Holomorphic functions in generalized Cayley-Dickson algebras

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Abstract. In this paper we investigated some properties of holomorphic functions (belonging to the kernel of the Dirac operator) defined on domains of the real Cayley-Dickson algebras. For this purpose, we study first some properties of these algebras, especially multiplication tables for certain elements of the basis. Using these properties, we provided an algorithm for constructing examples of the class of functions under consideration.

Keywords: Cayley-Dickson and generalized Cayley-Dickson algebras; Dirac operator; holomorphic functions.

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0. Introduction

The theory of quaternionic differentiable functions has many applications in different areas of mathematics, physics and in other applied sciences (see, for example, [3], [4]). This theory has its origins in the paper [7] in which the authors proposed, for the first time, an analogue of the Cauchy-Riemann conditions in three-dimensional case. For the four-dimensional case, an analogue of these conditions was considered in the paper [2] and, as a next step of this generalization, the differentiable functions in the octonionic algebra was considered in the papers [11], [12].

Generalization of the Cauchy-Riemann conditions in all algebras obtained by the Cayley-Dickson process (called Cayley-Dickson algebras) was done in the paper [6], where differentiable functions of variables belonging to Cayley-Dickson algebras were defined. For such functions, was established analogues results with the main results of complex analysis, results which can be successfully used in the further studies of special functions of variables with values in Cayley–Dickson algebras.

Comparing with [6], in the present paper, we investigate another class of differentiable functions (using the Dirac operator) in Cayley-Dickson algebras and, more important, we provide an example of this kind of functions and an algorithm to find such as examples. Since these functions are rather complicated objects, it is quite important to have a way to generate examples.

The paper is organized in two sections. In the first section, we briefly presented some properties of algebras obtained by the Cayley-Dickson process and the algorithm described by J. W. Bales regarding to an easy way to multiply the elements from a basis in such algebras (by using *exclusive or* operation and a *twist map).* In the second section, by description the multiplication tables for certain elements of the basis (Propositions 2.2 and 2.3), we obtained the main result of this work: an example of a left hyperholomorphic function in generalized Cayley-Dickson algebras (Theorem 2.12). Moreover, in the Thorem 2.10 we proved that for studying left A_t-holomorphic functions in generalized Cayley-Dickson algebras $A_t = \left(\frac{\gamma_1,\ldots,\gamma_t}{\mathbb{R}}\right)$ with $\gamma_1 < 0, \ldots, \gamma_t < 0$. it is suffices to consider left A_t-holomorphic functions only in the algebras $\left(\frac{-1,\ldots,-1}{\mathbb{R}}\right)$.

1. Preliminaries

Let K be a commutative field with $char K \neq 2$ and A be an algebra over the field K. A unitary algebra $A \neq K$ such that we have $x^2 + \alpha_x x + \beta_x = 0$, for each $x \in A$, with $\alpha_x, \beta_x \in K$, is called a *quadratic algebra.*

In the following, we briefly present the *Cayley-Dickson process* and the properties of the algebras obtained. For details about the Cayley-Dickson process, the reader is referred to [9] and [10].

Let A be a finite dimensional unitary algebra over a field K with a *scalar involution*

$$\overline{} : A \to A, \quad a \to \overline{a},$$

i. e. a linear map satisfying the following relations:

$$\overline{ab} = \overline{b}\overline{a}, \quad \overline{\overline{a}} = a,$$

and

$$a + \overline{a}, a\overline{a} \in K \cdot 1 \text{ for all } a, b \in A.$$

The element $\overline{a}$ is called the *conjugate* of the element a, the linear form

$$t : A \to K, \quad t(a) = a + \overline{a}$$

and the quadratic form

$$n : A \to K, \quad n(a) = a\overline{a}$$

are called the *trace* and the *norm* of the element a. Hence an algebra A with a scalar involution is quadratic.

Let $\gamma \in K$ be a fixed non-zero element. We define the following algebra multiplication on the vector space

$$A \oplus A : (a_1, a_2)(b_1, b_2) := \left(a_1 b_1 + \gamma b_2 \overline{a_2}, \overline{a_1} b_2 + b_1 a_2\right). \tag{1}$$

We obtain an algebra structure over $A \oplus A$, denoted by (A, γ) and called the *algebra obtained from A by the Cayley-Dickson process* or simply *generalized Cayley-Dickson algebra.* We have $\dim (A, \gamma) = 2 \dim A$.

Let $x \in (A, \gamma)$, $x = (a_1, a_2)$. The map

$$^{-} : (A, \gamma) \to (A, \gamma) \ , \ x \to \bar{x} = (\overline{a}_1, \text{-}a_2) \, ,$$

is a scalar involution of the algebra (A, γ), extending the involution $^{-}$ of the algebra A.

If we take $A = K$ and apply this process t times, $t \geq 1$, we obtain an algebra over K,

$$A_t = \left(\frac{\gamma_1, ..., \gamma_t}{K} \right).$$

By induction in this algebra, the set $\{e_0 = 1, e_1, ..., e_{n-1}\}, n = 2^t$, generates a basis with the properties:

$$e_i^2 = \gamma_i 1, \ \gamma_i \in K, \gamma_i \neq 0, \ i = 1, ..., n-1 \tag{2}$$

and

$$e_i e_j = -e_j e_i = \beta_{ij} e_k, \ \beta_{ij} \in K, \ \ \beta_{ij} \neq 0, \ \ i \neq j, \ \ i, j = \ 1, ...n-1, \tag{3}$$

β_{ij} and e_k being uniquely determined by e_i and e_j.

From [10], Lemma 4, it results that in an algebra A_t with the basis $B = \{e_0 = 1, e_1, ..., e_{n-1}\}$ satisfying relations (2) and (3) we have:

$$e_i \left(e_i x\right) = \gamma_i^2 x = (x e_i) e_i, \tag{4}$$

for all $i \in \{1, 2, ..., n-1\}$ and for every $x \in A$.

The algebras A_t, in general, are neither commutative and nor associative algebras, but are *flexible* (i. e. $x(yx) = (xy)x = xyx$, for all $x, y \in A_t$) quadratic and *power associative* (i. e. the subalgebra $< x >$ of A, generated by any element $x \in A$, is associative).

Remark 1.1. For $\gamma_1 = ... = \gamma_t = -1$ and $K = \mathbb{R}$, in [1], the author described how we can multiply the basis vectors in the algebra $A_t, \dim A_t = 2^t = n$. He used the binary decomposition for the subscript indices.

Let e_p, e_q be two vectors in the basis B with p, q representing the binary decomposition for the indices of the vectors, that means p, q are in $\mathbb{Z}_2^n$. We have that $e_p e_q = \gamma_n \left(p, q\right) e_{p \otimes q}$, where:

i) $p \otimes q$ are the sum of p and q in the group $\mathbb{Z}_2^n$ or, more precisely, the "*exclusive or*" for the binary numbers p and q;

ii) γ_n is a function $\gamma_n : \mathbb{Z}_2^n \times \mathbb{Z}_2^n \to \{-1, 1\}$.

The map γ_n is called the *twist map.*

The elements of the group $\mathbb{Z}_2^n$ can be considered as integers from 0 to $2^n - 1$ with multiplication "*exclusive or*" of the binary representations. Obviously, this operation is equivalent with the addition in $\mathbb{Z}_2^n$.

From now on, in whole the paper, we will consider $K=\mathbb{R}$. Using the same notations as in the Bales's paper, we consider the following matrices:

$$A_0 = A = \left(\begin{array}{cc} 1 & 1 \\ 1 & -1 \end{array} \right), \quad B = \left(\begin{array}{cc} 1 & -1 \\ 1 & 1 \end{array} \right), \quad C = \left(\begin{array}{cc} 1 & -1 \\ -1 & -1 \end{array} \right). \tag{5}$$

In the same paper [1], the author find the properties of the twist map γ_n and put the signs of this map in a table. He partitioned the twist table for $\mathbb{Z}_2^n$ into 2×2 matrices and obtained the following result:

Theorem 1.2. ([1], Theorem 2.2., p. 88-91) *For $n>0$, the Cayley-Dickson twist table γ_n can be partitioned in quadratic matrices of dimension 2 of the form $A, B, C, -B, -C$, defined in the relation (5). Relations between them can be found in the below twist trees*:

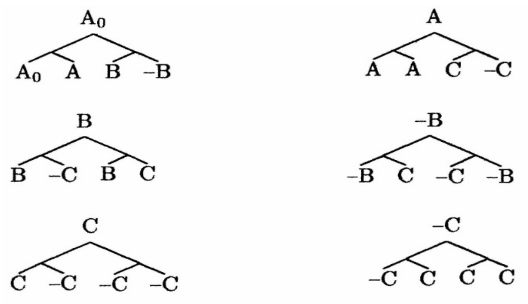

Fig. 1: Twist trees([1], Table 9)

□

Definition 1.3. Let $x = x_0, x_1, x_2, ...$ and $y = y_0, y_1, y_2, ...$ be two sequences of real numbers. The ordered pair

$$(x, y) = x_0, y_0, x_1, y_1, x_2, y_2, ...$$

is a sequence obtained by *shuffling* the sequences x and y.

Using Theorem 1.2, in [1], the author gave the below algorithm for find $\gamma_n(s,r)$, where $s, r \in \mathbb{Z}_2^n$:

i) We find the shuffling sequence (s,r).

ii) Starting with the root A_0, we can find $\gamma_n(s,r)$ using the twist tree. We remark that "00"= unchanged, "01" =left →right, "10"=right→ left, "11"=right → right.

Let $\mathbb{H}(\gamma_1, \gamma_2)$ be the generalized quaternion algebra and $\mathbb{H}(-1,-1)$ be the quaternion division algebra. Below, you can see the multiplication tables:

$\cdot$	1	e_1	e_2	e_3
1	1	e_1	e_2	e_3
e_1	e_1	γ_1	e_3	$\gamma_1 e_2$
e_2	e_2	$-e_3$	γ_2	$-\gamma_2 e_1$
e_3	e_3	$-\gamma_1 e_2$	$\gamma_2 e_1$	$-\gamma_1\gamma_2$

Multiplication table for the generalized quaternion algebra

$\cdot$	1	e_1	e_2	e_3
1	1	e_1	e_2	e_3
e_1	e_1	-1	e_3	$-e_2$
e_2	e_2	$-e_3$	-1	e_1
e_3	e_3	e_2	$-e_1$	-1

Multiplication table for the real division quaternion algebra

$\cdot$	1	e_1	e_2	e_3
1	1	1	1	1
e_1	1	-1	1	-1
e_2	1	-1	-1	1
e_3	1	1	-1	-1

Quaternion twist table

$$\begin{pmatrix} A_0 & A \\ B & -B \end{pmatrix}$$

Quaternion twist table using notations from Theorem 1.2.

Example 1.4. Let A_4 be the real sedenion algebra. That means $\dim A_4 = 16$ with $\{1, e_1, ..., e_{15}\}$ a basis in this algebra. Let compute $e_7 e_{13} = \gamma_4(7_2, 13_2)e_{7\otimes 13}$. We have the following binary decompositions:

$$\begin{aligned} 7_2 &= 0111, \text{ since } 7 = 2^2 + 2 + 1 \text{ and} \\ 13_2 &= 1101, \text{ since } 13 = 2^3 + 2^2 + 1. \end{aligned}$$

Since $0111 \otimes 1101 = 1010 (= 2^3 + 2 = 10)$, it results that $7 \otimes 13 = 10$.

Now, we compute $\gamma_4 (e_7, e_{13})$. First, we shuffle the sequences 0111 and 1101. We obtain 01 11 10 11. Starting with A_0, it results: $A_0 \overset{01}{\to} A \overset{11}{\to} -C \overset{10}{\to} C \overset{11}{\to} -C$, then $\gamma_4 (e_7, e_{13}) = -1$ and $e_7 e_{13} = -e_{10}$.

2. Main results

In this section, for a generalized Cayley-Dickson algebra A_t, writing the basis's elements in a convenient way, we can obtain multiplication tables for certain elements of the basis. Using these results, in Theorem 2.12 we provide an example of a left hyperholomorphic function in generalized Cayley-Dickson algebras.

2.1. Multiplication table in generalized Cayley-Dickson algebras.

Remark 2.1. i) In the generalized quaternion algebra, $\mathbb{H}(\gamma_1, \gamma_2)$, the basis can be written as

$$\{1 = e_0, e_1, e_2, e_1 e_2\}.$$

For the generalized octonion algebra, $\mathbb{O}(\gamma_1, \gamma_2, \gamma_3)$, the basis can be written

$$\{1 = e_0, e_1, e_2, e_1 e_2, e_4, e_1 e_4, e_2 e_4, (e_1 e_2) e_4\}.$$

Therefore $e_3 = e_1 e_2, e_7 = e_3 e_4 = (e_1 e_2) e_4$, $e_2 e_4 = e_6$ and, when compute them, in these products do not appear any of the elements $\gamma_1, \gamma_2, \gamma_3$, or products of some of them at the end.

We remark that in the algebra $A_t = \left(\frac{\gamma_1, \ldots, \gamma_t}{\mathbb{R}}\right)$ in the products of the form

$$e_1 e_2, (e_1 e_2) e_4, ..., ((e_{2^r} e_{2^{r+1}}) \ldots e_{2^k}) e_{2^i},$$

when compute them, do not appear any of the elements $\gamma_1, \gamma_2, ..., \gamma_t$ or products of some of them at the end.

ii) Using above remarks, the basis in the algebra $A_t = \left(\frac{\gamma_1, \ldots, \gamma_t}{\mathbb{R}}\right)$ can be written under the form

$$\{1 = e_0, e_1, e_2, \ldots, e_{2^{t-1}-1}, e_{2^{t-1}}, e_1 e_{2^{t-1}}, e_2 e_{2^{t-1}}, e_3 e_{2^{t-1}}, \ldots, e_{2^{t-1}-1} e_{2^{t-1}}\} \tag{6}$$

with

$$e_i e_{2^{t-1}} = -e_{2^{t-1}} e_i = e_{2^{t-1}} \overline{e}_i, \quad i \in \{1, 2, \ldots, 2^{t-1} - 1\}. \tag{7}$$

Proposition 2.2. *Let $A_t = \left(\frac{\gamma_1, \ldots, \gamma_t}{\mathbb{R}}\right)$ be an algebra obtained by the Cayley-Dickson process and $\{e_0 = 1, e_1, ..., e_{n-1}\}$, $n = 2^t$ be a basis. Let $r \geq 1$, $r < k \leq i < t$. Therefore*

$$((e_{2^r} e_{2^{r+1}}) \ldots e_{2^k}) e_{2^i} = (-1)^{k-r+2} e_T, \tag{8}$$

$$((e_1 e_{2^r}) e_{2^{r+1}}) \ldots e_{2^k}) e_{2^i} = (-1)^{k-r+3} e_{T+1}, \tag{9}$$

where $T = 2^r + 2^{r+1} + \ldots + 2^k + 2^i$ *and*

$$e_1 e_{2^i} = e_{2^i+1}. \tag{10}$$

.

Proof. From Remark 2.1, it results that we can use Theorem 1.2 for $\gamma_1, \gamma_2, \ldots, \gamma_t$ arbitrary. From Remark 1.1, it results $T = 2^r + 2^{r+1} + \ldots + 2^k + 2^i$. For T, we have the binary decomposition

$$T_2 = 1\underbrace{00\ldots0}_{i-k-1}\underbrace{111\ldots1}_{k-r+1}\underbrace{0\ldots0}_{r}.$$

Using the same remark, we obtain $e_{2^r} e_{2^{r+1}} = \gamma_n \left(\underbrace{01...0}_{r+2}, \underbrace{10\ldots0}_{r+2}\right) e_{2^r+2^{r+1}}$. We "shuffling" $\underbrace{01\ldots0}_{r+2}$ and $\underbrace{10\ldots0}_{r+2}$ and we obtain 01 10 $\underbrace{00\ 00\ldots00\ 00}_{r \text{ pairs}}$. Starting with A_0, it results:

$$A_0 \overset{01}{\to} A \overset{10}{\to} C,$$

then $\gamma_n \left(\underbrace{01\ldots0}_{r+2}, \underbrace{10\ldots0}_{r+2}\right) = 1$ and $e_{2^r} e_{2^{r+1}} = e_{2^r+2^{r+1}}$.

We compute $(e_{2^r} e_{2^{r+1}}) e_{2^{r+2}}$. We obtain

$$(e_{2^r} e_{2^{r+1}}) e_{2^{r+2}} = e_{2^r+2^{r+1}} e_{2^{r+2}} = \gamma_n \left(\underbrace{011...0}_{r+3}, \underbrace{10...0}_{r+3}\right) e_{2^r+2^{r+1}+2^{r+2}}.$$

Shuffling $\underbrace{011...0}_{r+3}$ and $\underbrace{10...0}_{r+3}$, we get 01 10 10$\underbrace{00\ 00...00\ 00}_{r \text{ pairs}}$. Starting with A_0, it results: $A_0 \overset{01}{\to} A \overset{10}{\to} C \overset{10}{\to} -C$, then

$$\gamma_n \left(\underbrace{011...0}_{r+3}, \underbrace{10...0}_{r+3}\right) = -1,$$

therefore $e_{2^r+2^{r+1}} e_{2^{r+2}} = -e_{2^r+2^{r+1}+2^{r+2}}$. Continuing this procedure, we remark that the number of "1" in the "shuffling" obtained influences the sign. Since $T = 2^r + 2^{r+1} + ... + 2^k + 2^i$ has binary decomposition

$$T_2 = 1\underbrace{00...0}_{i-k-1}\underbrace{111..1}_{k-r+1}\underbrace{0...0}_{r},$$

in which we have $k - r + 2$ elements equal with 1, we obtain relation (8). In the same way it results relations (9) and (10). □

Proposition 2.3. *With the same notations as in Proposition 2.2, for the algebra* $A_t = \left(\frac{-1,\ldots,-1}{\mathbb{R}}\right)$, *we have:*

$$\begin{array}{c|ll} \cdot & e_T & e_{T+1} \\ \hline e_{T_1} & (-1)^{k-r+1} e_{2^i} & -(-1)^{k-r+1} e_{2^i+1} \\ e_{T_1+1} & -(-1)^{k-r+1} e_{2^i+1} & -(-1)^{k-r+1} e_{2^i} \end{array} \qquad (11)$$

for $r < k$, *where* $T = 2^r + 2^{r+1} + \ldots + 2^k + 2^i$, $T_1 = 2^r + 2^{r+1} + \ldots + 2^k$ *and*

$$\begin{array}{c|ll} \cdot & e_T & e_{T+1} \\ \hline e_{2^k} & e_M & -e_{M+1} \\ e_{2^k+1} & -e_{M+1} & -e_M \end{array}, \qquad (12)$$

where $M = 2^k + 2^i$.

Proof. *Case 1*: $r < k$. We compute $e_{T_1} e_T$. We have $e_{T_1} e_T = \gamma(s,q)\, e_M$, where s, q are the binary decomposition of T_1 and T. The binary decomposition of M is $M_2 = T_1 \otimes T$. It results $M = 2^i$,

$$s = \underbrace{00...0}_{i-k}\underbrace{111...1}_{k-r+1}\underbrace{0...0}_{r}, \quad q = \underbrace{100...0}_{i-k}\underbrace{111...1}_{k-r+1}\underbrace{0...0}_{r}.$$

By "shuffling" $s \otimes q$, we obtain

$$\underbrace{01\ 00\ 00...00}_{(i-k)\text{ pairs}}\ \underbrace{11\ 11\ 11\ ...11}_{(k-r+1)\text{ pairs}}\ \underbrace{00\ 00\ ...00\ 00}_{r\text{ pairs}}.$$

Starting with A_0, we get:

$$\underbrace{A_0 \stackrel{01}{\to} A \stackrel{00}{\to} \ldots \stackrel{00}{\to}}_{i-k} \underbrace{A \stackrel{11}{\to} -C \stackrel{11}{\to} C \stackrel{11}{\to} -C \stackrel{11}{\to} C \stackrel{11}{\to} \ldots \stackrel{11}{\to} (-1)^{k-r+1} C}_{k-r+1} \underbrace{\stackrel{00}{\to} \ldots \stackrel{00}{\to} (-1)^{k-r+1} C}_{r}.$$

Therefore $\gamma(s,q) = (-1)^{k-r+1}$.

Now, we compute $e_{T_1} e_{T+1}$. For this, we will "shuffling" $\underbrace{00...0}_{i-k}\underbrace{111...1}_{k-r+1}\underbrace{0...0}_{r}$ with $\underbrace{100...0}_{i-k}\underbrace{111...1}_{k-r+1}\underbrace{0...1}_{r}$. It results

$$\underbrace{01\ 00\ 00...00}_{(i-k)\text{ pairs}}\ \underbrace{11\ 11\ 11...11}_{(k-r+1)\text{ pairs}}\ \underbrace{00\ 00...00\ 01}_{r\text{ pairs}}.$$

Starting with A_0, we get:

$$\underbrace{A_0 \stackrel{01}{\to} A \stackrel{00}{\to} \ldots \stackrel{00}{\to}}_{i-k} \underbrace{A \stackrel{11}{\to} -C \stackrel{11}{\to} C \stackrel{11}{\to} -C \stackrel{11}{\to} C \stackrel{11}{\to} \ldots \stackrel{11}{\to} (-1)^{k-r+1} C}_{k-r+1} \underbrace{\stackrel{00}{\to} \ldots \stackrel{01}{\to} -(-1)^{k-r+1} C}_{r}.$$

For $e_{T_1+1}e_T$, "shuffling" $\underbrace{00...0}_{i-k}\underbrace{111...1}_{k-r+1}\underbrace{0...1}_{r}$ with $\underbrace{100...0}_{i-k}\underbrace{111...1}_{k-r+1}\underbrace{0...0}_{r}$, it results

$$\underbrace{01\ 00\ 00...00}_{(i-k)\text{ pairs}}\ \underbrace{11\ 01\ 01...01}_{(k-r+1)\text{ pairs}}\ \underbrace{00\ 00...00\ 10}_{r\text{ pairs}}.$$

Starting with A_0,we get:

$$\underbrace{A_0 \overset{01}{\to} A \overset{00}{\to} ... \overset{00}{\to}}_{i-k}\underbrace{A \overset{11}{\to} -C \overset{11}{\to} C \overset{11}{\to} -C \overset{11}{\to} C \to ... \overset{11}{\to} (-1)^{k-r+1}C}_{k-r+1}\underbrace{\overset{00}{\to} ... \overset{10}{\to} -(-1)^{k-r+1}C}_{r}.$$

For $e_{T_1+1}e_{T+1}$, we compute first $(T_1+1)\otimes(T+1)$. We obtain:

$$\left(2^r+2^{r+1}+...+2^k+1\right)\otimes\left(2^r+2^{r+1}+...+2^k+2^i+1\right)=$$

$$=\left(\underbrace{00...0}_{i-k}\underbrace{111...1}_{k-r+1}\underbrace{0...1}_{r}\right)\otimes\left(\underbrace{100...0}_{i-k}\underbrace{111...1}_{k-r+1}\underbrace{0...1}_{r}\right)=$$

$$=\quad \underbrace{10...0}_{i-k}\underbrace{000...0}_{k-r+1}\underbrace{0...0}_{r}=2^i.$$

Now, "shuffling" $\underbrace{00...0}_{i-k}\underbrace{111...1}_{k-r+1}\underbrace{0...1}_{r}$ with $\underbrace{100...0}_{i-k}\underbrace{111...1}_{k-r+1}\underbrace{0...1}_{r}$, it results

$$\underbrace{01\ 00\ 00...00}_{(i-k)\text{ pairs}}\ \underbrace{11\ 01\ 01...01}_{(k-r+1)\text{ pairs}}\ \underbrace{00\ 00...00\ 11}_{r\text{ pairs}}.$$

Starting with A_0, we get:

$$\underbrace{A_0 \overset{01}{\to} A \overset{00}{\to} ... \overset{00}{\to}}_{i-k}\underbrace{A \overset{11}{\to} -C \overset{11}{\to} C \overset{11}{\to} -C \overset{11}{\to} C \overset{11}{\to} ... \overset{11}{\to} (-1)^{k-r+1}C}_{k-r+1}\underbrace{\overset{00}{\to} ... \overset{11}{\to} -(-1)^{k-r+1}C}_{r}.$$

Case 2: $r=k$. We have $M=2^k\otimes T=2^i+2^k$. For $e_{2^k}e_T$, "shuffling" $\underbrace{00...0}_{i-k}\underbrace{10...0}_{k+1}$ with $\underbrace{100...0}_{i-k}\underbrace{0...0}_{k+1}$, it results

$$\underbrace{01\ 00\ 00...00}_{(i-k)\text{ pairs}}\ \underbrace{10\ 00\ 00\ ...00}_{(k+1)\text{ pairs}}.$$

Starting with A_0, we get:

$$\underbrace{A_0 \overset{01}{\to} A \overset{00}{\to} ..... \overset{00}{\to}}_{i-k}\underbrace{A \overset{10}{\to} C \overset{00}{\to} C \overset{00}{\to} ... \overset{00}{\to} C}_{k+1}.$$

For $e_{2^k}e_{T+1}$, "shuffling" $\underbrace{00...0}_{i-k}\underbrace{10...0}_{k+1}$ with $\underbrace{100...0}_{i-k}\underbrace{0...1}_{k+1}$, it results

$$\underbrace{01\ 00\ 00...00}_{(i-k)\text{ pairs}}\ \underbrace{10\ 00\ 00\ ...01}_{(k+1)\text{ pairs}}.$$

Starting with A_0, we get:

$$\underbrace{A_0 \overset{01}{\to} A \overset{00}{\to}}_{i-k} ... \overset{00}{\to} \underbrace{A \overset{10}{\to} C \overset{00}{\to} C \overset{00}{\to} ... \overset{01}{\to} -C}_{k+1}.$$

etc.$\Box$

Proposition 2.4. *Let* $A_t = \left(\frac{\gamma_1,...,\gamma_t}{\mathbb{R}}\right)$ *be an algebra obtained by the Cayley-Dickson process. For any* $x_1, x_2, ..., x_t \in \mathbb{R} - \{0\}$, *we have that*

$$\left(\frac{\gamma_1, ..., \gamma_t}{\mathbb{R}}\right) \simeq \left(\frac{\gamma_1 x_1^2, ..., \gamma_t x_t^2}{\mathbb{R}}\right).$$

Proof. Let $A_t = \left(\frac{\gamma_1,...,\gamma_t}{\mathbb{R}}\right)$ with the basis $\{e_0 = 1, e_1, ..., e_{n-1}\},\ n = 2^t$ and let $A_t' = \left(\frac{\gamma_1 x_1^2,...,\gamma_t x_t^2}{\mathbb{R}}\right)$ with the basis $\{e_0' = 1, e_1', ..., e_{n-1}'\}$ such that $(e_i')^2 = \gamma_i x_i^2,\ i \in \{1, 2, ..., n-1\}$. We remark that $(x_i e_i)^2 = x_i^2 \gamma_i$ and from here, it results that the map $\tau : A_t' \to A_t,\ \tau(e_i') = e_i x_i$ is an algebra isomorphism. $\Box$

The above proposition generalized Proposition 1.1, p. 52 from [5].

Remark 2.5. From Proposition 2.4, it results that for each $n = 2^t$ there are only n non-isomorphic algebras A_t. These algebras are of the form $A_t = \left(\frac{\gamma_1,...,\gamma_t}{\mathbb{R}}\right)$, with $\gamma_1, ..., \gamma_t \in \{-1, 1\}$.

2.2. A_t-holomorphic functions.

Definition 2.6. Let $\{e_0 = 1, e_1, ..., e_{n-1}\}$ be a basis in $A_t = \left(\frac{\gamma_1,...,\gamma_t}{\mathbb{R}}\right)$, $n = 2^t$. To domain $\Omega \subset \mathbb{R}^{2^t-1}$ we will associate the domain $\Omega_\zeta := \{\zeta = x_1 e_1 + \ldots + x_{n-1}e_{n-1} : (x_1, x_2, \ldots, x_{n-1}) \in \Omega\}$ included in A_t.

Consider a function $\Phi : \Omega_\zeta \to A_t$ of the form

$$\Phi(\zeta) = \sum_{k=1}^{n-1} \Phi_k(x_1, x_2, \ldots, x_{n-1})e_k, \tag{13}$$

where $(x_1, x_2, \ldots, x_{n-1}) \in \Omega$ and $\Phi_k : \Omega \to \mathbb{R}$.

We say that a function of the form (13) is *left* A_t-*holomorphic* in a domain Ω_ζ if the first partial derivatives $\partial\Phi_k/\partial x_k$ exist in Ω and the following equality is fulfilled in every point of Ω_ζ:

$$D[\Phi](\zeta) = \sum_{k=1}^{2^t-1} e_k \frac{\partial \Phi}{\partial x_k} = 0.$$

The operator D is called *Dirac operator*. Note that if A_t is the generalized quaternion algebra, then the left A_t-holomorphic functions is also called *hyperholomorphic*. We also note that every hyperholomorphic function Φ in a domain Ω_ζ is a solution of the equation

$$\gamma_1 \frac{\partial^2 \Phi}{\partial x_1^2} + \gamma_2 \frac{\partial^2 \Phi}{\partial x_2^2} + \gamma_1\gamma_2 \frac{\partial^2 \Phi}{\partial x_3^2} = 0.$$

Remark 2.7. Let $\mathbb{H}(\gamma_1, \gamma_2)$ be the generalized quaternion algebra with the basis $\{1, e_1, e_2, e_3\}$, $\gamma_1 < 0$, $\gamma_2 < 0$ and $\mathbb{H}(-1, -1)$ be the usual quaternion division algebra with the basis $\{1, i, j, k\}$. Let Ω be a domain in $\mathbb{R}^3$, and let $\Omega_\zeta := \{\zeta = xi+yj+zk : (x, y, z) \in \Omega\}$ be a corresponding domain in $\mathbb{H}(-1, -1)$. The function $\Phi : \Omega_\zeta \to \mathbb{H}(-1, -1)$ of the form

$$\Phi(\zeta) = u_1(x, y, z) + u_2(x, y, z)\, i + u_3(x, y, z)\, j + u_4(x, y, z)\, k.$$

is hyperholomorphic in the domain Ω if

$$D[\Phi](\zeta) = i\frac{\partial \Phi}{\partial x} + j\frac{\partial \Phi}{\partial y} + k\frac{\partial \Phi}{\partial z} = 0.$$

For another domain $\Delta \subset \mathbb{R}^3$, we associate the domain $\Delta_{\widetilde{\zeta}} := \{\widetilde{\zeta} = \widetilde{x}e_1+\widetilde{y}e_2+\widetilde{z}e_3 : (\widetilde{x}, \widetilde{y}, \widetilde{z}) \in \Delta\}$ in the algebra $\mathbb{H}(\gamma_1, \gamma_2)$. The Dirac operator in $\mathbb{H}(\gamma_1, \gamma_2)$, denoted by $\widetilde{D}$, is

$$\widetilde{D} := e_1\frac{\partial}{\partial \widetilde{x}} + e_2\frac{\partial}{\partial \widetilde{y}} + e_3\frac{\partial}{\partial \widetilde{z}}.$$

The elements of bases in $\mathbb{H}(-1, -1)$ and $\mathbb{H}(\gamma_1, \gamma_2)$ satisfy the following equalities:

$$e_1 = i\sqrt{-\gamma_1}, \quad e_2 = j\sqrt{-\gamma_2}, \quad e_3 = k\sqrt{\gamma_1\gamma_2}. \tag{14}$$

Now we establish a connection between hyperholomorphic functions in the algebras $\mathbb{H}(-1, -1)$ and $\mathbb{H}(\gamma_1, \gamma_2)$, where $\gamma_1 < 0$, $\gamma_2 < 0$. For this, we denote

$$x = \frac{1}{\sqrt{-\gamma_1}}\widetilde{x}, \quad y = \frac{1}{\sqrt{-\gamma_2}}\widetilde{y}, \quad z = \frac{1}{\sqrt{\gamma_1\gamma_2}}\widetilde{z}.$$

These relations give us the operator equalities:

$$\frac{\partial}{\partial \widetilde{x}} = \frac{1}{\sqrt{-\gamma_1}}\frac{\partial}{\partial x}, \quad \frac{\partial}{\partial \widetilde{y}} = \frac{1}{\sqrt{-\gamma_2}}\frac{\partial}{\partial y}, \quad \frac{\partial}{\partial \widetilde{z}} = \frac{1}{\sqrt{\gamma_1\gamma_2}}\frac{\partial}{\partial z}. \tag{15}$$

Now, using relations (14) and (15), we obtain

$$\widetilde{D}[\Phi](\widetilde{\zeta}) = e_1\frac{\partial \Phi}{\partial \widetilde{x}} + e_2\frac{\partial \Phi}{\partial \widetilde{y}} + e_3\frac{\partial \Phi}{\partial \widetilde{z}} =$$

$$= i\frac{\partial\Phi}{\partial x}\frac{1}{\sqrt{-\gamma_1}}\sqrt{-\gamma_1} + j\frac{\partial\Phi}{\partial y}\frac{1}{\sqrt{-\gamma_2}}\sqrt{-\gamma_2} + k\frac{\partial\Phi}{\partial z}\frac{1}{\sqrt{\gamma_1\gamma_2}}\sqrt{\gamma_1\gamma_2} =$$

$$= i\frac{\partial\Phi}{\partial x} + j\frac{\partial\Phi}{\partial y} + k\frac{\partial\Phi}{\partial z} = D[\Phi](\zeta) = 0.$$

Using the above notations, we obtain the following theorem:

Theorem 2.8. *Let Ω be an arbitrary domain in $\mathbb{R}^3$ and Δ be a domain in $\mathbb{R}^3$ such that the coordinates of the corresponding points $\zeta = xi + yj + zk \in \Omega_\zeta$ and $\widetilde{\zeta} = \widetilde{x}e_1 + \widetilde{y}e_2 + \widetilde{z}e_3 \in \Delta_{\widetilde{\zeta}}$ satisfy the following relations:*

$$x = \frac{1}{\sqrt{-\gamma_1}}\widetilde{x},\;\; y = \frac{1}{\sqrt{-\gamma_2}}\widetilde{y},\;\; z = \frac{1}{\sqrt{\gamma_1\gamma_2}}\widetilde{z}.$$

Then if the function $\Phi : \Omega_\zeta \to \mathbb{H}(-1,-1)$ is hyperholomorphic in the domain Ω_ζ, then the same function Φ, of $\widetilde{\zeta}$, is hyperholomorphic in the domain $\Delta_{\widetilde{\zeta}} \in \mathbb{H}(\gamma_1,\gamma_2)$ with $\gamma_1 < 0$, $\gamma_2 < 0$. The converse is also true.

Proof. The result directly follows from Remark 2.7.□

Remark 2.9. (i) The above Theorem tell us that for studying hyperholomorphic functions in generalized quaternion algebras $\mathbb{H}(\gamma_1,\gamma_2)$ with $\gamma_1 < 0$, $\gamma_2 < 0$ it is suffices to consider hyperholomorphic functions only in the usual quaternion algebra $\mathbb{H}(-1,-1)$.

(ii) The result similar to the previous remark was established in the paper [8] (Theorem 5) in a three-dimensional commutative associative algebra.

Theorem 2.10. *Let $A_t = \left(\frac{\gamma_1,\ldots,\gamma_t}{\mathbb{R}}\right)$ be a generalized Cayley-Dickson algebra with $\gamma_1 < 0,\ldots,\gamma_t < 0$. Let Ω be an arbitrary domain in $\mathbb{R}^{2^t-1}$ and Δ be a domain in $\mathbb{R}^{2^t-1}$ such that the coordinates of the corresponding points $\zeta = x_1e_1 + \ldots + x_{2^t-1}e_{2^t-1} \in \Omega_\zeta$ and $\widetilde{\zeta} = \widetilde{x}_1\,\widetilde{e}_1 + \widetilde{x}_2\,\widetilde{e}_2 + \ldots + \widetilde{x}_{2^t-1}\,\widetilde{e}_{2^t-1} \in \Delta_{\widetilde{\zeta}}$ satisfy the following relations*

$$x_1 = \frac{1}{\sqrt{-\gamma_1}}\,\widetilde{x}_1, \quad x_2 = \frac{1}{\sqrt{-\gamma_2}}\,\widetilde{x}_2, \;\;\ldots,\;\; x_n = \frac{1}{\sqrt{(-1)^t\gamma_1\ldots\gamma_t}}\,\widetilde{x}_n.$$

If the function $\Phi : \Omega_\zeta \to \left(\frac{-1,\ldots,-1}{\mathbb{R}}\right)$ is left A_t-holomorphic in the domain Ω_ζ, then the same function Φ, but depending of $\widetilde{\zeta}$ is left A_t-holomorphic in the domain $\Delta_{\widetilde{\zeta}} \in A_t$. The converse is also true.

Proof. Let $\{1, e_1, ..., e_{n-1}\}$ be a basis in $\left(\frac{-1,\ldots,-1}{\mathbb{R}}\right)$ and $\{1, \widetilde{e}_1, ..., \widetilde{e}_{n-1}\}$ be a basis in $A_t = \left(\frac{\gamma_1,\ldots,\gamma_t}{\mathbb{R}}\right)$.

Since

$$\widetilde{e}_1 = e_1\sqrt{-\gamma_1}, \quad \widetilde{e}_2 = e_2\sqrt{-\gamma_2}, \ldots,$$
$$\ldots, \widetilde{e}_{n-1} = e_{n-1}\sqrt{(-1)^t\gamma_1\ldots\gamma_t}\,,$$

the result is obtained from a simple computation as in Remark 2.7.□

Remark 2.11. Using above Theorem, it is obvious that, for studying left A_t-holomorphic functions in generalized Cayley-Dickson algebras $A_t = \left(\frac{\gamma_1,\ldots,\gamma_t}{\mathbb{R}}\right)$ with $\gamma_1 < 0, \ldots, \gamma_t < 0$. it is suffices to consider left A_t-holomorphic functions only in the algebras $\left(\frac{-1,\ldots,-1}{\mathbb{R}}\right)$.

Now we consider another class of differentiable functions. Let $A_t = \left(\frac{\gamma_1,\ldots,\gamma_t}{\mathbb{R}}\right)$, with $\gamma_1 = ... = \gamma_t = -1$, and the domain $\Omega \subset \mathbb{R}^{2^t}$. We denote with $\Omega_\zeta := \{\zeta = x_0 + x_1 e_1 + \ldots + x_{n-1} e_{n-1} : (x_0, x_1, \ldots, x_{n-1}) \in \Omega\}$ a domain in A_t. This domain is called *congruent* with the domain Ω.

We consider a function $\Phi : \Omega_\zeta \to A_t$ of the form

$$\Phi(\zeta) = \sum_{k=0}^{n-1} \Phi_k(x_0, x_1, \ldots, x_{n-1}) e_k, \tag{16}$$

where $(x_0, x_1, \ldots, x_{n-1}) \in \Omega$ and $\Phi_k : \Omega \to \mathbb{R}$.

We say that a function of the form (16) is *left A_t-hyperholomorphic* in a domain Ω_ζ if the first partial derivatives $\partial\Phi_k/\partial x_k$ exist in Ω and the following equality is fulfilled in every point of Ω_ζ:

$$\sum_{k=0}^{2^t-1} e_k \frac{\partial \Phi}{\partial x_k} = 0.$$

In the following, we will provide an algorithm to constructing a left A_t-hyperholomorphic functions. Using the above notations, let $v\,(x, y)$ be a rational function defined in a domain $G \subset \mathbb{R}^2$. In the following, using some ideas given in Theorem 3 from [12], we will give an example of left A_t-hyperholomorphic function, for all $t \geq 1,\ t \in \mathbb{N}$. For this, we consider the following functions:

$$\phi_1 = x_0 + e_1 x_1, \quad \phi_2 = \frac{1}{e_1}(x_0 + e_1 x_1),$$

$$\rho_{2s-1} = x_{2s} - e_1 x_{2s+1}, \quad \rho_{2s} = -\frac{1}{e_1}(x_{2s} - e_1 x_{2s+1}), \ \ s \in \{1, 2, ..., 2^{t-1} - 1\},$$

$$F_t\,(\zeta) = v\,(\phi_1, \phi_2) + v\,(\rho_1, \rho_2)\, e_2 + v\,(\rho_3, \rho_4)\, e_4 + [v\,(\rho_5, \rho_6)\, e_2]\, e_4 +$$

$$+ v\,(\rho_7, \rho_8)\, e_8 + (v\,(\rho_9, \rho_{10})\, e_2)\, e_8 + (v\,(\rho_{11}, \rho_{12})\, e_4)\, e_8 + [(v\,(\rho_{13}, \rho_{14})\, e_2)\, e_4]\, e_8 + ...$$

$$... + \sum_{i=4}^{t-1}(\sum_{k=1}^{i}(\sum_{r=1}^{k-1} v\,(\rho_{M_{rki}-1}, \rho_{M_{rki}})\, e_{2^r}) e_{2^{r+1}} ...) e_{2^k}) e_{2^i}) + \sum_{i=1}^{t-1}\left(v\,(\rho_{2^i-1}, \rho_{2^i})\, e_{2^i}\right),$$

where $M_{rki} = 2^r + 2^{r+1} + ... + 2^k + 2^i$.

It results

$$F_t\,(\zeta) = v\,(\phi_1, \phi_2) +$$

$$+ \sum_{i=1}^{t-1}(\sum_{k=1}^{i}(\sum_{r=1}^{k-1} v\,(\rho_{M_{rki}-1}, \rho_{M_{rki}})\, e_{2^r}) e_{2^{r+1}} ...) e_{2^k}) e_{2^i}) + \sum_{i=1}^{t-1}\left(v\,(\rho_{2^i-1}, \rho_{2^i})\, e_{2^i}\right),$$

or

$$F_t(\zeta) = F_{t-1}(\zeta) +$$

$$+(\sum_{k=1}^{t-2}(\sum_{r=1}^{k-1} v\left(\rho_{M_{rk(t-1)}-1}, \rho_{M_{rk(t-1)}}\right) e_{2^r}) e_{2^{r+1}} ...) e_{2^k}) e_{2^{t-1}}) + v\left(\rho_{2^{t-1}-1}, \rho_{2^{t-1}}\right) e_{2^{t-1}}.$$

We denote with $\mathbb{C}_{2s}$ the "complex" planes $\{x_{2s} + e_1 x_{2s+1} : x_{2s}, x_{2s+1} \in \mathbb{R}\}$ and with $D_{2s} := \{(x_{2s}, x_{2s+1}) : x_{2s} + e_1 x_{2s+1} \in \mathbb{C}_{2s}\}$, $s \in \{0, 1, 2, ..., 2^{t-1} - 1\}$ the Euclidian planes. Let G_{2s} be a domains in $\mathbb{C}_{2s}$ and let $\widetilde{G}_{2s}$ be the corresponded domains in D_{2s}. We have the following theorem:

Theorem 2.12. *With the above notations, we consider the functions $v(\phi_1, \phi_2)$ and $v(\rho_{2s-1}, \rho_{2s})$ defined in the corresponding domains $G_0 \subset \mathbb{C}_0$ and $G_{2s} \subset \mathbb{C}_{2s}$, $s \in \{1, 2, ..., 2^{t-1} - 1\}$. Then the map $F_t(\zeta)$ is a left A_t-hyperholomorphic function in the domain $\Theta \subset A_t$ which is congruent with the domain $\widetilde{G}_0 \times \widetilde{G}_2 \times \widetilde{G}_4 \times ... \times \widetilde{G}_{2^{t-1}-1} \subset \mathbb{R}^{2^t}$, for $t \geq 1$.*

Proof. For $t = 1$, we have $F_1(\zeta) = v(\phi_1, \phi_2)$, which is an holomorphic function in $D_0 \subset \mathbb{C}_0$, as we can see in [12], Theorem 3.

For $t = 2$, we obtain $F_2(\zeta) = v(\phi_1, \phi_2) + v(\rho_1, \rho_2) e_2$ and for $t = 3$, we get $F_3(\zeta) = v(\phi_1, \phi_2) + v(\rho_1, \rho_2) e_2 + v(\rho_3, \rho_4) e_4$. $F_2(\zeta)$ and $F_3(\zeta)$ are hyperholomorphic, respectively octonionic hyperholomorphic function, from Remark 2.1 and Theorem 3 from [12].

For $t \geq 4$, using induction steps, supposing that $F_{t-1}(\zeta)$ is a left A_{t-1}-hyperholomorphic function, we will prove that $F_t(\zeta)$ is A_t-hyperholomorphic. That means $D[F_t] = 0$. From relations (6) and (7), we have that

$$D[F_t] = \sum_{k=0}^{2^t-1} e_k \frac{\partial F_t}{\partial x_k} = \sum_{k=0}^{2^{t-1}-1} e_k \frac{\partial F_t}{\partial x_k} + \sum_{k=2^{t-1}}^{2^t-1} e_k \frac{\partial F_t}{\partial x_k} =$$

$$= D[F_{t-1}] + e_{2^{t-1}} \sum_{k=0}^{2^{t-1}-1} \overline{e}_k \frac{\partial F_t}{\partial x_{k+2^{t-1}}}.$$

From induction steps, we obtain $D[F_{t-1}] = 0$. We will prove that $\sum_{k=0}^{2^{t-1}-1} \overline{e}_k \frac{\partial F_t}{\partial x_{2^{t-1}+k}} = 0$. This sum has 2^{t-1} terms. First two terms are:

$$\left(\frac{\partial F_t}{\partial x_{2^{t-1}}} - e_1 \frac{\partial F_t}{\partial x_{2^{t-1}+1}}\right) =$$

$$= \frac{\partial v}{\partial \rho_{2^{t-1}-1}} \frac{\partial \rho_{2^{t-1}-1}}{\partial x_{2^{t-1}}} + \frac{\partial v}{\partial \rho_{2^{t-1}}} \frac{\partial \rho_{2^{t-1}}}{\partial x_{2^{t-1}}} - e_1 \left(\frac{\partial v}{\partial \rho_{2^{t-1}-1}} \frac{\partial \rho_{2^{t-1}-1}}{\partial x_{2^{t-1}+1}} + \frac{\partial v}{\partial \rho_{2^{t-1}}} \frac{\partial \rho_{2^{t-1}}}{\partial x_{2^{t-1}+1}}\right) =$$

$$= \frac{\partial v}{\partial \rho_{2^{t-1}-1}} + \frac{\partial v}{\partial \rho_{2^{t-1}}} \left(\frac{-1}{e_1} \right) - e_1 \left(\frac{\partial v}{\partial \rho_{2^{t-1}-1}} (-e_1) + \frac{\partial v}{\partial \rho_{2^{t-1}}} \right) =$$

$$= \frac{\partial v}{\partial \rho_{2^{t-1}-1}} + \frac{\partial v}{\partial \rho_{2^{t-1}}} e_1 - \frac{\partial v}{\partial \rho_{2^{t-1}-1}} - e_1 \frac{\partial v}{\partial \rho_{2^{t-1}}} = 0.$$

Since $e_1^2 = \gamma_1$, $\gamma_1^2 = 1$, $\frac{\partial v}{\partial \rho_{2^{t-1}-1}}$ and $\frac{\partial v}{\partial \rho_{2^{t-1}}}$ can be written as $a_{2^{t-1}-1}(\zeta) + b_{2^{t-1}-1}(\zeta) e_1$, respectively $a_{2^{t-1}}(\zeta) + b_{2^{t-1}}(\zeta) e_1$ where $a_{2^{t-1}-1}(\zeta), b_{2^{t-1}-1}(\zeta)$, $a_{2^{t-1}}(\zeta)$, $b_{2^{t-1}}(\zeta)$ are real valued functions.

Case 1: $r < k$. In the general case, we denote $T = 2^r + 2^{r+1} + ... + 2^k + 2^{t-1}$ and $T_1 = 2^r + 2^{r+1} + ... + 2^k$, for $r < k$. We will compute the terms

$$-e_{T_1} \frac{\partial F_t}{\partial x_T} - e_{T_1+1} \frac{\partial F_t}{\partial x_{T+1}}.$$

We compute first $\frac{\partial F_t}{\partial x_T}$. It results

$$\frac{\partial F_t}{\partial x_T} = \left(... \left(\frac{\partial v}{\partial \rho_{T-1}} \frac{\partial \rho_{T-1}}{\partial x_T} + \frac{\partial v}{\partial \rho_T} \frac{\partial \rho_T}{\partial x_T} \right) e_{2^r}) e_{2^{r+1}}) ... e_{2^k}) e_{2^{t-1}} =$$

$$= \left(... \left(\frac{\partial v}{\partial \rho_{T-1}} + \frac{\partial v}{\partial \rho_T} \frac{-1}{e_1} \right) e_{2^r}) e_{2^{r+1}}) ... e_{2^k}) e_{2^{t-1}} =$$

$$= \left(... \left(\frac{\partial v}{\partial \rho_{T-1}} + \frac{\partial v}{\partial \rho_T} e_1 \right) e_{2^r}) e_{2^{r+1}}) ... e_{2^k}) e_{2^{t-1}}.$$

Since we can write $\frac{\partial v}{\partial \rho_{T-1}}$ under the form $a_{T-1}(\zeta) + b_{T-1}(\zeta) e_1$ and $\frac{\partial v}{\partial \rho_T}$ under the form $a_T(\zeta) + b_T(\zeta) e_1$, where a_{T-1}, b_{T-1}, a_T, b_T are real valued functions, using Proposition 2.2, we obtain:

$$\frac{\partial F_t}{\partial x_T} = \left(... \left(\frac{\partial v}{\partial \rho_{T-1}} + \frac{\partial v}{\partial \rho_T} e_1 \right) e_{2^r}) e_{2^{r+1}}) ... e_{2^k}) e_{2^{t-1}} =$$

$$= (...(a_{T-1}(\zeta) e_{2^r}) e_{2^{r+1}}) ... e_{2^k}) e_{2^{t-1}} + (...(b_{T-1}(\zeta) e_1) e_{2^r}) e_{2^{r+1}}) ... e_{2^k}) e_{2^{t-1}} +$$

$$+ (...(a_T(\zeta) e_1) e_{2^r}) e_{2^{r+1}}) ... e_{2^k}) e_{2^{t-1}} + (...(b_T(\zeta) e_1) e_1) e_{2^r}) e_{2^{r+1}}) ... e_{2^k}) e_{2^{t-1}} =$$

$$= a_{T-1}(\zeta)(-1)^{k-r+2} e_T + b_{T-1}(\zeta)(-1)^{k-r+3} e_{T+1} +$$

$$+ a_T(\zeta)(-1)^{k-r+3} e_{T+1} - b_T(\zeta)(-1)^{k-r+2} e_T.$$

Using Proposition 2.3, relation (11), we compute $-e_{T_1}\frac{\partial F_t}{\partial x_T}$.

$$-e_{T_1}\frac{\partial F_t}{\partial x_T}=-e_{T_1}\bigg(a_{T-1}(\zeta)(-1)^{k-r+2}e_T+b_{T-1}(\zeta)(-1)^{k-r+3}e_{T+1}+$$

$$+a_T(\zeta)(-1)^{k-r+3}e_{T+1}-b_T(\zeta)(-1)^{k-r+2}e_T\bigg)=$$

$$=-\bigg(a_{T-1}(\zeta)(-1)^{k-r+2}(-1)^{k-r+1}e_{2^i}-b_{T-1}(\zeta)(-1)^{k-r+3}(-1)^{k-r+1}e_{2^i+1}\bigg)-$$

$$-\bigg(-a_T(\zeta)(-1)^{k-r+3}(-1)^{k-r+1}e_{2^i+1}-b_T(\zeta)(-1)^{k-r+2}(-1)^{k-r+1}e_{2^i}\bigg)=$$

$$=-\bigg(a_{T-1}(\zeta)(-1)^{2k-2r+3}e_{2^i}-b_{T-1}(\zeta)(-1)^{2k-2r+4}e_{2^i+1}\bigg)-$$

$$-\bigg(-a_T(\zeta)(-1)^{2k-2r+4}e_{2^i+1}-b_T(\zeta)(-1)^{2k-2r+3}e_{2^i}\bigg).$$

Now, we compute $\frac{\partial F_t}{\partial x_{T+1}}$. We obtain

$$\frac{\partial F_t}{\partial x_{T+1}}=\bigg(...\bigg(\frac{\partial v}{\partial \rho_{T-1}}\frac{\partial \rho_{T-1}}{\partial x_{T+1}}+\frac{\partial v}{\partial \rho_T}\frac{\partial \rho_T}{\partial x_{T+1}}\bigg)e_{2^r})e_{2^{r+1}})...e_{2^k})e_{2^{t-1}}=$$

$$=\bigg(...\bigg(-\frac{\partial v}{\partial \rho_{T-1}}e_1+\frac{\partial v}{\partial \rho_T}\bigg)e_{2^r})e_{2^{r+1}})...e_{2^k})e_{2^{t-1}}.$$

Since we can write $\frac{\partial v}{\partial \rho_{T-1}}$ under the form $a_{T-1}(\zeta)+b_{T-1}(\zeta)e_1$ and $\frac{\partial v}{\partial \rho_T}$ under the form $a_T(\zeta)+b_T(\zeta)e_1$, where a_{T-1}, b_{T-1}, a_T, b_T are real valued functions, using Proposition 2.2, we obtain:

$$\frac{\partial F_t}{\partial x_{T+1}}=\bigg(...\bigg(-\frac{\partial v}{\partial \rho_{T-1}}e_1+\frac{\partial v}{\partial \rho_T}\bigg)e_{2^r})e_{2^{r+1}})...e_{2^k})e_{2^{t-1}}=$$

$$=(...(-a_{T-1}(\zeta)e_1)e_{2^r})e_{2^{r+1}})...e_{2^k})e_{2^{t-1}}-(...(b_{T-1}(\zeta)e_1e_1)e_{2^r})e_{2^{r+1}})...e_{2^k})e_{2^{t-1}}+$$

$$+(...(a_T(\zeta))e_{2^r})e_{2^{r+1}})...e_{2^k})e_{2^{t-1}}+(...(b_T(\zeta)e_1))e_{2^r})e_{2^{r+1}})...e_{2^k})e_{2^{t-1}}=$$

$$=-a_{T-1}(\zeta)(-1)^{k-r+3}e_{T+1}+b_{T-1}(\zeta)(-1)^{k-r+2}e_T+$$

$$+a_T(\zeta)(-1)^{k-r+2}e_T + b_T(\zeta)(-1)^{k-r+3}e_{T+1}.$$

Using Proposition 2.3, we compute $-e_{T_1+1}\frac{\partial F_t}{\partial x_{T+1}}$.

$$-e_{T_1+1}\frac{\partial F_t}{\partial x_{T+1}} = -e_{T_1+1}\Big(- a_{T-1}(\zeta)(-1)^{k-r+3}e_{T+1} + b_{T-1}(\zeta)(-1)^{k-r+2}e_T +$$

$$+a_T(\zeta)(-1)^{k-r+2}e_T + b_T(\zeta)(-1)^{k-r+3}e_{T+1}\Big) =$$

$$= -\Big(a_{T-1}(\zeta)(-1)^{k-r+3}(-1)^{k-r+1}e_{2^i} - b_{T-1}(\zeta)(-1)^{k-r+2}(-1)^{k-r+1}e_{2^i+1}\Big) -$$

$$-\Big(- a_T(\zeta)(-1)^{k-r+2}(-1)^{k-r+1}e_{2^i+1} - b_T(\zeta)(-1)^{k-r+3}(-1)^{k-r+1}e_{2^i}\Big) =$$

$$= -\Big(a_{T-1}(\zeta)(-1)^{2k-2r+4}e_{2^i} - b_{T-1}(\zeta)(-1)^{2k-2r+3}e_{2^i+1}\Big) -$$

$$-\Big(- a_T(\zeta)(-1)^{2k-2r+3}e_{2^i+1} - b_T(\zeta)(-1)^{2k-2r+4}e_{2^i}\Big).$$

Now, we can compute $-e_{T_1}\frac{\partial F_t}{\partial x_T} - e_{T_1+1}\frac{\partial F_t}{\partial x_{T+1}}$. It results

$$-e_{T_1}\frac{\partial F_t}{\partial x_T} - e_{T_1+1}\frac{\partial F_t}{\partial x_{T+1}} =$$

$$= -\Big(a_{T-1}(\zeta)(-1)^{2k-2r+3}e_{2^i} - b_{T-1}(\zeta)(-1)^{2k-2r+4}e_{2^i+1}\Big) -$$

$$-\Big(- a_T(\zeta)(-1)^{2k-2r+4}e_{2^i+1} - b_T(\zeta)(-1)^{2k-2r+3}e_{2^i}\Big) -$$

$$-\Big(a_{T-1}(\zeta)(-1)^{2k-2r+4}e_{2^i} - b_{T-1}(\zeta)(-1)^{2k-2r+3}e_{2^i+1}\Big) -$$

$$-\Big(- a_T(\zeta)(-1)^{2k-2r+3}e_{2^i+1} - b_T(\zeta)(-1)^{2k-2r+4}e_{2^i}\Big) = 0.$$

Case 2: $r = k$, we use Proposition 2.2 and Proposition 2.3, relation (12) and it easy to show that

$$-e_{2^k}\frac{\partial F_t}{\partial x_T} - e_{2^k+1}\frac{\partial F_t}{\partial x_{T+1}} = 0.$$

□

Remark 2.13. The above proposition generalizes Theorem 3 from [12].

The Algorithm

1) Input t.
2) Input functions v, ϕ_1, ϕ_2.
3) For $i \in \{1, ..., t-1\}$, $k \in \{1, ..., i\}$, $r \in \{1, ..., k-1\}$, compute $M_{rki} = 2^r + ... + 2^k + 2^i$, $v\left(\rho_{M_{rki}-1}, \rho_{M_{rki}}\right) = \alpha_{M_{rki}} + \beta_{M_{rki}} e_1$.
4) For $i \in \{1, ..., t-1\}$, $k \in \{1, ..., i\}$, $r \in \{1, ..., k-1\}$,

-if $r < k$, we compute

$$(...\left(\alpha_{M_{rki}} + \beta_{M_{rki}} e_1\right) e_{2^r}) e_{2^{r+1}} ...) e_{2^k}) e_{2^i}) =$$

$$= (-1)^{k-r+2} \left(\alpha_{M_{rki}} e_{M_{rki}} - \beta_{M_{rki}} e_{M_{rki}-1}\right)$$

-if $r = k$, we compute

$$v\left(\rho_{2^i-1}, \rho_{2^i}\right) e_{2^i} = (\alpha_{2^i-1} + \beta_{2^i-1} e_1) e_{2^i} =$$

$$= \alpha_{2^i-1} e_{2^i} + \beta_{2^i-1} e_{2^i+1}.$$

5) Output function

$$F_t(\zeta) = v(\phi_1, \phi_2) + \sum_{i=4}^{t-1} \left(\sum_{k=1}^{i} \left(\sum_{r=1}^{k-1} (-1)^{k-r+2} \left(\alpha_{M_{rki}}(\zeta) e_{M_{rki}} - \beta_{M_{rki}}(\zeta) e_{M_{rki}-1}\right)\right)\right) +$$

$$+ \sum_{i=1}^{t-1} \left(\alpha_{2^i-1}(\zeta) e_{2^i} + \beta_{2^i-1}(\zeta) e_{2^i+1}\right).$$

Conclusion. In this paper, we generalized the notion of left A_t-holomorphic functions from quaternions to all algebras obtained by the Cayley-Dickson process and we provided an algorithm to find examples of left A_t-hyperholomorphic functions, using the *shuffling* procedure given by Bales in [1].

The theory of the right A_t-holomorphic functions and the theory of the right A_t-hyperholomorphic functions are similarly to the corresponding theories for the left functions and can be easy treated, using the above ideas and procedures.

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