\ell x a_\ell + c = 0 \tag{3}$$

is reduced to a system of four non-linear equations with four unknowns.

In this article we are interested in the structure of the solutions set of the quadratic quaternionic equation (4). With the help of theorem 1, we shall reduce this equation to system (9) of four equations of the second degree. This system will be investigated more carefully for the incomplete quaternionic equations of the forms (23) and (34). For these equations simple but important propositions are proved.

Moreover in this paper some equations are solved and structures of the set of their roots are found. As it was specified above, in the articles [2]– [4] the equations of more simple type are solved, and the set of their roots consisted of points, spheres, and for the general linear equation (see [4]) the set is a straight line, a plane, or a three-dimensional space. But given in this article examples show that the set of the solutions of a quadratic quaternionic equation can be other, a hyperbola, two crossed straight lines, a cylindrical surface, and other surface.

2. Notations of the paper

For a quaternion we use the standard notations:

$$x=x_0+x_1i+x_2j+x_3k,\quad x_0,x_1,x_2,x_3\in\mathbb{R},$$

where for the imaginary units i,j,k, the following equalities are true:

$$i^2=j^2=k^2=-1, ij=-ji=k, jk=-kj=i, ki=-ik=j.$$

Note also that we deal with only real quaternions, i. e., their components are real; we use the word "quaternion" only for real one and denote the system of all real quaternions by $\mathbb{H}$. The number $\mathrm{Sc}(x) := x_0$ is referred to as the scalar part of the quaternion x; $x_1i+x_2j+x_3k$ is called the vector part of x and is denoted by $\mathrm{Vec}(x)$ or $\vec{x}$;

$$\sqrt{x_0^2+x_1^2+x_2^2+x_3^2}$$

is called the modulus of x and is denoted by $|x|$; we shall denote by $\vec{\mathbb{H}}$ the set $\{x\in\mathbb{H}:\mathrm{Sc}(x)=0\}$;

$$x_0 - x_1 i - x_2 j - x_3 k$$

is called the conjugate number with respect to x and is denoted by $\bar{x}$. Let

$$y = y_0 + y_1 i + y_2 j + y_3 k,\ y_0, y_1, y_2, y_3 \in \mathbb{R},$$

then the real number

$$(x, y) := x_0 y_0 + x_1 y_1 + x_2 y_2 + x_3 y_3$$

is referred to as the scalar product of x and y. According to the introduced definitions

$$|\vec{x}| = \sqrt{x_1^2 + x_2^2 + x_3^2}, \quad \overrightarrow{(\vec{x})} = \vec{x}.$$

We shall agree that top index in brackets denote corresponding coordinates of coefficients of equations, that is

$$a = a^{(0)} + a^{(1)} i + a^{(2)} j + a^{(3)} k,$$

where

$$a^{(0)}, a^{(1)}, a^{(2)}, a^{(3)} \in \mathbb{R}$$

(though for unknowns we use subindices).

3. The basic results

In this article we consider quadratic quaternionic equations of the form:

$$\sum_{p=1}^{n} a_p x^2 b_p + \sum_{m=1}^{r} c_m x d_m x + \sum_{t=1}^{\ell} f_t x g_t = h, \tag{4}$$

where $\{a_p; b_p; c_m; d_m; f_t; g_t; h; x\} \subset \mathbb{H}$.

Lemma 1. *For any $a, b \in \mathbb{H}$ the following equality is true:*

$$\vec{a}\vec{b} + \vec{b}\vec{a} = -2(\vec{a}, \vec{b}). \tag{5}$$

Proof. Using rules of multiplication of quaternions we easily conclude:

$$\vec{a}\vec{b} = -(\vec{a}, \vec{b}) + A,$$
$$\vec{b}\vec{a} = -(\vec{a}, \vec{b}) - A,$$

where $A \in \vec{\mathbb{H}}$. Then having combined last two equality, we get the formula (5).

Lemma 2. *For any $a, b \in \mathbb{H}$ the following equality is true:*

$$ab = ba - 2\vec{b}\vec{a} - 2(\vec{a}, \vec{b}). \tag{6}$$

Proof. We shall consider products:

(7) $$ab = (a^{(0)} + \vec{a})(b^{(0)} + \vec{b}) = a^{(0)}b^{(0)} + a^{(0)}\vec{b} + \vec{a}b^{(0)} + \vec{a}\vec{b}$$

and

(8) $$ba = (b^{(0)} + \vec{b})(a^{(0)} + \vec{a}) = b^{(0)}a^{(0)} + b^{(0)}\vec{a} + \vec{b}a^{(0)} + \vec{b}\vec{a}.$$

Having combined equality (7), (8), in view of the formula (5), we get the equality (6).

Theorem 1. *Any equation of the form* (4) *can be rewritten in the form:*

(9) $$A_1x_0^2 + A_2|\vec{x}|^2 + A_3x_0x_1 + A_4x_0x_2 + A_5x_0x_3 - 2\sum_{m=1}^{r} c_m\vec{x}(\vec{x}, \vec{d}_m) + \\ + A_6x_0 + A_7x_1 + A_8x_2 + A_9x_3 = h,$$

where

(10) $$\begin{aligned}
&A_1 := \sum_{p=1}^{n} a_pb_p + \sum_{m=1}^{r} c_md_m,\ A_2 := +2\sum_{m=1}^{r} c_m\vec{d}_m - \sum_{m=1}^{r} c_md_m - \sum_{p=1}^{n} a_pb_p,\\
&B_0 := 2\sum_{p=1}^{n} a_pb_p + 2\sum_{m=1}^{r} c_md_m - 4\sum_{p=1}^{n} a_p\vec{b}_p - 2\sum_{m=1}^{r} c_m\vec{d}_m,\\
&B_{q+2} := -4\sum_{p=1}^{n} a_pb_p^{(q)} - 2\sum_{m=1}^{r} c_md_m^{(q)},\quad q = 1, 2, 3,\\
&A_3 := B_3 + B_0i, A_4 := B_4 + B_0j, A_5 := B_5 + B_0k,\\
&A_6 := \sum_{t=1}^{\ell} f_tg_t,\ B_6 := \sum_{t=1}^{\ell} f_tg_t - 2\sum_{t=1}^{\ell} f_t\vec{g}_t,\ B_{u+6} := -2\sum_{t=1}^{\ell} f_tg_t^{(u)},\\
&u = 1, 2, 3,\quad A_7 := B_6i + B_7, A_8 := B_6j + B_8, A_9 := B_6k + B_9.
\end{aligned}$$

Proof. We shall transform every addend of the equation (4) , using equality (6). So for the first addend, using the formula $x^2 = 2x_0x - |x|^2$, we have the following:

$$\sum_{p=1}^{n} a_px^2b_p = \sum_{p=1}^{n} a_p\left(2x_0x - |x|^2\right)b_p = 2x_0\sum_{p=1}^{n} a_pxb_p - |x|^2\sum_{p=1}^{n} a_pb_p =$$

$$= 2x_0\sum_{p=1}^{n} a_p\left(b_px - 2\vec{b}_p\vec{x} - 2(\vec{b}_p, \vec{x})\right) - |x|^2A =$$

$$= 2Ax_0x - 4Bx_0\vec{x} - 4x_0\sum_{p=1}^{n} a_p(\vec{b}_p, \vec{x}) - x_0^2A - |\vec{x}|^2\mathbf{A}.$$

Hence

(11) $$\sum_{p=1}^{n} a_px^2b_p = Ax_0^2 + (2A - 4B)\,x_0\vec{x} - A|\vec{x}|^2 - 4x_0\sum_{p=1}^{n} a_p(\vec{b}_p, \vec{x}),$$

where

$$A := \sum_{p=1}^{n} a_p b_p, \; B := \sum_{p=1}^{n} a_p \vec{b}_p. \tag{12}$$

We shall consider the second addend of equality (4):

$$\sum_{m=1}^{r} c_m x d_m x = \sum_{m=1}^{r} c_m x \left(x d_m - 2\vec{x}\vec{d}_m - 2(\vec{x}, \vec{d}_m) \right)$$

or

$$\sum_{m=1}^{r} c_m x d_m x = \sum_{m=1}^{r} c_m x^2 d_m - 2\sum_{m=1}^{r} c_m x\vec{x}\vec{d}_m - 2\sum_{m=1}^{r} c_m x(\vec{x}, \vec{d}_m). \tag{13}$$

We can write down he first addend of equality (13) similarly to expression (11):

$$\sum_{m=1}^{r} c_m x^2 d_m = D x_0^2 + (2D - 4C)\, x_0 \vec{x} - D|\vec{x}|^2 - 4x_0 \sum_{m=1}^{r} c_m (\vec{d}_m, \vec{x}), \tag{14}$$

where

$$D := \sum_{m=1}^{r} c_m d_m, \; C := \sum_{m=1}^{r} c_m \vec{d}_m. \tag{15}$$

For the second addend of expression (13), we have the following:

$$-2\sum_{m=1}^{r} c_m x\vec{x}\vec{d}_m = -2\sum_{m=1}^{r} c_m (x_0 + \vec{x})\vec{x}\vec{d}_m = -2x_0 \sum_{m=1}^{r} c_m \vec{x}\vec{d}_m + 2|\vec{x}|^2 C =$$

$$= -2x_0 \sum_{m=1}^{r} c_m \left(\vec{d}_m \vec{x} - 2\vec{d}_m \vec{x} - 2(\vec{d}_m, \vec{x}) \right) + 2|\vec{x}|^2 C.$$

Hence

$$-2\sum_{m=1}^{r} c_m x\vec{x}\vec{d}_m = 2x_0 C\vec{x} + 4x_0 \sum_{m=1}^{r} c_m (\vec{d}_m, \vec{x}) + 2C|\vec{x}|^2. \tag{16}$$

And, for the last addend of equality (13), we have:

$$-2\sum_{m=1}^{r} c_m x(\vec{x}, \vec{d}_m) = -2x_0 \sum_{m=1}^{r} c_m (\vec{x}, \vec{d}_m) - 2\sum_{m=1}^{r} c_m \vec{x}(\vec{x}, \vec{d}_m). \tag{17}$$

Then in view of equalities (14), (16) and (18) for the second addend of equation (4) we have the following equality:

$$\begin{aligned} \sum_{m=1}^{r} c_m x d_m x = D x_0^2 + (2D - 2C)\, x_0 \vec{x} + (+2C - D)\, |\vec{x}|^2 - \\ -2x_0 \sum_{m=1}^{r} c_m (\vec{x}, \vec{d}_m) - 2\sum_{m=1}^{r} c_m \vec{x}(\vec{x}, \vec{d}_m). \end{aligned} \tag{18}$$

We shall transform a linear part of equation (4):

$$\sum_{t=1}^{\ell} f_t x g_t = \sum_{t=1}^{\ell} f_t\left(g_t x - 2\vec{g}_t\vec{x} - 2(\vec{g}_t,\vec{x})\right) = \sum_{t=1}^{\ell} f_t g_t x - 2\sum_{t=1}^{\ell} f_t\vec{g}_t\vec{x} - 2\sum_{t=1}^{\ell} f_t(\vec{g}_t,\vec{x})$$

or

$$\sum_{t=1}^{\ell} f_t x g_t = Ex_0 + (E-2F)\,\vec{x} - 2\sum_{t=1}^{\ell} f_t(\vec{g}_t,\vec{x}), \tag{19}$$

where

$$E := \sum_{t=1}^{\ell} f_t g_t,\ F := \sum_{t=1}^{\ell} f_t\vec{g}_t. \tag{20}$$

It is necessary to transform three more addends, namely:

$$\begin{aligned} -2\sum_{t=1}^{\ell} f_t(\vec{g}_t,\vec{x}) &= x_1M_1 + x_2M_2 + x_3M_3,\\ -2\sum_{m=1}^{r} c_m(\vec{d}_m,\vec{x}) &= x_1N_1 + x_2N_2 + x_3N_3,\\ -4\sum_{p=1}^{n} a_p(\vec{b}_p,\vec{x}) &= x_1K_1 + x_2K_2 + x_3K_3, \end{aligned} \tag{21}$$

where

$$M_s := -2\sum_{t=1}^{\ell} f_t g_t^{(s)},\ N_s := -2\sum_{m=1}^{r} c_m d_m^{(s)},\ K_s := -4\sum_{p=1}^{n} a_p b_p^{(s)},\ s = 1,2,3. \tag{22}$$

To end the proof of theorem it is enough to combine equalities (11), (18), (19), (21) and to take into account designations (12), (15), (20), and (22).

It is not difficult to write out the system of equations corresponding to (9), but it would be very long and would demand many new notations. Therefore for the solution of certain examples one has to construct a system of equations, which turns out from certain conditions. Using the theorem proved above, we can receive many interesting results for incomplete quadratic quaternionic equations.

4. The solution of incomplete quadratic equations of the first form

In this section we shall study properties of equations of the form

$$\sum_{p=1}^{n} a_p x^2 b_p + \sum_{t=1}^{\ell} f_t x g_t = h. \tag{23}$$

In this section and in the next one we shall be interested in the following question: if x is a root of the equation then under what conditions the opposite $-x$ is a root of this equation, and the same question about the conjugate $\bar{x}$.

Proposition 1. *Let x be a root of equation (23), then $-x$ is a root of this equation if and only if*

$$\sum_{t=1}^{\ell} f_t x g_t = 0. \tag{24}$$

Proof. If equality (24) is true it is obvious, that $-x$ is also a root. Let now x and $-x$ be roots of the equation (23). We shall prove that equality (24) holds true. The condition and (23) imply:

$$\sum_{p=1}^{n} a_p x^2 b_p + \sum_{t=1}^{\ell} f_t x g_t = h, \quad \sum_{p=1}^{n} a_p x^2 b_p - \sum_{t=1}^{\ell} f_t x g_t = h.$$

The difference between last two equalities implies (24).

Under condition of (24), equation (23) takes the form:

$$\sum_{p=1}^{n} a_p x^2 b_p = h. \tag{25}$$

According to theorem 1, the last equation is equivalent to the following equation:

$$\begin{aligned} &Ax_0^2 - A|\vec{x}|^2 + (-4K_1 + (2A - 4B)i)\,x_0x_1 + (-4K_2 + \\ &+(2A-4B)j)\,x_0x_2 + (-4K_3 + (2A-4B)k)\,x_0x_3 = h, \end{aligned} \tag{26}$$

where A, B are defined by equality (12), and K_s by equality (22). Moreover if all $a_p, b_p \in \vec{\mathbb{H}}$, then $A = B$.

Remark 1. Vectorially it means, that the set of roots of equation (23) is symmetric with respect to point 0 only under condition (24) (see Fig. 1 for Example 1).

Example 1. Solve in $\mathbb{H}$ the following equation:

$$jx^2k + kx^2j = h. \tag{27}$$

Easily we get: $A = B = K_1 = 0$, $K_2 = -4k$, $K_3 = -4j$. Then equation (26) is equivalent to the following:

$$-4x_0x_3j - 4x_0x_2k = h. \tag{28}$$

- Let $h = 0$, then the set of the solutions of the equation (27) is either subspace $\vec{\mathbb{H}}$, that is set $\{x = x_1i + x_2j + x_3k : x_1, x_2, x_3 \in \mathbb{R}\}$, or a coordinate plane x_0Ox_1, that is set $\{x = x_0 + x_1i : x_0, x_1 \in \mathbb{R}\}$.

- Let $h = -4j - 4k$, then the set of the solutions of the equation (27) is a set $\{x = x_0 + x_1i + x_2(j+k) : x_0 = 1/x_2, \{x_0; x_2\} \subset \mathbb{R} \setminus \{0\}, x_1 \in \mathbb{R}\}$. Then

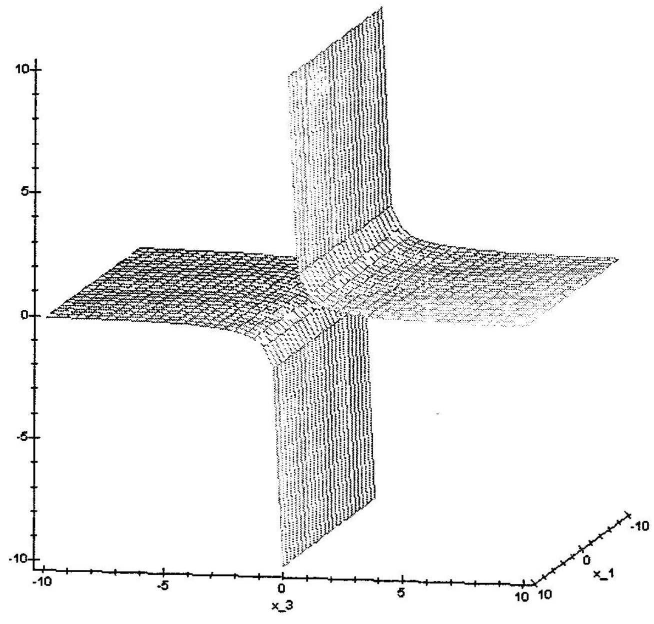

Fig. 1:

the corresponding set of points (x_0, x_1, x_2) constitute a cylindric surface (see Fig. 1).

- Let $h = -4j$ (or $-4k$), then the set of the solutions of the equation (27) is set of points $x = x_0 + x_1 i + x_3 k$ (or $x = x_0 + x_1 i + x_2 j$), where (x_0, x_1, x_3) (or (x_0, x_1, x_2)) constitute a cylindrical surface $x_0 x_3 = 1$ (or $x_0 x_2 = 1$) on three-dimensional space $x_0 x_1 x_3$ (or $x_0 x_1 x_2$). See Fig. 1.

Note that Proposition 1 is true in all these cases.

Proposition 2. *If x is a root of the equation*

$$\sum_{p=1}^{n} a_p x^2 b_p + \sum_{t=1}^{\ell} f_t x g_t = 0 \tag{29}$$

and

$$A = E = 0, \tag{30}$$

then its conjugate $\bar{x}$ is also a root of this equation.

Proof. Taking into account conditions of proposition, we conclude that in equality (9) $A_1 = A_2 = A_6 = 0$. Then if in equality (9) one changes x_1, x_2, x_3 by $-x_1, -x_2, -x_3$, then one obtains an identity.

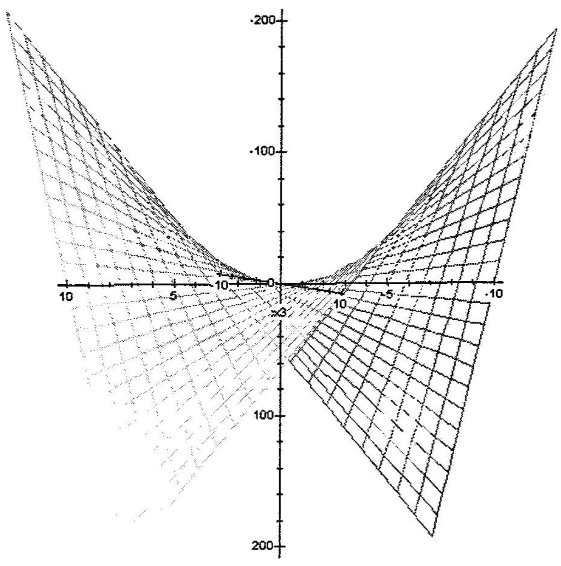

Fig. 2:

Remark 2. Vectorially it means, that the set of roots of the equation (29) is symmetric with respect to the real axis under condition (30). See, for example, the the set of solutions of equation (31) on Fig. 2.

Example 2. Solve in $\mathbb{H}$ the following equation:

$$jx^2k + kx^2j + jxi + ixj = 0. \tag{31}$$

We calculate coefficients of equation (9):

$$A_1 = A_2 = A_3 = A_6 = 0, \quad A_4 = -4k, \quad A_5 = -4j, \quad A_7 = -2j, \quad A_8 = -2i.$$

Then the equation (9) will be rewritten as follows:

$$2ix_2 + 2j(x_1 + 2x_0x_3) + 4kx_0x_2 = 0.$$

The solution of the last equation is $x = x_0 + x_1i + x_3k$, where the corresponding points (x_0, x_1, x_3) constitute on surface $x_1 + 2x_0x_3 = 0$ (see Fig. 2). This fact implies, that conjugate $\bar{x}$ is also a root of the equation (31) (proposition 2).

The proposition, contrary to proposition 2, generally speaking is not true. We can prove only the following

Proposition 3. *If for every root x of* (29) *its conjugate $\bar{x}$ is a root of* (29) *then the coefficients E and A satisfy the following relation:*

1. $E = A\frac{x_0^2 - |\vec{x}|^2}{x_0}$, *if* $x_0 \neq 0$,
2. $A = 0$, *if* $x_0 = 0$.

Proof. If in the formula (1) we substitute values x and $\bar{x}$, and then add the received equality, we shall have:

$$A\left(x_0^2 - |\vec{x}|^2\right) + Ex_0 = 0,$$

hence the proposition is proved.

For more detailed study of structure of the set of roots of the equation (23), we shall consider two more examples.

Example 3a. Solve in $\mathbb{H}$ the following equation:

$$ix^2k + jxi + ixj = 0. \tag{32}$$

We calculate coefficients:

$$A_1 = -A_2 = j, \quad A_3 = -2k, \quad A_4 = -2, \quad A_5 = -2i,$$
$$A_6 = A_9 = 0, \quad A_7 = -2j, \quad A_8 = -2i.$$

Hence, equation (32) is equivalent to the following equation:

$$-2x_0x_2 - 2i(x_2 + 2x_0x_3) + j(-x_0^2 + |\vec{x}|^2 - 2x_1) - 2x_0x_1k = 0,$$

which is equivalent to system:

$$\begin{cases} x_0x_2 = 0, \\ x_2 + 2x_0x_3 = 0, \\ -x_0^2 + x_1^2 + x_2^2 + x_3^2 - 2x_1 = 0, \\ x_0x_1 = 0. \end{cases}$$

The solution of the last system is circle

$$(x_1 - 1)^2 + x_3^2 = 1,$$

that is set

$$\{x = x_1i + x_3k : x_1, x_3 \in \mathbb{R}, (x_1 - 1)^2 + x_3^2 = 1\}$$

and point $x = 0$.

Example 3b. Solve in $\mathbb{H}$ the following equation:

$$kx^2i + jxi + ixj = 0. \tag{33}$$

Using the data of the previous example, we get system:

$$\begin{cases} x_0x_2 = 0, \\ x_2 + 2x_0x_3 = 0, \\ x_0^2 - x_1^2 - x_2^2 - x_3^2 - 2x_1 = 0, \\ x_0x_1 = 0. \end{cases}$$

The solution of the last system is circle

$$(x_1 + 1)^2 + x_3^2 = 1,$$

that is set

$$\{x = x_1 i + x_3 k : x_1, x_3 \in \mathbb{R}, (x_1 + 1)^2 + x_3^2 = 1\}$$

and point $x = 0$.

5. The solution of incomplete quadratics equations of the second form

In this section we shall study properties and solve equations of the form

$$\sum_{p=1}^{n} a_p x^2 b_p + \sum_{m=1}^{r} c_m x d_m x = h. \tag{34}$$

If $h = 0$, then equation (34) takes the form:

$$\sum_{p=1}^{n} a_p x^2 b_p + \sum_{m=1}^{r} c_m x d_m x = 0. \tag{35}$$

According to theorem 1 equation (34) is equivalent to the following one:

$$A_1 x_0^2 + A_2 |\vec{x}|^2 + A_3 x_0 x_1 + A_4 x_0 x_2 + A_5 x_0 x_3 - 2 \sum_{m=1}^{r} c_m \vec{x}(\vec{x}, \vec{d}_m) = h. \tag{36}$$

Proposition 4. *If x is a root of equation* (35), *then $-x$ is also a root of this equation.*

It follows from the form of equation (35).

Remark 3. The set of solutions of equation (35) is always symmetrical with respect to point 0.

Proposition 5. *If x is a root of equation* (35) *and $A_1 = A_2 = 0$, then its conjugate $\bar{x}$ is also a root of this equation.*

It follows from decomposition (36).

Example 4. Solve in $\mathbb{H}$ the following equation:

$$jx^2k + kx^2j + ixjx = q. \tag{37}$$

We calculate coefficients of equation (36), and we have:

$$A_1 = k, A_2 = +3k, A_3 = 0, A_4 = -4k - 2i, A_5 = -4j,$$

$$-2\sum_{m=1}^{r} c_m\vec{x}(\vec{x}, \vec{d}_m) = 2x_1x_2 - 2kx_2^2 + 2jx_2x_3.$$

Then equation (37) is equivalent to the following equation:

$$2x_1x_2 - 2ix_0x_2 + 2j(x_2x_3 - 2x_0x_3) + k(x_0^2 - 5x_2^2 + 3x_1^2 + 3x_3^2 - 4x_0x_2) = q. \tag{38}$$

We consider cases.

- If $q = 0$, then the set of the solution is two straight lines $x_0^2 - 3x_1^2 = 0$ in the plane x_0Ox_1, that is set

$$\{x = x_0 + x_1 i : x_0, \quad x_1 \in \mathbb{R}, \quad x_0^2 - 3x_1^2 = 0\}.$$

- If $q = Rk$, $R \in \mathbb{R}\backslash\{0\}$, then equation (38) is equivalent to system

$$\begin{cases} x_1x_2 = 0, \\ x_0x_2 = 0, \\ x_2x_3 - 2x_0x_3 = 0, \\ x_0^2 - 5x_2^2 + 3x_1^2 + 3x_3^2 - 4x_0x_2 = R. \end{cases} \tag{39}$$

The set of the solutions of system (39) is one of the following sets:

1) the hyperbola

$$x_0^2 - 3x_1^2 = R$$

in the plane x_0Ox_1, that is set

$$\{x = x_0 + x_1 i : x_0, x_1 \in \mathbb{R}, x_0^2 - 3x_1^2 = R\};$$

2) the circle

$$x_1^2 + x_3^2 = -R/3$$

with $R < 0$ in the plane x_1Ox_3, that is set

$$\{x = x_1 i + x_3 k : x_1, x_3 \in \mathbb{R}, x_1^2 + x_3^2 = -R/3\};$$

3) two points:

$$x_2 = \pm\sqrt{-R/5}$$

with $R < 0$, that is set

$$\{x = \pm\sqrt{-R/5}j : R \in (-\infty, 0)\}.$$

Example 5. Solve in $\mathbb{H}$ the following equation:

$$jx^2k + kx^2j + ixjx + jxix = s. \tag{40}$$

We calculate coefficients of equation (36). We have:

$$A_1 = A_2 = 0, A_3 = -2j, A_4 = -2i - 4k, A_5 = -4j,$$

$$2\sum_{m=1}^{r} c_m\vec{x}(\vec{x}, \vec{d}_m) = 4x_1x_2 + 2k(x_1^2 - x_2^2) + 2jx_2x_3 - 2ix_1x_3.$$

Then we have equation:

$$4x_1x_2 - 2i(x_0x_2 + x_1x_3) + 2j(x_2x_3 - x_0x_1 - 2x_0x_3) + 2k(x_1^2 - x_2^2 - 2x_0x_2) = s.$$

- Let $s = 0$, then we have a system

$$\begin{cases} x_1x_2 = 0, \\ x_0x_2 + x_1x_3 = 0, \\ x_2x_3 - x_0x_1 - 2x_0x_3 = 0, \\ x_1^2 - x_2^2 - 2x_0x_2 = 0, \end{cases} \tag{41}$$

  The solutions of system (41) are $x = 0$ and $x = \pm k$.

- If $s = 2k$, then $x = \pm i$.

- If $s = 2j$, then the set of solutions is a hyperbole

$$x_0 = -\frac{1}{2x_3}$$

  in the plane x_0Ox_3, that is set

$$\left\{x = x_0 + x_3k : x_0, x_3 \in \mathbb{R}, x_0 = -\frac{1}{2x_3}\right\}.$$

- If $s = -2i$, then equation (40) has no solution.

As we see, the set of every possible set of solutions of the equations (37), (40) is symmetric concerning a real axis.

It is interesting to note, that sometimes a quadratic equation has exactly three solutions.

Acknowledgement

The author is grateful to Dr. D. A. Mierzejewski for useful discussions and attention to work.

Partially supported by State Foundation of Fundamental Researches of Ukraine (project no. 25.1/084).

References

[1] D. Mierzejewski, *Investigation of quaternionic quadratic equations I. Factorization and passing to a system of real equations*, Bull. Soc. Sci. Lettres Łódź **58** Sér. Rech. Déform. **56** (2008), 17–26.

[2] D. Mierzejewski, V. Szpakowski, *On solutions of some types of quaternionic quadratic equations*, Bull. Soc. Sci. Lettres Łódź **58** Sér. Rech. Déform. **55** (2008), 23–32.

[3] A. Pogorui, M. Shapiro, *On the structure of the set of the zeros of quaternionic polynomials*, Complex Variables and Elliptic Equations, **49** no. 6 (2004), 379–389.

[4] V. Szpakowski, *Solution of general linear quaternionic equations* (in Ukrainian), The XI Kravchuk International Scientific Conference, Kyiv (Kiev), Ukraine, 2006, 661.

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Presented by Leon Mikołajczyk at the Session of the Mathematical-Physical Commission of the Łódź Society of Sciences and Arts on June 25, 2009

ROZWIĄZANIE OGÓLNEGO KWATERNIONOWEGO RÓWNANIA KWADRATOWEGO

S t r e s z c z e n i e

Rozważamy pewną szeroką klasę kwaternionowych równań kwadratowych. W artykule [3] badane były równania postaci $\sum_{\ell=0}^{n} a_\ell x^\ell = 0$ i zostało tam wykazane, że ich pierwiastki są bądź punktami odosobnionymi bądź układają się w sfery. W obecnym artykule dowodzimy, że pierwiastki ogólnego kwaternionowego równania kwadratowego mogą być nie tylko punktami odosobnionymi lub układać się w sfery, lecz także tworzyć okrąg, hiperbolę, powierzchnię walcową względnie parę płaszczyzn (które nie pokrywają się z płaszczyznami współrzędnych) lub inne powierzchnie. Opisujemy przykłady równań, których pierwiastki tworzą wspomniane struktury.