

Constructive description of monogenic functions in a finite-dimensional commutative associative algebra

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Abstract. Let \mathbb{A}_n^m be an arbitrary n -dimensional commutative associative algebra over the field of complex numbers with m idempotents. Let $e_1 = 1, e_2, e_3$ be elements of \mathbb{A}_n^m which are linearly independent over the field of real numbers. We consider monogenic (i. e. continuous and differentiable in the sense of Gateaux) functions of the variable $xe_1 + ye_2 + ze_3$, where x, y, z are real, and obtain a constructive description of all mentioned functions by means of holomorphic functions of complex variables. It follows from this description that monogenic functions have Gateaux derivatives of all orders.

Keywords: Commutative associative algebra; monogenic function; constructive description.

1 Introduction.

An effectiveness of the analytic function methods in the complex plane for researching plane potential fields inspires mathematicians to develop analogous methods for spatial fields.

Apparently, W. Hamilton (1843) made the first attempts to construct an algebra associated with the three-dimensional Laplace equation

$$\Delta_3 u(x, y, z) := \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) u(x, y, z) = 0 \quad (1)$$

meaning that components of hypercomplex functions satisfy the equation (1).

He constructed an algebra of noncommutative quaternions over the field of real numbers \mathbb{R} and made a base for developing the hypercomplex analysis.

C. Segre [1] constructed an algebra of commutative quaternions over the field \mathbb{R} that can be considered as a two-dimensional commutative semi-simple algebra of bicomplex numbers over the field of complex numbers \mathbb{C} . F. Ringleb [2] and J. Riley [3] obtained a constructive description of analytic function of a bicomplex variable, namely, they proved that such an analytic function can be constructed with an use of two holomorphic functions of complex variables.

Furthermore, F. Ringleb [2] considered an arbitrary finite-dimensional associative (commutative or not) semi-simple algebra over the field \mathbb{R} . For analytic functions which maps the mentioned algebra onto itself, he obtained a constructive description by means of real and complex analytic functions.

A relation between spatial potential fields and analytic functions given in commutative algebras was established by P. W. Ketchum [4] who shown that every analytic function $\Phi(\zeta)$ of the variable $\zeta = xe_1 + ye_2 + ze_3$ satisfies the equation (1) in the case where the elements e_1, e_2, e_3 of a commutative algebra satisfy the condition

$$e_1^2 + e_2^2 + e_3^2 = 0, \quad (2)$$

because

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} \equiv \Phi''(\zeta) (e_1^2 + e_2^2 + e_3^2) = 0, \quad (3)$$

where $\Phi'' := (\Phi')'$ and $\Phi'(\zeta)$ is defined by the equality $d\Phi = \Phi'(\zeta)d\zeta$.

We say that a commutative associative algebra \mathbb{A} is *harmonic* (cf. [4, 5, 7]) if in \mathbb{A} there exists a triad of linearly independent vectors $\{e_1, e_2, e_3\}$ satisfying the equality (2) with $e_k^2 \neq 0$ for $k = 1, 2, 3$. We say also that such a triad $\{e_1, e_2, e_3\}$ is *harmonic*.

P. W. Ketchum [4] considered the C. Segre algebra of quaternions [1] as an example of harmonic algebra.

I. P. Mel'nichenko [5] noticed that doubly differentiable in the sense of Gateaux functions form the largest algebra of functions Φ satisfying identically the equalities (3), where Φ'' is the Gateaux second derivative of function Φ . He proved that there exist exactly 3 three-dimensional harmonic algebras with unit over the field \mathbb{C} (see [5, 6, 7]).

Constructive descriptions of monogenic (i. e. continuous and differentiable in the sense of Gateaux) functions taking values in the mentioned three-dimensional harmonic algebras by means three corresponding holomorphic functions of the complex variable are obtained in the papers [8, 9, 10]. Such descriptions make it possible to prove the infinite differentiability in the sense of Gateaux of monogenic functions and integral theorems for these functions that

are analogous to classical theorems of the complex analysis (see, e. g., [11, 12]).

Furthermore, constructive descriptions of monogenic functions taking values in special n -dimensional commutative algebras by means n holomorphic functions of complex variables are obtained in the papers [13, 14].

In this paper we obtaine a constructive description of monogenic functions taking values in an arbitrary finite-dimensional commutative associative algebra with unit by means of holomorphic functions of complex variables. It follows from this description that the monogenic functions have the Gateaux derivatives of all orders.

2 The algebra \mathbb{A}_n^m .

Let \mathbb{N} be the set of natural numbers. We fix the numbers $m, n \in \mathbb{N}$ such that $m \leq n$. Let \mathbb{A}_n^m be an arbitrary commutative associative algebra with unit over the field of complex number \mathbb{C} . E. Cartan [15, p. 33] proved that there exist a basis $\{I_k\}_{k=1}^n$ in \mathbb{A}_n^m satisfying the following multiplication rules:

1. $\forall r, s \in [1, m] \cap \mathbb{N} : \quad I_r I_s = \begin{cases} 0 & \text{if } r \neq s, \\ I_r & \text{if } r = s; \end{cases}$
2. $\forall r, s \in [m+1, n] \cap \mathbb{N} : \quad I_r I_s = \sum_{k=\max\{r,s\}+1}^n \Upsilon_{r,k}^s I_k ;$
3. $\forall s \in [m+1, n] \cap \mathbb{N} \quad \exists! u_s \in [1, m] \cap \mathbb{N} \quad \forall r \in [1, m] \cap \mathbb{N} :$

$$I_r I_s = \begin{cases} 0 & \text{if } r \neq u_s, \\ I_s & \text{if } r = u_s. \end{cases} \quad (4)$$

Moreover, the structure constants $\Upsilon_{r,k}^s \in \mathbb{C}$ satisfy the associativity conditions:

$$(A 1). \quad (I_r I_s) I_p = I_r (I_s I_p) \quad \forall r, s, p \in [m+1, n] \cap \mathbb{N};$$

$$(A 2). \quad (I_u I_s) I_p = I_u (I_s I_p) \quad \forall u \in [1, m] \cap \mathbb{N} \quad \forall s, p \in [m+1, n] \cap \mathbb{N}.$$

Obviously, the first m basic vectors $\{I_u\}_{u=1}^m$ are idempotents and form a semi-simple subalgebra of the algebra \mathbb{A}_n^m . The vectors $\{I_r\}_{r=m+1}^n$ form a nilpotent subalgebra of the algebra \mathbb{A}_n^m . The element $1 = \sum_{u=1}^m I_u$ is the unit

of \mathbb{A}_n^m .

In the cases where \mathbb{A}_n^m has some specific properties, the following propositions are true.

Proposition 1. *If there exists the unique $u_0 \in [1, m] \cap \mathbb{N}$ such that $I_{u_0} I_s = I_s$ for all $s = m + 1, \dots, n$, then the associativity condition (A 2) is satisfied.*

Proof. In the associativity condition (A 2), two cases are possible:

- 1) $I_u \neq I_{u_0}$, then $I_u I_s = 0 \quad \forall s = m + 1, \dots, n$;
- 2) $I_u = I_{u_0}$, then $I_u I_s = I_s \quad \forall s = m + 1, \dots, n$.

In the first case, the condition (A 2) takes the form

$$0 \cdot I_p = I_u \sum_{k=\max\{s,p\}+1}^n \Upsilon_{s,k}^p I_k = 0,$$

because $I_u I_k = 0$ for all $k = \max\{s, p\} + 1, \dots, n$.

In the second case, the condition (A 2) takes the form

$$I_s I_p = I_u \sum_{k=\max\{s,p\}+1}^n \Upsilon_{s,k}^p I_k.$$

It is equivalent to the equality

$$\sum_{k=\max\{s,p\}+1}^n \Upsilon_{s,k}^p I_k = \sum_{k=\max\{s,p\}+1}^n \Upsilon_{s,k}^p I_k,$$

because $I_u I_k = I_k$ for all $k = \max\{s, p\} + 1, \dots, n$. The proposition is proved.

Thus, under the conditions of Proposition 1, the associativity condition (A 1) is only required. It means that the nilpotent subalgebra of \mathbb{A}_n^m with the basis $\{I_r\}_{r=m+1}^n$ can be an arbitrary commutative associative nilpotent algebra of dimension $n - m$. Note that such nilpotent algebras are fully described for the dimensions 1, 2, 3, 4 in the paper [16].

Proposition 2. *If all u_r are different in the multiplication rule 3, then $I_s I_p = 0$ for all $s, p = m + 1, \dots, n$.*

Proof. Let $s \in [m + 1, n] \cap \mathbb{N}$. We choose I_u such that $I_u I_s = I_s$. Then from the associativity condition (A 2) we obtain the identity

$$I_s I_p = I_u \sum_{k=\max\{s,p\}+1}^n \Upsilon_{s,k}^p I_k = 0,$$

because by assumption of theorem $I_u I_k = 0$ for all $k = \max\{s, p\} + 1, \dots, n$. The proposition is proved.

Thus, under the conditions of Proposition 2, the multiplication table of the nilpotent subalgebra of \mathbb{A}_n^m with the basis $\{I_r\}_{r=m+1}^n$ consists only of zeros, and all associativity conditions are satisfied.

The algebra \mathbb{A}_n^m contains m maximal ideals

$$\mathcal{I}_u := \left\{ \sum_{k=1, k \neq u}^n \lambda_k I_k : \lambda_k \in \mathbb{C} \right\}, \quad u = 1, 2, \dots, m,$$

and their intersection is the radical

$$\mathcal{R} := \left\{ \sum_{k=m+1}^n \lambda_k I_k : \lambda_k \in \mathbb{C} \right\}.$$

Consider m linear functionals $f_u : \mathbb{A}_n^m \rightarrow \mathbb{C}$ satisfying the equalities

$$f_u(I_u) = 1, \quad f_u(\omega) = 0 \quad \forall \omega \in \mathcal{I}_u, \quad u = 1, 2, \dots, m.$$

Inasmuch as the kernel of functional f_u is the maximal ideal \mathcal{I}_u , this functional is also continuous and multiplicative (see [17, p. 147]).

3 Monogenic functions.

We consider the vectors $e_1 = 1, e_2, e_3$ in \mathbb{A}_n^m which are linearly independent over the field of real numbers \mathbb{R} . It means that the equality

$$\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 = 0, \quad \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R},$$

holds if and only if $\alpha_1 = \alpha_2 = \alpha_3 = 0$.

Let the vectors $e_1 = 1, e_2, e_3$ have the following decompositions with respect to the basis $\{I_k\}_{k=1}^n$:

$$e_1 = \sum_{r=1}^m I_r, \quad e_2 = \sum_{r=1}^n a_r I_r, \quad e_3 = \sum_{r=1}^n b_r I_r, \quad (5)$$

where $a_r, b_r \in \mathbb{C}$.

Let $\zeta := x e_1 + y e_2 + z e_3$, where $x, y, z \in \mathbb{R}$. It is obvious that $\xi_u := f_u(\zeta) = x + y a_u + z b_u$, $u = 1, 2, \dots, m$. Let $E_3 := \{\zeta = x e_1 + y e_2 + z e_3 : x, y, z \in \mathbb{R}\}$ be the linear span of vectors e_1, e_2, e_3 over the field \mathbb{R} .

Let Ω be a domain in \mathbb{R}^3 . Associate with Ω the domain $\Omega_\zeta := \{\zeta = xe_1 + ye_2 + ze_3 : (x, y, z) \in \Omega\}$ in E_3 .

We say that a continuous function $\Phi : \Omega_\zeta \rightarrow \mathbb{A}_n^m$ is *monogenic* in Ω_ζ if Φ is differentiable in the sense of Gateaux in every point of Ω_ζ , i. e. if for every $\zeta \in \Omega_\zeta$ there exists an element $\Phi'(\zeta) \in \mathbb{A}_n^m$ such that

$$\lim_{\varepsilon \rightarrow 0+0} (\Phi(\zeta + \varepsilon h) - \Phi(\zeta)) \varepsilon^{-1} = h\Phi'(\zeta) \quad \forall h \in E_3. \quad (6)$$

$\Phi'(\zeta)$ is the *Gateaux derivative* of the function Φ in the point ζ .

In the scientific literature the denomination of monogenic function is used else for functions satisfying certain conditions similar to the classical Cauchy – Riemann conditions (see, e. g., [18, 19]). Such functions are also called regular functions (see [20]) or hyperholomorphic functions (see, e. g., [21, 22]).

Consider the decomposition of a function $\Phi : \Omega_\zeta \rightarrow \mathbb{A}_n^m$ with respect to the basis $\{I_k\}_{k=1}^n$:

$$\Phi(\zeta) = \sum_{k=1}^n U_k(x, y, z) I_k. \quad (7)$$

In the case where the functions $U_k : \Omega \rightarrow \mathbb{C}$ are \mathbb{R} -differentiable in Ω , i. e. for every $(x, y, z) \in \Omega$,

$$U_k(x + \Delta x, y + \Delta y, z + \Delta z) - U_k(x, y, z) = \frac{\partial U_k}{\partial x} \Delta x + \frac{\partial U_k}{\partial y} \Delta y + \frac{\partial U_k}{\partial z} \Delta z + o\left(\sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}\right), \quad (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 \rightarrow 0,$$

the function Φ is monogenic in the domain Ω_ζ if and only if the following Cauchy – Riemann conditions are satisfied in Ω_ζ :

$$\frac{\partial \Phi}{\partial y} = \frac{\partial \Phi}{\partial x} e_2, \quad \frac{\partial \Phi}{\partial z} = \frac{\partial \Phi}{\partial x} e_3. \quad (8)$$

Below, it will be shown that all components U_k of the monogenic function (7) are infinitely \mathbb{R} -differentiable in Ω .

4 An expansion of the resolvent.

Lemma 1. *An expansion of the resolvent is of the form*

$$(te_1 - \zeta)^{-1} = \sum_{r=1}^n A_r I_r \quad \forall t \in \mathbb{C} : t \neq \xi_u, \quad u = 1, 2, \dots, m,$$

with the coefficients A_r are determined by the following recurrence relations:

$$A_u = \frac{1}{t - \xi_u}, \quad u = 1, 2, \dots, m, \quad A_{m+1} = \frac{T_{m+1}}{(t - \xi_{u_{m+1}})^2}, \quad (9)$$

$$A_p = \frac{T_p}{(t - \xi_{u_p})^2} + \frac{1}{t - \xi_{u_p}} \sum_{r=m+1}^{p-1} A_r B_{r,p}, \quad p = m+2, m+3, \dots, n,$$

where

$$T_p := ya_p + zb_p, \quad p = m+1, m+2, \dots, n, \quad (10)$$

$$B_{r,p} := \sum_{s=m+1}^{p-1} T_s \Upsilon_{r,p}^s, \quad p = m+2, m+3, \dots, n, \quad (11)$$

and the natural numbers u_p are defined in the rule 3 of the multiplication table of algebra \mathbb{A}_n^m .

Proof. Let's find $t \in \mathbb{C}$ for which there exists the element $(te_1 - \zeta)^{-1}$ in the algebra \mathbb{A}_n^m and let's find the coefficients A_r of decomposition of this element with respect to the basis $\{I_k\}_{k=1}^n$:

$$(te_1 - \zeta)^{-1} = \sum_{r=1}^n A_r I_r.$$

Taking into account the decompositions (5) and the multiplication table of algebra \mathbb{A}_n^m , we obtain

$$\begin{aligned} & \sum_{u=1}^m I_u = 1 = (te_1 - \zeta)^{-1}(te_1 - \zeta) = \\ & = \left(\sum_{u=1}^m A_u I_u + \sum_{r=m+1}^n A_r I_r \right) \left(\sum_{u=1}^m (t - \xi_u) I_u - \sum_{r=m+1}^n (ya_r + zb_r) I_r \right) = \\ & = \sum_{u=1}^m A_u (t - \xi_u) I_u + \left(A_{m+1} (t - \xi_{u_{m+1}}) + A_{u_{m+1}} (-ya_{m+1} - zb_{m+1}) \right) I_{m+1} + \end{aligned}$$

$$+ \sum_{p=m+2}^n \left(A_{u_p}(-ya_p - zb_p) + A_p(t - \xi_{u_p}) + \sum_{r=m+1}^{p-1} A_r \sum_{s=m+1}^{p-1} (-ya_s - zb_s) \Upsilon_{r,p}^s \right) I_p.$$

Now, to determine the coefficients A_r , we have the system of equations:

$$\begin{aligned} A_u(t - \xi_u) &= 1, \quad u = 1, 2, \dots, m, \quad A_{m+1}(t - \xi_{u_{m+1}}) - A_{u_{m+1}}T_{m+1} = 0, \\ -A_{u_p}T_p + A_p(t - \xi_{u_p}) - \sum_{r=m+1}^{p-1} A_r B_{r,p} &= 0, \quad p = m+2, m+3, \dots, n, \end{aligned}$$

where the denotations (10), (11) are used. As immediate consequences of this system, we obtain the equalities (9). The lemma is proved.

It follows from Lemma 1 that the points $(x, y, z) \in \mathbb{R}^3$ corresponding to the noninvertible elements $\zeta = xe_1 + ye_2 + ze_3$ form the straight lines

$$L_u : \begin{cases} x + y\operatorname{Re} a_u + z\operatorname{Re} b_u = 0, \\ y\operatorname{Im} a_u + z\operatorname{Im} b_u = 0, \end{cases} \quad u = 1, 2, \dots, m,$$

in the three-dimensional space \mathbb{R}^3 .

Lemma 2. *If there exists $p \in [m+2, n] \cap \mathbb{N}$ and $r \in [m+1, p-1] \cap \mathbb{N}$ such that $B_{r,p} \neq 0$, then $u_p = u_r$.*

Proof. Since $B_{r,p} \neq 0$, at least one of the numbers $\Upsilon_{m+1,p}^r, \Upsilon_{m+2,p}^r, \dots, \Upsilon_{p-1,p}^r$ is different from zero. Let $\Upsilon_{k,p}^r \neq 0$, where k is one of the numbers $m+1, m+2, \dots, p-1$. The associativity condition of the algebra implies the equality

$$(I_{u_r} I_r) I_k = I_{u_r} (I_r I_k),$$

which is equivalent to the equality

$$\sum_{\ell=\max\{k,r\}+1}^n \Upsilon_{k,\ell}^r I_\ell = \sum_{\ell=\max\{k,r\}+1}^n \Upsilon_{k,\ell}^r I_{u_r} I_\ell. \quad (12)$$

Note that by the multiplication rule (4), for each $\ell = m+1, m+2, \dots, n$ the product $I_{u_r} I_\ell$ is equal to either zero or I_ℓ . Therefore, since $\Upsilon_{k,p}^r \neq 0$, the equality (12) implies the equality $\Upsilon_{k,p}^r I_p = \Upsilon_{k,p}^r I_{u_r} I_p$, i. e. $I_p = I_{u_r} I_p$, and it means that $u_r = u_p$. The lemma is proved.

Lemma 3. *For any $s = m+1, m+2, \dots, n$ the coefficients A_s can be*

expressed in the form

$$A_s = \sum_{k=2}^{s-m+1} \frac{Q_{k,s}}{(t - \xi_{u_s})^k}, \quad (13)$$

where $Q_{k,s}$ are determined by the following recurrence relations:

$$Q_{2,s} := T_s, \quad Q_{k,s} = \sum_{r=m+1}^{s-1} Q_{k-1,r} B_{r,s}, \quad k = 3, 4, \dots, s - m + 1. \quad (14)$$

Proof. Let us prove the expression (13) by mathematical induction. First, the expression (13) coincides with the equality (9) for $s = m + 1$.

Further, suppose the correctness of the formula (13) for all $A_{m+1}, A_{m+2}, \dots, A_{s-1}$ and prove that A_s is also expressed by the formula (13). For this purpose, we use the equality (9) for $p = s$. Substituting the expressions (13) for A_r in the equality (9), we obtain:

$$\begin{aligned} A_s &= \frac{T_s}{(t - \xi_{u_s})^2} + \sum_{r=m+1}^{s-1} \frac{A_r B_{r,s}}{t - \xi_{u_s}} = \\ &= \frac{T_s}{(t - \xi_{u_s})^2} + \sum_{r=m+1}^{s-1} \sum_{k=2}^{r-m+1} \frac{Q_{k,r} B_{r,s}}{(t - \xi_{u_s})(t - \xi_{u_r})^k}. \end{aligned} \quad (15)$$

If all $B_{r,s} = 0$ for $r = m + 1, m + 2, \dots, s - 1$, then the formula (15) takes the form (13) with $Q_{2,m+2} = T_{m+2}$ and $Q_{k,s} = 0$. Furthermore, for every $r = m + 1, m + 2, \dots, s - 1$ for which $B_{r,s} \neq 0$, by Lemma 2 $u_r = u_s$, and we get again the formula (13), where $Q_{k,s}$ are determined by the equalities (14). The lemma is proved.

As a consequence of Lemmas 1, 3, we obtain the following expansion of resolvent:

$$(te_1 - \zeta)^{-1} = \sum_{u=1}^m \frac{1}{t - \xi_u} I_u + \sum_{s=m+1}^n \sum_{k=2}^{s-m+1} \frac{Q_{k,s}}{(t - \xi_{u_s})^k} I_s. \quad (16)$$

5 A constructive description of monogenic functions.

Denote $f_u(E_3) := \{f_u(\zeta) : \zeta \in E_3\}$. In what follows, we make the following essential assumption: $f_u(E_3) = \mathbb{C}$ for all $u = 1, 2, \dots, m$. Obviously, it holds

if and only if for every fixed $u = 1, 2, \dots, m$ at least one of the numbers a_u or b_u belongs to $\mathbb{C} \setminus \mathbb{R}$.

Lemma 4. *Suppose that a domain $\Omega \subset \mathbb{R}^3$ is convex in the direction of the straight lines L_u and $f_u(E_3) = \mathbb{C}$ for all $u = 1, 2, \dots, m$. Suppose also that a function $\Phi : \Omega_\zeta \rightarrow \mathbb{A}_n^m$ is monogenic in the domain Ω_ζ . If points $\zeta_1, \zeta_2 \in \Omega_\zeta$ such that $\zeta_2 - \zeta_1 \in \{\zeta = xe_1 + ye_2 + ze_3 : (x, y, z) \in L_u\}$, then*

$$\Phi(\zeta_1) - \Phi(\zeta_2) \in \mathcal{I}_u. \quad (17)$$

The proof of Lemma 4 is similar to the proof of Lemma 1 [8], where one must take the straight line L_u and the functional f_u instead of the straight line L and the functional f , respectively.

Let a domain $\Omega \subset \mathbb{R}^3$ be convex in the direction of the straight lines L_u , $u = 1, 2, \dots, m$. By D_u we denote that domain in \mathbb{C} onto which the domain Ω_ζ is mapped by the functional f_u .

We introduce the linear operators A_u , $u = 1, 2, \dots, m$, which assign holomorphic functions $F_u : D_u \rightarrow \mathbb{C}$ to every monogenic function $\Phi : \Omega_\zeta \rightarrow \mathbb{A}_n^m$ by the formula

$$F_u(\xi_u) = f_u(\Phi(\zeta)), \quad (18)$$

where $\xi_u = f_u(\zeta) \equiv x + ya_u + zb_u$ and $\zeta \in \Omega_\zeta$. It follows from Lemma 4 that the value $F_u(\xi_u)$ does not depend on a choice of a point ζ for which $f_u(\zeta) = \xi_u$.

Lemma 5. *Suppose that a domain $\Omega \subset \mathbb{R}^3$ is convex in the direction of the straight lines L_u and $f_u(E_3) = \mathbb{C}$ for all $u = 1, 2, \dots, m$. Suppose also that for any fixed $u = 1, 2, \dots, m$, a function $F_u : D_u \rightarrow \mathbb{C}$ is holomorphic in a domain D_u and Γ_u is a closed Jordan rectifiable curve in D_u which surrounds the point ξ_u and contains no points ξ_q , $q = 1, 2, \dots, m$, $q \neq u$. Then the function*

$$\Psi_u(\zeta) := I_u \int_{\Gamma_u} F_u(t)(te_1 - \zeta)^{-1} dt \quad (19)$$

is monogenic in the domain Ω_ζ .

Proof. Let $\zeta \in \Omega_\zeta$. First, for any $h \in E_3$ and any $\varepsilon > 0$, it follows from

the equality (16) that

$$\begin{aligned}
I_u(te_1 - \zeta - \varepsilon h)^{-1} &= \sum_{u=1}^m \frac{1}{t - \xi_u - \varepsilon f_u(h)} I_u + \\
&+ \sum_{s=m+1}^n \sum_{k=2}^{s-m+1} \frac{Q_{k,s}}{(t - \xi_{u_s} - \varepsilon f_{u_s}(h))^k} I_s I_u. \tag{20}
\end{aligned}$$

Note that for any natural k , a complex-value function $1/(t - \xi_u - \varepsilon f_u(h))^k$ tends to the function $1/(t - \xi_u)^k$ uniformly for all $t \in \Gamma_u$ when $\varepsilon \rightarrow 0$. Therefore, for any $h \in E_3$ the function $I_u(te_1 - \zeta - \varepsilon h)^{-1}$ tends to the function $I_u(te_1 - \zeta)^{-1}$ uniformly for all $t \in \Gamma_u$ when $\varepsilon \rightarrow 0$.

Further, let us prove the existence of Gateaux derivative $\Psi'_u(\zeta)$ by the definition (6). Taking into account the Hilbert identity (see, e. g., Theorem 4.8.2 [17, p. 140])

$$(te_1 - a)^{-1} - (te_1 - b)^{-1} = (te_1 - a)^{-1}(te_1 - b)^{-1}(a - b) \quad \forall a, b \in E_3,$$

we have

$$\begin{aligned}
\Lambda &:= I_u \lim_{\varepsilon \rightarrow 0+0} \frac{1}{\varepsilon} \left(\int_{\Gamma_u} F_u(t) (te_1 - (\zeta + \varepsilon h))^{-1} dt - \int_{\Gamma_u} F_u(t) (te_1 - \zeta)^{-1} dt \right) = \\
&= I_u h \lim_{\varepsilon \rightarrow 0+0} \int_{\Gamma_u} F_u(t) (te_1 - (\zeta + \varepsilon h))^{-1} (te_1 - \zeta)^{-1} dt.
\end{aligned}$$

Now, since the function $I_u(te_1 - \zeta - \varepsilon h)^{-1}$ converges uniformly to the function $I_u(te_1 - \zeta)^{-1}$ when $\varepsilon \rightarrow 0$, we obtain the equality

$$\Lambda = I_u h \int_{\Gamma_u} F_u(t) ((te_1 - \zeta)^{-1})^2 dt$$

which means the existence of the Gateaux derivative

$$\Psi'_u(\zeta) := I_u \int_{\Gamma_u} F_u(t) ((te_1 - \zeta)^{-1})^2 dt.$$

Finally, in view of the equality (16), components of the expansion of function $\Psi_u(\zeta)$ with respect to the basis $\{I_k\}_{k=1}^n$ are continuous functions. Therefore,

the function $\Psi'_u(\zeta)$ is also continuous, and the function $\Psi'_u(\zeta)$ is monogenic in Ω_ζ . Lemma is proved.

Lemma 6. *Suppose that a domain $\Omega \subset \mathbb{R}^3$ is convex in the direction of the straight lines L_u and $f_u(E_3) = \mathbb{C}$ for all $u = 1, 2, \dots, m$. Suppose also that a function $V : \Omega \rightarrow \mathbb{C}$ satisfies the equalities*

$$\frac{\partial V}{\partial y} = \frac{\partial V}{\partial x} a_u, \quad \frac{\partial V}{\partial z} = \frac{\partial V}{\partial x} b_u \quad (21)$$

in Ω . Then V is a holomorphic function of the variable $\xi_u = f_u(\zeta) = x + ya_u + zb_u$ in the domain D_u .

Proof. We first separate the real and the imaginary part of the expression

$$\xi_u = x + y \operatorname{Re} a_u + z \operatorname{Re} b_u + i(y \operatorname{Im} a_u + z \operatorname{Im} b_u) =: \tau + i\eta \quad (22)$$

and note that the equalities (21) yield

$$\frac{\partial V}{\partial \eta} \operatorname{Im} a_u = i \frac{\partial V}{\partial \tau} \operatorname{Im} a_u, \quad \frac{\partial V}{\partial \eta} \operatorname{Im} b_u = i \frac{\partial V}{\partial \tau} \operatorname{Im} b_u. \quad (23)$$

It follows from the condition $f_u(E_3) = \mathbb{C}$ that at least one of the numbers $\operatorname{Im} a_u$ or $\operatorname{Im} b_u$ is not equal to zero. Therefore, using (23), we get

$$\frac{\partial V}{\partial \eta} = i \frac{\partial V}{\partial \tau}. \quad (24)$$

We prove that $V(x_1, y_1, z_1) = V(x_2, y_2, z_2)$ for points $(x_1, y_1, z_1), (x_2, y_2, z_2) \in \Omega$ such that the segment that connects these points is parallel to the straight line L_u . To this end, in the domain Ω , we construct two surfaces with common edge, namely a surface Q that contains the point (x_1, y_1, z_1) and a surface Σ that contains the point (x_2, y_2, z_2) , such that the restrictions of the functional f_u to the corresponding subsets $Q_\zeta := \{\zeta \in E_3 : (x, y, z) \in Q\}$ and $\Sigma_\zeta := \{\zeta \in E_3 : (x, y, z) \in \Sigma\}$ of the domain Ω_ζ are bijections of these subsets to the same domain D_u of the complex plane.

As the surface Q in the domain Ω , we take a fixed equilateral triangle with vertices $A_{u,1}, A_{u,2}$, and $A_{u,3}$ centered at the point (x_1, y_1, z_1) the plane of which is perpendicular to the straight line L_u . We now continue the construction of the surface Σ .

Consider the triangle with vertices $A'_{u,1}$, $A'_{u,2}$, and $A'_{u,3}$ centered at the point (x_2, y_2, z_2) , lying in the domain Ω , and such that its sides $A'_{u,1}A'_{u,2}$, $A'_{u,2}A'_{u,3}$, and $A'_{u,1}A'_{u,3}$ are parallel to the segments $A_{u,1}A_{u,2}$, $A_{u,2}A_{u,3}$, and $A_{u,1}A_{u,3}$, respectively, and have smaller lengths than the sides of the triangle $A_{u,1}A_{u,2}A_{u,3}$. Since the domain Ω is convex in the direction of the straight line L_u , we conclude that the prism with vertices $A'_{u,1}$, $A'_{u,2}$, $A'_{u,3}$, $A''_{u,1}$, $A''_{u,2}$, and $A''_{u,3}$ such that the points $A''_{u,1}$, $A''_{u,2}$, and $A''_{u,3}$ lie in the plane of the triangle $A_{u,1}A_{u,2}A_{u,3}$ and its edges $A'_{u,m}A''_{u,m}$, $m = \overline{1,3}$, are parallel to the straight line L_u is completely contained in Ω .

We now fix a triangle with vertices $B_{u,1}$, $B_{u,2}$, and $B_{u,3}$ such that the point $B_{u,m}$ lies on the segment $A'_{u,m}A''_{u,m}$ for $m = \overline{1,3}$ and the truncated pyramid with vertices $A_{u,1}$, $A_{u,2}$, $A_{u,3}$, $B_{u,1}$, $B_{u,2}$, and $B_{u,3}$ and lateral edges $A_{u,m}B_{u,m}$, $m = \overline{1,3}$, is completely contained in the domain Ω .

Finally, in the plane of the triangle $A'_{u,1}A'_{u,2}A'_{u,3}$, we fix a triangle T with vertices $C_{u,1}$, $C_{u,2}$, and $C_{u,3}$ such that its sides $C_{u,1}C_{u,2}$, $C_{u,2}C_{u,3}$, and $C_{u,1}C_{u,3}$ are parallel to the segments $A'_{u,1}A'_{u,2}$, $A'_{u,2}A'_{u,3}$, and $A'_{u,1}A'_{u,3}$, respectively, and have smaller lengths than the sides of the triangle $A'_{u,1}A'_{u,2}A'_{u,3}$. By construction, the truncated pyramid with vertices $B_{u,1}$, $B_{u,2}$, $B_{u,3}$, $C_{u,1}$, $C_{u,2}$, and $C_{u,3}$ and lateral edges $B_{u,m}C_{u,m}$, $m = \overline{1,3}$, is completely contained in the domain Ω .

Let Σ denote the surface formed by the triangle T and the lateral surfaces of the truncated pyramids $A_{u,1}A_{u,2}A_{u,3}B_{u,1}B_{u,2}B_{u,3}$ and $B_{u,1}B_{u,2}B_{u,3}C_{u,1}C_{u,2}C_{u,3}$.

Since the surfaces Q and Σ have a common edge, the sets Q_ζ and Σ_ζ are mapped by the functional f_u onto the same domain D_u of the complex plane. In the domain D_u , we introduce two complex-valued functions H_u and W_u in D_u as follows:

$$H_u(\xi_u) = V(x, y, z) \quad \text{for } (x, y, z) \in Q,$$

$$W_u(\xi_u) = V(x, y, z) \quad \text{for } (x, y, z) \in \Sigma,$$

where the correspondence between the points (x, y, z) and $\xi_u \in D_u$ is described by relation (22).

By virtue of equality (24) and Theorem 6 in [23], the functions H_u and W_u are holomorphic in the domain D_u . According to the definition of the functions H_u and W_u , we have $H_u(\xi_u) \equiv W_u(\xi_u)$ on the boundary of the domain D_u . By virtue of the holomorphy of the functions H_u and W_u in the domain D_u , the identity $H_u(\xi_u) \equiv W_u(\xi_u)$ holds everywhere in D_u . Therefore, the equality $V(x_1, y_1, z_1) = V(x_2, y_2, z_2)$ is proved.

Thus, a function $V : \Omega \rightarrow \mathbb{C}$ of the form $V(x, y, z) := F(\xi_u)$, where $F(\xi_u)$ is an arbitrary function holomorphic in the domain D_u , is a general solution of the system (21). The lemma is proved.

Theorem 1. *Let a domain $\Omega \subset \mathbb{R}^3$ be convex in the direction of the straight lines L_u and $f_u(E_3) = \mathbb{C}$ for all $u = 1, 2, \dots, m$. Then every monogenic function $\Phi : \Omega_\zeta \rightarrow \mathbb{A}_n^m$ can be expressed in the form*

$$\Phi(\zeta) = \sum_{u=1}^m I_u \frac{1}{2\pi i} \int_{\Gamma_u} F_u(t)(te_1 - \zeta)^{-1} dt + \sum_{s=m+1}^n I_s \frac{1}{2\pi i} \int_{\Gamma_{u_s}} G_s(t)(te_1 - \zeta)^{-1} dt, \quad (25)$$

where F_u and G_s are certain holomorphic functions in the domains D_u and D_{u_s} , respectively, and Γ_q is a closed Jordan rectifiable curve in D_q which surrounds the point ξ_q and contains no points ξ_ℓ , $\ell, q = 1, 2, \dots, m$, $\ell \neq q$.

Proof. We set

$$F_u := A_u \Phi, \quad u = 1, 2, \dots, m. \quad (26)$$

Let us show that the values of monogenic function

$$\Phi_0(\zeta) := \Phi(\zeta) - \sum_{u=1}^m I_u \frac{1}{2\pi i} \int_{\Gamma_u} F_u(t)(te_1 - \zeta)^{-1} dt \quad (27)$$

belong to the radical \mathcal{R} , i. e. $\Phi_0(\zeta) \in \mathcal{R}$ for all $\zeta \in \Omega_\zeta$. As a consequence of the equality (16), we have the equality

$$\begin{aligned} I_u \frac{1}{2\pi i} \int_{\Gamma_u} F_u(t)(te_1 - \zeta)^{-1} dt &= I_u \frac{1}{2\pi i} \int_{\Gamma_u} \frac{F_u(t)}{t - \xi_u} dt + \\ &+ \frac{1}{2\pi i} \sum_{s=m+1}^n \sum_{k=2}^{s-m+1} \int_{\Gamma_u} \frac{F_u(t) Q_{k,s}}{(t - \xi_{u_s})^k} dt I_s I_u, \end{aligned}$$

from which we obtain the equality

$$f_u \left(\sum_{u=1}^m I_u \frac{1}{2\pi i} \int_{\Gamma_u} F_u(t)(te_1 - \zeta)^{-1} dt \right) = F_u(\xi_u). \quad (28)$$

Operating onto the equality (27) by the functional f_u and taking into account the relations (18), (26), (28), we get the equality

$$f_u(\Phi_0(\zeta)) = F_u(\xi_u) - F_u(\xi_u) = 0$$

for all $u = 1, 2, \dots, m$, i. e. $\Phi_0(\zeta) \in \mathcal{R}$.

Therefore, the function Φ_0 is of the form

$$\Phi_0(\zeta) = \sum_{s=m+1}^n V_s(x, y, z) I_s, \quad (29)$$

where $V_s : \Omega \rightarrow \mathbb{C}$, and the Cauchy – Riemann conditions (8) are satisfied with $\Phi = \Phi_0$. Substituting the expressions (5), (29) into the equality (8), we obtain

$$\begin{aligned} \sum_{s=m+1}^n \frac{\partial V_s}{\partial y} I_s &= \sum_{s=m+1}^n \frac{\partial V_s}{\partial x} I_s \sum_{r=1}^n a_r I_r, \\ \sum_{s=m+1}^n \frac{\partial V_s}{\partial z} I_s &= \sum_{s=m+1}^n \frac{\partial V_s}{\partial x} I_s \sum_{r=1}^n b_r I_r. \end{aligned} \quad (30)$$

Equating the coefficients of I_{m+1} in these equalities, we obtain the following system of equations for determining the function $V_{m+1}(x, y, z)$:

$$\frac{\partial V_{m+1}}{\partial y} = \frac{\partial V_{m+1}}{\partial x} a_{u_{m+1}}, \quad \frac{\partial V_{m+1}}{\partial z} = \frac{\partial V_{m+1}}{\partial x} b_{u_{m+1}}.$$

It follows from Lemma 6 that $V_{m+1}(x, y, z) \equiv G_{m+1}(\xi_{u_{m+1}})$, where G_{m+1} is a function holomorphic in the domain $D_{u_{m+1}}$. Therefore,

$$\Phi_0(\zeta) = G_{m+1}(\xi_{u_{m+1}}) I_{m+1} + \sum_{s=m+2}^n V_s(x, y, z) I_s. \quad (31)$$

Due to the expansion (16), we have the representation

$$I_{m+1} \frac{1}{2\pi i} \int_{\Gamma_{u_{m+1}}} G_{m+1}(t)(te_1 - \zeta)^{-1} dt = G_{m+1}(\xi_{u_{m+1}}) I_{m+1} + \Psi(\zeta), \quad (32)$$

where $\Psi(\zeta)$ is a function with values in the set $\left\{ \sum_{k=m+2}^n \alpha_k I_k : \alpha_k \in \mathbb{C} \right\}$.

Now, consider the function

$$\Phi_1(\zeta) := \Phi_0(\zeta) - I_{m+1} \frac{1}{2\pi i} \int_{\Gamma_{u_{m+1}}} G_{m+1}(t)(te_1 - \zeta)^{-1} dt.$$

In view of the relations (31), (32), Φ_1 can be represented in the form

$$\Phi_1(\zeta) = \sum_{s=m+2}^n \tilde{V}_s(x, y, z) I_s,$$

where $\tilde{V}_s : \Omega \rightarrow \mathbb{C}$.

Inasmuch as Φ_1 is a monogenic function in Ω_ζ , the functions $\tilde{V}_{m+2}, \tilde{V}_{m+3}, \dots, \tilde{V}_n$ satisfy the system (30), where $V_{m+1} \equiv 0$, $V_s = \tilde{V}_s$ for $s = m+2, m+3, \dots, n$. Therefore, similarly to the function $V_{m+1}(x, y, z) \equiv G_{m+1}(\xi_{u_{m+1}})$, the function \tilde{V}_{m+2} satisfies the equations

$$\frac{\partial \tilde{V}_{m+2}}{\partial y} = \frac{\partial \tilde{V}_{m+2}}{\partial x} a_{u_{m+2}}, \quad \frac{\partial \tilde{V}_{m+2}}{\partial z} = \frac{\partial \tilde{V}_{m+2}}{\partial x} b_{u_{m+2}}$$

and is of the form $\tilde{V}_{m+2}(x, y, z) \equiv G_{m+2}(\xi_{u_{m+2}})$, where G_{m+2} is a function holomorphic in the domain $D_{u_{m+2}}$.

In such a way, step by step, considering the functions

$$\Phi_j(\zeta) := \Phi_{j-1}(\zeta) - I_{m+j} \frac{1}{2\pi i} \int_{\Gamma_{u_{m+j}}} G_{m+j}(t)(te_1 - \zeta)^{-1} dt$$

for $j = 2, 3, \dots, n - m - 1$, we get the representation (25) of the function Φ . The theorem is proved.

Taking into account the expansion (16), one can rewrite the equality (25) in the following equivalent form:

$$\begin{aligned} \Phi(\zeta) &= \sum_{u=1}^m F_u(\xi_u) I_u + \sum_{s=m+1}^n \sum_{k=2}^{s-m+1} \frac{1}{(k-1)!} Q_{k,s} F_{u_s}^{(k-1)}(\xi_{u_s}) I_s + \\ &+ \sum_{q=m+1}^n G_q(\xi_{u_q}) I_q + \sum_{q=m+1}^n \sum_{s=m+1}^n \sum_{k=2}^{s-m+1} \frac{1}{(k-1)!} Q_{k,s} G_q^{(k-1)}(\xi_{u_q}) I_q I_s. \end{aligned} \quad (33)$$

Thus, the equalities (25) and (33) specify methods to construct explicitly any monogenic functions $\Phi : \Omega_\zeta \rightarrow \mathbb{A}_n^m$ using n corresponding holomorphic functions of complex variables.

The following statement follows immediately from the equality (33) in which the right-hand side is a monogenic function in the domain $\Pi_\zeta := \{\zeta \in E_3 : f_u(\zeta) = D_u, u = 1, 2, \dots, m\}$.

Theorem 2. *Let a domain Ω be convex in the directions of the straight lines L_u and $f_u(E_3) = \mathbb{C}$ for all $u = 1, 2, \dots, m$. Then every monogenic function $\Phi : \Omega_\zeta \rightarrow \mathbb{A}_n^m$ can be continued to a function monogenic in the domain Π_ζ .*

The next statement is a fundamental consequence of the equality (33), and it is true for an arbitrary domain Ω_ζ .

Theorem 3. *Let $f_u(E_3) = \mathbb{C}$ for all $u = 1, 2, \dots, m$. Then for every monogenic function $\Phi : \Omega_\zeta \rightarrow \mathbb{A}_n^m$ in an arbitrary domain Ω_ζ , the Gateaux r -th derivatives $\Phi^{(r)}$ are monogenic functions in Ω_ζ for all r .*

The proof is completely analogous to the proof of Theorem 4 [8].

Using the integral expression (25) of monogenic function $\Phi : \Omega_\zeta \rightarrow \mathbb{A}_n^m$ in the case where a domain Ω is convex in the directions of the straight lines L_u , $u = 1, 2, \dots, m$, we obtain the following expression for the Gateaux r -th derivative $\Phi^{(r)}$:

$$\begin{aligned} \Phi^{(r)}(\zeta) &= \sum_{u=1}^m I_u \frac{r!}{2\pi i} \int_{\Gamma_u} F_u(t) \left((t - \zeta)^{-1} \right)^{r+1} dt + \\ &+ \sum_{s=m+1}^n I_s \frac{r!}{2\pi i} \int_{\Gamma_{u_s}} G_s(t) \left((t - \zeta)^{-1} \right)^{r+1} dt \quad \forall \zeta \in \Omega_\zeta. \end{aligned}$$

6 Special cases.

Note that in the cases where the algebra \mathbb{A}_n^m has some specific properties (for instance, properties described in Propositions 1 and 2), it is easy to simplify the form of the equality (33).

1. In the case considered in Proposition 1, the following equalities hold:

$$u_{m+1} = u_{m+2} = \dots = u_n =: \eta.$$

In this case the representation (33) takes the form

$$\begin{aligned} \Phi(\zeta) &= \sum_{u=1}^m F_u(\xi_u) I_u + \sum_{s=m+1}^n \sum_{k=2}^{s-m+1} \frac{1}{(k-1)!} Q_{k,s} F_\eta^{(k-1)}(\xi_\eta) I_s + \\ &+ \sum_{s=m+1}^n G_s(\xi_\eta) I_s + \sum_{q=m+1}^n \sum_{s=m+1}^n \sum_{k=2}^{s-m+1} \frac{1}{(k-1)!} Q_{k,s} G_q^{(k-1)}(\xi_\eta) I_s I_q. \end{aligned} \quad (34)$$

The formula (34) generalizes representations of monogenic functions in both three-dimensional harmonic algebras (see [8, 9, 10]) and specific n -dimensional algebras (see [13, 14]) to the case of algebras more general form.

2. In the case considered in Proposition 2, the functions $B_{r,p}$ from the equalities (11) are identically equal to zero. In this case the representation (16) takes the form

$$(te_1 - \zeta)^{-1} = \sum_{u=1}^m \frac{1}{t - \xi_u} I_u + \sum_{s=m+1}^n \frac{T_s}{(t - \xi_{u_s})^2} I_s, \quad (35)$$

and as a consequence of the equalities (25), (35), we obtain the following representation of monogenic function:

$$\Phi(\zeta) = \sum_{u=1}^m F_u(\xi_u) I_u + \sum_{s=m+1}^n G_s(\xi_{u_s}) I_s + \sum_{s=m+1}^n T_s F'_{u_s}(\xi_{u_s}) I_s. \quad (36)$$

The formula (36) generalizes representations of monogenic functions in both a three-dimensional harmonic algebra with one-dimensional radical (see [9]) and semi-simple algebras (see [10, 14]) to the case of algebras more general form.

3. In the case where $n = m$, the algebra \mathbb{A}_n^n is semi-simple and contains no nilpotent subalgebra. Then the formulas (34), (36) take the form

$$\Phi(\zeta) = \sum_{u=1}^n F_u(\xi_u) I_u,$$

because there are no vectors $\{I_k\}_{k=m+1}^n$. This formula was obtained in the paper [14].

7 The relations between monogenic functions and partial differential equations.

Consider the following linear partial differential equation with constant coefficients:

$$\mathcal{L}_N U(x, y, z) := \sum_{\alpha+\beta+\gamma=N} C_{\alpha,\beta,\gamma} \frac{\partial^N U}{\partial x^\alpha \partial y^\beta \partial z^\gamma} = 0, \quad C_{\alpha,\beta,\gamma} \in \mathbb{R}. \quad (37)$$

If a function $\Phi(\zeta)$ is N -times differentiable in the sense of Gateaux in every point of Ω_ζ , then

$$\frac{\partial^{\alpha+\beta+\gamma} \Phi}{\partial x^\alpha \partial y^\beta \partial z^\gamma} = e_1^\alpha e_2^\beta e_3^\gamma \Phi^{(\alpha+\beta+\gamma)}(\zeta) = e_2^\beta e_3^\gamma \Phi^{(N)}(\zeta).$$

Therefore, due to the equality

$$\mathcal{L}_N \Phi(\zeta) = \Phi^{(N)}(\zeta) \sum_{\alpha+\beta+\gamma=N} C_{\alpha,\beta,\gamma} e_2^\beta e_3^\gamma, \quad (38)$$

every N -times differentiable in the sense of Gateaux in Ω_ζ function Φ satisfies the equation $\mathcal{L}_N \Phi(\zeta) = 0$ everywhere in Ω_ζ if and only if

$$\sum_{\alpha+\beta+\gamma=N} C_{\alpha,\beta,\gamma} e_2^\beta e_3^\gamma = 0. \quad (39)$$

Accordingly, if the condition (39) is satisfied, then the real-valued components $\operatorname{Re} U_k(x, y, z)$ and $\operatorname{Im} U_k(x, y, z)$ of the decomposition (7) are solutions of the equation (37).

In the case where $f_u(E_3) = \mathbb{C}$ for all $u = 1, 2, \dots, m$, it follows from Theorem 3 that the equality (38) holds for every monogenic function $\Phi : \Omega_\zeta \rightarrow \mathbb{A}_n^m$.

Thus, to construct solutions of the equation (37) in the form of components of monogenic functions, we must to find a triad of linearly independent over the field \mathbb{R} vectors (5) satisfying the characteristic equation (39) and to verify the condition: $f_u(E_3) = \mathbb{C}$ for all $u = 1, 2, \dots, m$. Then, the formula (25) gives a constructive description of all mentioned monogenic functions.

In the next theorem, we assign a special class of equations (37) for which $f_u(E_3) = \mathbb{C}$ for all $u = 1, 2, \dots, m$. Let us introduce the polynomial

$$P(a, b) := \sum_{\alpha+\beta+\gamma=N} C_{\alpha,\beta,\gamma} a^\beta b^\gamma. \quad (40)$$

Theorem 4. *Suppose that there exist linearly independent over the field \mathbb{R} vectors e_1, e_2, e_3 in \mathbb{A}_n^m of the form (5) that satisfy the equality (39). If $P(a, b) \neq 0$ for all real a and b , then $f_u(E_3) = \mathbb{C}$ for all $u = 1, 2, \dots, m$.*

Proof. Using the multiplication table of \mathbb{A}_n^m , we obtain the equalities

$$e_2^\beta = \sum_{u=1}^m a_u^\beta I_u + \Psi_{\mathcal{R}}, \quad e_3^\gamma = \sum_{u=1}^m b_u^\gamma I_u + \Theta_{\mathcal{R}},$$

where $\Psi_{\mathcal{R}}, \Theta_{\mathcal{R}} \in \mathcal{R}$. Now the equality (39) takes the form

$$\sum_{\alpha+\beta+\gamma=N} C_{\alpha,\beta,\gamma} \left(\sum_{u=1}^m a_u^\beta b_u^\gamma I_u + \tilde{\Psi}_{\mathcal{R}} \right) = 0, \quad (41)$$

where $\tilde{\Psi}_{\mathcal{R}} \in \mathcal{R}$. Moreover, due to the assumption that the vectors e_1, e_2, e_3 of the form (5) satisfy the equality (39), there exist complex coefficients a_k, b_k for $k = 1, 2, \dots, n$ that satisfy the equality (41).

It follows from the equality (41) that

$$\sum_{\alpha+\beta+\gamma=n} C_{\alpha,\beta,\gamma} a_u^\beta b_u^\gamma = 0, \quad u = 1, 2, \dots, m. \quad (42)$$

Since $P(a, b) \neq 0$ for all $a, b \in \mathbb{R}$, the equalities (42) can be satisfied only if for each $u = 1, 2, \dots, m$ at least one of the numbers a_u or b_u belongs to $\mathbb{C} \setminus \mathbb{R}$ that implies the relation $f_u(E_3) = \mathbb{C}$ for all $u = 1, 2, \dots, m$. The theorem is proved.

Note that if $P(a, b) \neq 0$ for all $a, b \in \mathbb{R}$, then $C_{N,0,0} \neq 0$ because otherwise $P(a, b) = 0$ for $a = b = 0$.

Since the function $P(a, b)$ is continuous on \mathbb{R}^2 , the condition $P(a, b) \neq 0$ means either $P(a, b) > 0$ or $P(a, b) < 0$ for all $a, b \in \mathbb{R}$. Therefore, it is obvious that for any equation (37) of elliptic type, the condition $P(a, b) \neq 0$ is always satisfied for all $a, b \in \mathbb{R}$. At the same time, there are equations (37) for which $P(a, b) > 0$ for all $a, b \in \mathbb{R}$, but which are not elliptic. For example, such are the equations

$$\frac{\partial^3 u}{\partial x^3} + \frac{\partial^3 u}{\partial x \partial y^2} + \frac{\partial^3 u}{\partial x \partial z^2} = 0 \quad \text{and} \quad \frac{\partial^5 u}{\partial x^5} + \frac{\partial^5 u}{\partial x^3 \partial y^2} + \frac{\partial^5 u}{\partial x \partial y^2 \partial z^2} = 0$$

considered in \mathbb{R}^3 , in particular.

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