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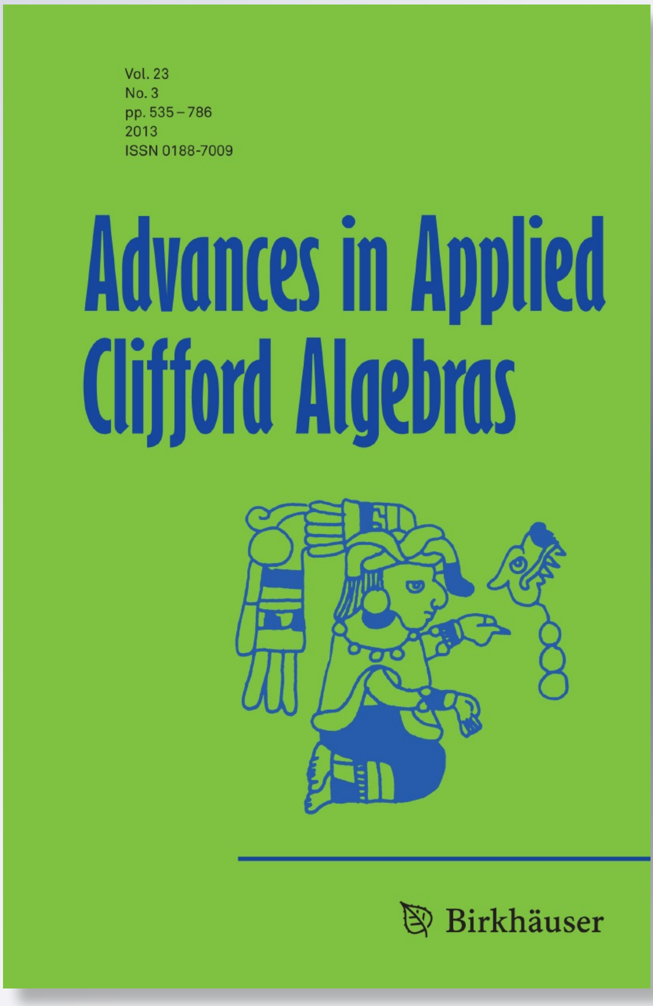

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On Generalized Fibonacci Quaternions and Fibonacci-Narayana Quaternions

Cristina Flaut* and Vitalii Shpakivskyi

Abstract. In this paper, we investigate some properties of generalized Fibonacci quaternions and Fibonacci-Narayana quaternions in a generalized quaternion algebra.

Keywords. Fibonacci quaternions, generalized Fibonacci quaternions, Fibonacci-Narayana quaternions.

0. Introduction

The Fibonacci numbers was introduced by *Leonardo of Pisa* (1170-1240) in his book *Liber abbaci*, book published in 1202 AD (see [9], p. 1, 3). These numbers were used as a model for investigate the growth of rabbit populations (see [4]). The Latin name of Leonardo was *Leonardus Pisanus*, also called *Leonardus filius Bonaccii*, shortly *Fibonacci*. This name is attached to the following sequence of numbers

$$0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots,$$

with the nth term given by the formula:

$$f_n = f_{n-1} + f_{n-2}, \; n \geq 2,$$

where $f_0 = 0, f_1 = 1$.

Fibonacci numbers was known in India before Leonardo's time and used by the Indian authorities on metrical sciences (see [10], p. 230). These numbers have many properties which were studied by many authors (see [6], [2], [10], [9]).

Narayana was an outstanding Indian mathematician of the XIV century. From him came to us the manuscript "Bidzhahanity" (incomplete), written in the middle of the XIV century. For Narayana was interesting summation

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of arithmetic series and magic squares. In the middle of the XIV century he proved a more general summation. Using the following sums

$$1+2+3+\ldots+n=S_n^{(1)},$$

$$S_1^{(1)}+S_2^{(1)}+\ldots+S_n^{(1)}=S_n^{(2)},$$

$$S_1^{(2)}+S_2^{(2)}+\ldots+S_n^{(2)}=S_n^{(2)},\ \ldots\ ,$$

Narayana calculated that

$$S_n^{(m)}=\frac{n(n+1)(n+2)\ldots(n+m)}{1\cdot 2\cdot 3\cdot\ldots\cdot(m+1)}\ . \qquad (*)$$

Narayana applied its rules to the problem of a herd of cows and heifers (see [16], [12], [13], [1]).

Narayana problem ([1]). *A cow annually brings heifers. Every heifer, beginning from the fourth year of his life also brings heifer. How many cows and calves will be after* 20 *years?*

Narayana's calculation is in the following:

1) a cow within 20 years brings 20 heifers of the first generation;
2) the first heifer of the first generation brings 17 heifers second generation, the second heifer of the first generation brings 16 heifers second generation etc. The total in the second generation will be $17+16+\ldots+1=S_{17}^{(1)}$ cows and calves;
3) the first heifer of the seventeen heifers of the second generation brings 14 heifers of the third generation, the second heifer of the seventeen heifers of the second generation brings 13 heifers of third generation, etc. The total heifers of the first generation brings $13+12+\ldots+1=S_{13}^{(1)}$ heads. Now, all heifers of the second generation brings $S_{14}^{(1)}+S_{13}^{(1)}+\ldots+S_1^{(1)}=S_{14}^{(2)}$ heads in the third generation.

Similarly, Narayana calculated total number in the herd after 20 years:

$$n=1+20+S_{17}^{(1)}+S_{14}^{(2)}+\ldots+S_2^{(6)}.$$

Using formula (*), he obtained:

$$n=1+20+\frac{17\cdot 18}{1\cdot 2}+\frac{14\cdot 15\cdot 16}{1\cdot 2\cdot 3}+\frac{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}=2745.$$

This problem can be solved in the same way that Fibonacci solved its problem about rabbits (see [8], [9], [12], [13]).

In the beginning of the first year were 1 cow and 1 heifer which born. We have now 2 heads. In the beginning of the second year and in the beginning of the third year the number of heads increased by one. Therefore the number of heads are 3 and 4, respectively. From the fourth year, the number of heads in the herd is defined by recurrence formulae:

$$x_4=x_3+x_1, x_5=x_4+x_2,\ldots,x_n=x_{n-1}+x_{n-3},$$

since the number of cows for any year is equal with the number of cows of the previous year plus the number of heifers which was born (= number of heads that were three years ago) (see [1]).

We have the sequence

$$2,3,4,6,9,\ldots,\ u_{n+1}=u_n+u_{n-2}.$$

Computing, we obtain that $u_{20}=2745$ (see [8], [9], [12], [13], [1]).

Now, we can consider the sequence

$$1,1,1,2,3,4,6,9,\ldots,\ u_{n+1}=u_n+u_{n-2},$$

with $n\geq 2, u_0=0, u_1=1, u_2=1$. These numbers are called the *Fibonacci-Narayana numbers* (see [3]).

In the same paper [Di, St; 03], authors proved some basic properties of Fibonacci-Narayana numbers, namely:

1) $u_1+u_2+\ldots+u_n=u_{n+3}-1$.
2) $u_1+u_4+u_7+\ldots+u_{3n-2}=u_{3n-1}$.
3) $u_2+u_5+u_8+\ldots+u_{3n-1}=u_{3n}$.
4) $u_3+u_6+u_9+\ldots+u_{3n}=u_{3n+1}-1$.
5) $u_{n+m}=u_{n-1}u_{m+2}+u_{n-2}u_m+u_{n-3}u_{m+1}$.
6) $u_{2n}=u_{n+1}^2+u_{n-1}^2-u_{n-2}^2$.
7) If in the sequences $\{u_n\}$, $n=7k+4$, $n=7k+6$, $n=7k$, when $k=0,1,2,\ldots$, then u_n is even.
8) If in the sequences $\{u_n\}$ $n=8k$, $n=8k-1$, $n=8k-3$, when $k=0,1,2,\ldots$, then $3\mid u_n$.

Another property of Fibonacci-Narayana numbers was proved in [14]. For all natural $n\geq 2$, we have

$$u_n=\sum_{m=0}^{[n/3]}\mathrm{C}_{[n/3]}^m u_{n-[n/3]-2m},$$

where $[a]$ is an integer part of a and $\mathrm{C}_n^k=\frac{n!}{k!(n-k)!}$, $k!=1\cdot 2\cdot 3\cdot\ldots\cdot k, k\in\mathbb{N}$.

Let $\mathbb{H}(\beta_1,\beta_2)$ be the generalized real quaternion algebra, the algebra of the elements of the form $a=a_1\cdot 1+a_2e_2+a_3e_3+a_4e_4$, where $a_i\in\mathbb{R}, i\in\{1,2,3,4\}$, and the basis elements $\{1,e_2,e_3,e_4\}$ satisfy the following multiplication table:

$\cdot$	1	e_2	e_3	e_4
1	1	e_2	e_3	e_4
e_2	e_2	$-\beta_1$	e_4	$-\beta_1 e_3$
e_3	e_3	$-e_4$	$-\beta_2$	$\beta_2 e_2$
e_4	e_4	$\beta_1 e_3$	$-\beta_2 e_2$	$-\beta_1\beta_2$

We denote by $\boldsymbol{t}(a)$ and $\boldsymbol{n}(a)$ the trace and the norm of a real quaternion a. The norm of a generalized quaternion has the following expression $\boldsymbol{n}(a)=a_1^2+\beta_1a_2^2+\beta_2a_3^2+\beta_1\beta_2a_4^2$. For $\beta_1=\beta_2=1$, we obtain the real division algebra $\mathbb{H}$.

1. Preliminaries

In the present days, several mathematicians studied properties of the Fibonacci sequence. In [6], the author generalized Fibonacci numbers and gave many properties of them:

$$h_n = h_{n-1} + h_{n-2}, \quad n \geq 2,$$

where $h_0 = p, h_1 = q$, with p, q being arbitrary integers. In the same paper [6], relation (7), the following relation between Fibonacci numbers and generalized Fibonacci numbers was obtained:

$$h_{n+1} = pf_n + qf_{n+1}. \tag{1.1}$$

The same author, in [7], defined and studied Fibonacci quaternions and generalized Fibonacci quaternions in the real division quaternion algebra and found a lot of properties of them. For the generalized real quaternion algebra, the Fibonacci quaternions and generalized Fibonacci quaternions are defined in the same way:

$$F_n = f_n \cdot 1 + f_{n+1}e_2 + f_{n+2}e_3 + f_{n+3}e_4,$$

for the nth Fibonacci quaternions, and

$$H_n = h_n \cdot 1 + h_{n+1}e_2 + h_{n+2}e_3 + h_{n+3}e_4,$$

for the nth generalized Fibonacci quaternions.

In the same paper, we find the norm formula for the nth Fibonacci quaternions:

$$\boldsymbol{n}\left(F_n\right) = F_n\overline{F}_n = 3f_{2n+3}, \tag{1.2}$$

where $\overline{F}_n = f_n \cdot 1 - f_{n+1}e_2 - f_{n+2}e_3 - f_{n+3}e_4$ is the conjugate of the F_n in the algebra $\mathbb{H}$. After that, many authors studied Fibonacci and generalized Fibonacci quaternions in the real division quaternion algebra giving more and surprising new properties (for example, see [15], [11] and [5]).

M. N. S. Swamy, in [15], formula (17), obtained the norm formula for the nth generalized Fibonacci quaternions:

$$\begin{aligned} \boldsymbol{n}\left(H_n\right) &= H_n\overline{H}_n \\ &= 3(2pq - p^2)f_{2n+2} + (p^2 + q^2)f_{2n+3}, \end{aligned}$$

where $\overline{H}_n = h_n \cdot 1 - h_{n+1}e_2 - h_{n+2}e_3 - h_{n+3}e_4$ is the conjugate of the H_n in the algebra $\mathbb{H}$.

Similar to A. F. Horadam, we define the Fibonacci-Narayana quaternions as

$$U_n = u_n \cdot 1 + u_{n+1}e_2 + u_{n+2}e_3 + u_{n+3}e_4,$$

where u_n are the nth Fibonacci-Narayana number.

In this paper, we give some properties of generalized Fibonacci quaternions and Fibonacci-Narayana quaternions.

2. Generalized Fibonacci Quaternions

As in the case of Fibonacci numbers, numerous results between Fibonacci generalized numbers can be deduced. In the following, we will study some properties of the generalized Fibonacci quaternions in the generalized real quaternion algebra $\mathbb{H}(\beta_1, \beta_2)$. Let $F_n = f_n \cdot 1 + f_{n+1}e_2 + f_{n+2}e_3 + f_{n+3}e_4$ be the nth Fibonacci quaternion and $H_n = h_n \cdot 1 + h_{n+1}e_2 + h_{n+2}e_3 + h_{n+3}e_4$ be the nth generalized Fibonacci quaternion. A first question which can arise is what algebraic structure have these elements? The answer will be found in the below theorem, denoting first a nth generalized Fibonacci number and a nth generalized Fibonacci element with $h_n^{p,q}$, respectively $H_n^{p,q}$. In this way, we emphasis the starting integers p and q.

Theorem 2.1. *The set $\mathcal{H}_n = \{H_n^{p,q} \ / \ p, q \in \mathbb{Z}\} \cup \{0\}$ is a $\mathbb{Z}$-module.*

Proof. Indeed, $aH_n^{p,q} + bH_n^{p',q'} = H_n^{ap+bp',aq+bq'} \in \mathcal{H}_n$, where $a, b, p, q, p', q' \in \mathbb{Z}$. □

Theorem 2.2. i) *For the Fibonacci quaternion elements, we have*

$$\sum_{m=1}^{n} (-1)^{m+1} F_m = (-1)^{n+1} F_{n-1} + 1 + e_3 + e_4. \tag{2.1}$$

ii) *For the generalized Fibonacci quaternion elements, the following relation is true,*

$$\sum_{m=1}^{n} (-1)^{m+1} H_m^{p,q} = (-1)^{n+1} H_{n-1}^{p,q} - p + q + pe_2 + qe_3 + pe_4 + qe_4. \tag{2.2}$$

Proof. i) From [2], we know that

$$\sum_{m=1}^{n} (-1)^{m+1} f_m = (-1)^{n+1} f_{n-1} + 1. \tag{2.3}$$

It results:

$$\begin{aligned}
&\sum_{m=1}^{n} (-1)^{m+1} F_m \\
&= \sum_{m=1}^{n} (-1)^{m+1} f_m + e_2 \sum_{m=1}^{n} (-1)^{m+1} f_{m+1} \\
&\quad + e_3 \sum_{m=1}^{n} (-1)^{m+1} f_{m+2} + e_4 \sum_{m=1}^{n} (-1)^{m+1} f_{m+3} \\
&= (-1)^{n+1} f_{n-1} + 1 - e_2[(-1)^{n+1} f_{n-1} + (-1)^{n+2} f_{n+1}] \\
&\quad + e_3 \left[(-1)^{n+1} f_{n-1} + 1 + (-1)^{n} f_{n+1} + (-1)^{n+1} f_{n+2}\right] \\
&\quad - e_4[(-1)^{n+1} f_{n-1} - 1 + (-1)^{n+2} f_{n+1} + (-1)^{n+3} f_{n+2} + (-1)^{n+4} f_{n+3}] \\
&= (-1)^{n+1} f_{n-1} + 1 + (-1)^{n+1} e_2 f_n + e_3 (-1)^{n+1} \left[f_{n+1} + (-1)^{n+1}\right]
\end{aligned}$$

$$
\begin{aligned}
&- e_4 (-1)^{n+1} \left[-f_{n+2} - (-1)^{n+1}\right] \\
&= (-1)^{n+1} \left(f_{n-1} + f_n e_2 + f_{n+1} e_3 + f_{n+2} e_4\right) + 1 + e_3 + e_4 \\
&= (-1)^{n+1} F_{n-1} + 1 + e_3 + e_4.
\end{aligned}
$$

ii) Using relations (1.1) and (2.3) ,we have

$$
\begin{aligned}
&\sum_{m=1}^{n} (-1)^{m+1} H_m^{p,q} \\
&= \sum_{m=1}^{n} (-1)^{m+1} h_m^{p,q} + e_2 \sum_{m=1}^{n} (-1)^{m+1} h_{m+1}^{p,q} \\
&\quad + e_3 \sum_{m=1}^{n} (-1)^{m+1} h_{m+2}^{p,q} + e_4 \sum_{m=1}^{n} (-1)^{m+1} h_{m+3}^{p,q} \\
&= \sum_{m=1}^{n} (-1)^{m+1} p f_{m-1} + \sum_{m=1}^{n} (-1)^{m+1} q f_m \\
&\quad + e_2 \sum_{m=1}^{n} (-1)^{m+1} p f_m + e_2 \sum_{m=1}^{n} (-1)^{m+1} q f_{m+1} \\
&\quad + e_3 \sum_{m=1}^{n} (-1)^{m+1} p f_{m+1} + e_3 \sum_{m=1}^{n} (-1)^{m+1} q f_{m+2} \\
&\quad + e_4 \sum_{m=1}^{n} (-1)^{m+1} p f_{m+2} + e_4 \sum_{m=1}^{n} (-1)^{m+1} q f_{m+3} \\
&= p (-1)^{n+1} f_{n-2} - p + q (-1)^{n+1} f_{n-1} + q \\
&\quad + e_2 p (-1)^{n+1} f_{n-1} + p e_2 + e_2 q \left[(-1)^{n+1} f_{n+1} - (-1)^{n+1} f_{n-1}\right] \\
&\quad + e_3 p \left[(-1)^{n+1} f_{n+1} - (-1)^{n+1} f_{n-1}\right] \\
&\quad + e_3 q \left[(-1)^{n+1} f_{n-1} + 1 + (-1)^{n} f_{n+1} + (-1)^{n+1} f_{n+2}\right] \\
&\quad + e_4 p \left[(-1)^{n+1} f_{n-1} + 1 + (-1)^{n} f_{n+1} + (-1)^{n+1} f_{n+2}\right] \\
&\quad - e_4 q [(-1)^{n+1} f_{n-1} - 1 + (-1)^{n+2} f_{n+1} + (-1)^{n+3} f_{n+2} + (-1)^{n+4} f_{n+3}] \\
&= p (-1)^{n+1} f_{n-2} - p + q (-1)^{n+1} f_{n-1} + q \\
&\quad + e_2 p (-1)^{n+1} f_{n-1} + p e_2 + e_2 q (-1)^{n+1} f_n + e_3 p (-1)^{n+1} f_n \\
&\quad + e_3 q (-1)^{n+1} [f_{n-1} + (-1)^{n+1} - f_{n+1} + f_{n+2}] \\
&\quad + e_4 p (-1)^{n+1} \left[f_{n-1} + (-1)^{n+1} - f_{n+1} + f_{n+2}\right] \\
&\quad - e_4 q (-1)^{n+1} \left[f_{n-1} - (-1)^{n+1} - f_{n+1} + f_{n+2} - f_{n+3}\right] \\
&= p (-1)^{n+1} f_{n-2} - p + q (-1)^{n+1} f_{n-1} + q
\end{aligned}
$$

$$
\begin{aligned}
&+ e_2 p (-1)^{n+1} f_{n-1} + pe_2 + e_2 q (-1)^{n+1} f_n + e_3 p (-1)^{n+1} f_n \\
&+ e_3 q (-1)^{n+1} \left[f_{n+1} + (-1)^{n+1} \right] + e_4 p (-1)^{n+1} \left[(-1)^{n+1} + f_{n+1} \right] \\
&- e_4 q (-1)^{n+1} \left[-f_{n+2} - (-1)^{n+1} \right] \\
= &(-1)^{n+1} H_{n-1}^{p,q} - p + q + pe_2 + qe_3 + pe_4 + qe_4. \qquad \square
\end{aligned}
$$

From the above Theorem, we can remark that all identities valid for the Fibonacci quaternions can be easy adapted in an approximative similar expression for the generalized Fibonacci quaternions, if we use relation (1.1), a true relation in the both algebras $\mathbb{H}(\beta_1, \beta_2)$ and $\mathbb{H}$.

Proposition 2.3. *If $h_{n+1} = pf_n + qf_{n+1} = 0$, then we have:*

$$
H_{n+1}^2 = 3\frac{q^2}{f_n^2} \left[f_{2n+1}^2 - f_{n+1} f_{n-2} f_{2n+2} \right], \tag{2.4}
$$

where $H_{n+1}^2 \in \mathbb{H}(\beta_1, \beta_2)$.

Proof. Since $h_{n+1} = 0$, it results that $\boldsymbol{t}(H_{n+1}) = h_{n+1} = 0$, therefore $\boldsymbol{n}(H_{n+1}) = H_{n+1}^2$. From $h_n = pf_n + qf_{n+1} = 0$, we have $p = -\frac{qf_{n+1}}{f_n}$ and we obtain:

$$
p^2 + 2pq = \frac{q^2 f_{n+1}^2}{f_n^2} - 2q^2 \frac{f_{n+1}}{f_n} = -\frac{q^2 f_{n+1} f_{n-2}}{f_n^2}
$$

and

$$
p^2 + q^2 = \frac{q^2 f_{n+1}^2}{f_n^2} + q^2 = q^2 \frac{f_{n+1}^2 + f_n^2}{f_n^2} = q^2 \frac{f_{2n+1}}{f_n^2},
$$

since $f_{n+1}^2 + f_n^2 = f_{2n+1}$.

It results

$$
\begin{aligned}
\boldsymbol{n}(H_{n+1}) &= 3[(p^2 + 2pq) f_{2n+2} + (p^2 + q^2) f_{2n+1}] \\
&= 3\frac{q^2}{f_n^2} [-f_{n+1} f_{n-2} f_{2n+2} + f_{2n+1}^2]. \qquad \square
\end{aligned}
$$

In the following, we will compute the norm of a Fibonacci quaternion and of a generalized Fibonacci quaternion in the algebra $\mathbb{H}(\beta_1, \beta_2)$.

Let $F_n = f_n \cdot 1 + f_{n+1} e_2 + f_{n+2} e_3 + f_{n+3} e_4$ be the nth Fibonacci quaternion, then its norm is

$$
\boldsymbol{n}(F_n) = f_n^2 + \beta_1 f_{n+1}^2 + \beta_2 f_{n+2}^2 + \beta_1 \beta_2 f_{n+3}^2.
$$

Using recurrence of Fibonacci numbers and relations

$$
f_n^2 + f_{n-1}^2 = f_{2n-1}, \quad n \in \mathbb{N}, \tag{2.5}
$$

$$
f_{2n} = f_n^2 + 2 f_n f_{n-1}, \quad n \in \mathbb{N}, \tag{2.6}
$$

from [2], we have

$$
\begin{aligned}
\boldsymbol{n}\left(F_n\right) &= f_n^2+\beta_1 f_{n+1}^2+\beta_2 f_{n+2}^2+\beta_1\beta_2 f_{n+3}^2 \\
&= f_n^2+\beta_1 f_{n+1}^2+\beta_2\left(f_{n+2}^2+\beta_1 f_{n+3}^2\right) \\
&= f_{2n+1}+\left(\beta_1-1\right) f_{n+1}^2+\beta_2\left(f_{2n+5}+\left(\beta_1-1\right) f_{n+3}^2\right) \\
&= f_{2n+1}+\beta_2 f_{2n+5}+\left(\beta_1-1\right)\left(f_{n+1}^2+\beta_2 f_{n+3}^2\right) \\
&= \left(1+2\beta_2\right) f_{2n+1}+3\beta_2 f_{2n+2}+\left(\beta_1-1\right)\left(f_{n+1}^2+\beta_2 f_{n+3}^2\right) \\
&= h_{2n+2}^{1+2\beta_2,3\beta_2}+\left(\beta_1-1\right)\left(f_{n+1}^2+\beta_2 f_{n+3}^2\right) \\
&= h_{2n+2}^{1+2\beta_2,3\beta_2}+\left(\beta_1-1\right)\left(f_{2n+2}-2 f_n f_{n+1}+\beta_2 f_{2n+6}-2\beta_2 f_{n+2} f_{n+3}\right) \\
&= h_{2n+2}^{1+2\beta_2,3\beta_2}+\left(\beta_1-1\right)\left[f_{2n+2}+\beta_2 f_{2n+6}-2\left(f_n f_{n+1}+\beta_2 f_{n+2} f_{n+3}\right)\right] \\
&= h_{2n+2}^{1+2\beta_2,3\beta_2}+\left(\beta_1-1\right)\left[f_{2n+2}+\beta_2 f_{2n+6}\right. \\
&\quad \left.-2\left(f_n f_{n+1}+\beta_2 f_{n+2}^2+\beta_2 f_{n+1} f_{n+2}\right)\right] \\
&= h_{2n+2}^{1+2\beta_2,3\beta_2}+\left(\beta_1-1\right)\left[f_{2n+2}+\beta_2 f_{2n+6}\right. \\
&\quad -2\left(f_n f_{n+1}+\beta_2 f_{n+2}^2+\beta f_{n+1}^2+\beta_2 f_n f_{n+1}\right) \\
&= h_{2n+2}^{1+2\beta_2,3\beta_2}+\left(\beta_1-1\right)\left[f_{2n+2}+\beta_2 f_{2n+6}-2\left(1+\beta_2\right) f_n f_{n+1}-2\beta_2 f_{2n+3}\right] \\
&= h_{2n+2}^{1+2\beta_2,3\beta_2}+\left(\beta_1-1\right)\left[f_{2n+2}+\beta_2 f_{2n+4}+\beta_2 f_{2n+3}+\beta_2 f_{2n+4}-2\beta_2 f_{2n+3}\right. \\
&\quad \left.-2\left(1+\beta_2\right) f_n f_{n+1}\right] \\
&= h_{2n+2}^{1+2\beta_2,3\beta_2}+\left(\beta_1-1\right)\left[f_{2n+2}+2\beta_2 f_{2n+4}-\beta_2 f_{2n+3}-2\left(1+\beta_2\right) f_n f_{n+1}\right] \\
&= h_{2n+2}^{1+2\beta_2,3\beta_2}+\left(\beta_1-1\right)\left[f_{2n+2}+2\beta_2 f_{2n+2}+2\beta_2 f_{2n+3}-\beta_2 f_{2n+3}\right. \\
&\quad \left.-2\left(1+\beta_2\right) f_n f_{n+1}\right] \\
&= h_{2n+2}^{1+2\beta_2,3\beta_2}+\left(\beta_1-1\right)\left[\left(1+2\beta_2\right) f_{2n+2}+\beta_2 f_{2n+3}-2\left(1+\beta_2\right) f_n f_{n+1}\right] \\
&= h_{2n+2}^{1+2\beta_2,3\beta_2}+\left(\beta_1-1\right)\left[h_{2n+3}^{1+2\beta_2,\beta_2}-2\left(1+\beta_2\right) f_n f_{n+1}\right] \\
&= h_{2n+2}^{1+2\beta_2,3\beta_2}+\left(\beta_1-1\right) h_{2n+3}^{1+2\beta_2,\beta_2}-2\left(\beta_1-1\right)\left(1+\beta_2\right) f_n f_{n+1}.
\end{aligned}
$$

We just proved

Theorem 2.4. *The norm of the nth Fibonacci quaternion F_n in a generalized quaternion algebra is*

$$\boldsymbol{n}\left(F_n\right)=h_{2n+2}^{1+2\beta_2,3\beta_2}+\left(\beta_1-1\right) h_{2n+3}^{1+2\beta_2,\beta_2}-2\left(\beta_1-1\right)\left(1+\beta_2\right) f_n f_{n+1}. \tag{2.7}$$

Using formula (2.7) and relation (1.1) when $\beta_1=\beta_2=1$, we obtain formula (1.2).

Using the above theorem and relations (2.5) and (2.6), we can compute the norm of a generalized Fibonacci quaternion in a generalized quaternion algebra. Let $H_n=h_n\cdot 1+h_{n+1}e_2+h_{n+2}e_3+h_{n+3}e_4$ be the nth generalized Fibonacci quaternion. The norm is

$$\begin{aligned}
&\boldsymbol{n}\left(H_n^{p,q}\right) = h_n^2 + \beta_1 h_{n+1}^2 + \beta_2 h_{n+2}^2 + \beta_1\beta_2 h_{n+3}^2 \\
&= \left(pf_{n-1}+qf_n\right)^2 + \beta_1\left(pf_n+qf_{n+1}\right)^2 + \beta_2\left(pf_{n+1}+qf_{n+2}\right)^2 \\
&\quad + \beta_1\beta_2\left(pf_{n+2}+qf_{n+3}\right)^2 \\
&= p^2\left(f_{n-1}^2 + \beta_1 f_n^2 + \beta_2 f_{n+1}^2 + \beta_1\beta_2 f_{n+2}^2\right) \\
&\quad + q^2\left(f_n^2 + \beta_1 f_{n+1}^2 + \beta_2 f_{n+2}^2 + \beta_1\beta_2 f_{n+3}^2\right) \\
&\quad + 2pq\left(f_{n-1}f_n + \beta_1 f_n f_{n+1} + \beta_2 f_{n+1}f_{n+2} + \beta_1\beta_2 f_{n+3}f_{n+2}\right) \\
&= p^2 h_{2n}^{1+2\beta_2,3\beta_2} + p^2(\beta_1-1)h_{2n+1}^{1+2\beta_2,\beta_2} - 2p^2(\beta_1-1)\left(1+\beta_2\right)f_{n-1}f_n \\
&\quad + q^2 h_{2n+2}^{1+2\beta_2,3\beta_2} + q^2(\beta_1-1)h_{2n+3}^{1+2\beta_2,\beta_2} - 2q^2(\beta_1-1)\left(1+\beta_2\right)f_n f_{n+1} \\
&\quad + 2pq\left(1\text{-}\beta_1\right)f_n f_{n-1} + 2pq\beta_1 f_{2n} + 2pq\beta_2\left(1\text{-}\beta_1\right)f_{n+1}f_{n+2} + 2pq\beta_1\beta_2 f_{2n+4} \\
&= p^2 h_{2n}^{1+2\beta_2,3\beta_2} + p^2(\beta_1-1)h_{2n+1}^{1+2\beta_2,\beta_2} + q^2 h_{2n+2}^{1+2\beta_2,3\beta_2} + q^2(\beta_1-1)h_{2n+3}^{1+2\beta_2,\beta_2} \\
&\quad - 2p\left(\beta_1-1\right)\left(p\beta_2+p+q\right)f_{n-1}f_n - 2q^2(\beta_1-1)\left(1+\beta_2\right)f_n f_{n+1} \\
&\quad + h_{2n+1}^{2pq\beta_1,2pq\beta_1\beta_2} + 2pq\beta_1\beta_2(f_{2n}+f_{2n+3}) + 2pq\beta_2\left(1-\beta_1\right)f_{n+1}f_{n+2}.
\end{aligned}$$

From the above, we proved

Theorem 2.5. *The norm of the nth generalized Fibonacci quaternion $H_n^{p,q}$ in a generalized quaternion algebra is*

$$\begin{aligned}
\boldsymbol{n}\left(H_n^{p,q}\right) &= p^2 h_{2n}^{1+2\beta_2,3\beta_2} + p^2(\beta_1-1)h_{2n+1}^{1+2\beta_2,\beta_2} + q^2 h_{2n+2}^{1+2\beta_2,3\beta_2} \\
&\quad + q^2(\beta_1-1)h_{2n+3}^{1+2\beta_2,\beta_2} - 2p\left(\beta_1-1\right)\left(p\beta_2+p+q\right)f_{n-1}f_n \\
&\quad - 2q^2(\beta_1-1)\left(1+\beta_2\right)f_n f_{n+1} + h_{2n+1}^{2pq\beta_1,2pq\beta_1\beta_2} \\
&\quad + 2pq\beta_1\beta_2(f_{2n}+f_{2n+3}) + 2pq\beta_2\left(1-\beta_1\right)f_{n+1}f_{n+2}.
\end{aligned} \tag{2.8}$$

It is known that the expression for the nth term of a Fibonacci element is

$$f_n = \frac{1}{\sqrt{5}}[\alpha^n - \beta^n] = \frac{\alpha^n}{\sqrt{5}}[1 - \frac{\beta^n}{\alpha^n}], \tag{2.9}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.

From the above, we can compute the following,

$$\begin{aligned}
\lim_{n\to\infty}\boldsymbol{n}\left(F_n\right) &= \lim_{n\to\infty}(f_n^2 + \beta_1 f_{n+1}^2 + \beta_2 f_{n+2}^2 + \beta_1\beta_2 f_{n+3}^2) \\
&= \lim_{n\to\infty}(\frac{\alpha^{2n}}{5} + \beta_1\frac{\alpha^{2n+2}}{5} \underset{n\to\infty}{+} \beta_2\frac{\alpha^{2n+4}}{5} + \beta_1\beta_2\frac{\alpha^{2n+6}}{5}) \\
&= \operatorname{sgn}E(\beta_1,\beta_2)\cdot\infty
\end{aligned}$$

where

$$\begin{aligned}
E(\beta_1,\beta_2) &= (\frac{1}{5} + \frac{\beta_1}{5}\alpha^2 + \frac{\beta_2}{5}\alpha^4 + \frac{\beta_1\beta_2}{5}\alpha^6) \\
&= \frac{1}{5}\left(1 + \beta_1\left(\alpha+1\right) + \beta_2\left(3\alpha+2\right) + \beta_1\beta_2\left(8\alpha+5\right)\right) \\
&= \frac{1}{5}[1 + \beta_1 + 2\beta_2 + 5\beta_1\beta_2 + \alpha\left(\beta_1 + 3\beta_2 + 8\beta_1\beta_2\right)],
\end{aligned}$$

since $\alpha^2 = \alpha + 1$.

If $E(\beta_1, \beta_2) > 0$, there exist a number $n_1 \in \mathbb{N}$ such that for all $n \geq n_1$ we have

$$h_{2n+2}^{1+2\beta_2, 3\beta_2} + (\beta_1 - 1) h_{2n+3}^{1+2\beta_2, \beta_2} - 2(\beta_1 - 1)(1 + \beta_2) f_n f_{n+1} > 0.$$

In the same way, if $E(\beta_1, \beta_2) < 0$, there exist a number $n_2 \in \mathbb{N}$ such that for all $n \geq n_2$ we have

$$h_{2n+2}^{1+2\beta_2, 3\beta_2} + (\beta_1 - 1) h_{2n+3}^{1+2\beta_2, \beta_2} - 2(\beta_1 - 1)(1 + \beta_2) f_n f_{n+1} < 0.$$

Therefore for all $\beta_1, \beta_2 \in \mathbb{R}$ with $E(\beta_1, \beta_2) \neq 0$, in the algebra $\mathbb{H}(\beta_1, \beta_2)$ there is a natural number $n_0 = \max\{n_1, n_2\}$ such that $\boldsymbol{n}(F_n) \neq 0$, hence F_n is an invertible element for all $n \geq n_0$. Using the same arguments, we can compute

$$\begin{aligned}
\lim_{n\to\infty} (\boldsymbol{n}(H_n^{p,q})) &= \lim_{n\to\infty} \left(h_n^2 + \beta_1 h_{n+1}^2 + \beta_2 h_{n+2}^2 + \beta_1\beta_2 h_{n+3}^2\right) \\
&= \lim_{n\to\infty} [(pf_{n-1} + qf_n)^2 + \beta_1 (pf_n + qf_{n+1})^2 + \beta_2 (pf_{n+1} + qf_{n+2})^2 \\
&\quad + \beta_1\beta_2 (pf_{n+2} + qf_{n+3})^2] \\
&= \operatorname{sgn} E'(\beta_1, \beta_2) \cdot \infty
\end{aligned}$$

where

$$\begin{aligned}
&E'(\beta_1, \beta_2) \\
&= \frac{1}{5}[(p + \alpha q)^2 + \beta_1 \left(p\alpha + \alpha^2 q\right)^2 + \beta_2 \left(p\alpha^2 + \alpha^3 q\right)^2 + \beta_1\beta_2 \left(p\alpha^3 + \alpha^4 q\right)^2] \\
&= \frac{1}{5}(p + \alpha q)^2 [1 + \beta_1 \alpha^2 + \beta_2 \alpha^4 + \beta_1\beta_2 \alpha^6] \\
&= \frac{1}{5}(p + \alpha q)^2 E(\beta_1, \beta_2).
\end{aligned}$$

Therefore for all $\beta_1, \beta_2 \in \mathbb{R}$ with $E'(\beta_1, \beta_2) \neq 0$ in the algebra $\mathbb{H}(\beta_1, \beta_2)$ there exist a natural number n_0' such that $\boldsymbol{n}(H_n^{p,q}) \neq 0$, hence $H_n^{p,q}$ is an invertible element for all $n \geq n_0'$.

Now, we proved

Theorem 2.6. *For all $\beta_1, \beta_2 \in \mathbb{R}$ with $E'(\beta_1, \beta_2) \neq 0$, there exists a natural number n' such that for all $n \geq n'$ Fibonacci elements F_n and generalized Fibonacci elements $H_n^{p,q}$ are invertible elements in the algebra $\mathbb{H}(\beta_1, \beta_2)$.*

Remark 2.7. Algebra $\mathbb{H}(\beta_1, \beta_2)$ is not always a division algebra, and sometimes can be difficult to find an example of invertible element. Above Theorem provides us infinite sets of invertible elements in this algebra, namely Fibonacci elements and generalized Fibonacci elements.

3. Fibonacci-Narayana Quaternions

In this section, we will study some properties of Fibonacci-Narayana elements in the algebra $\mathbb{H}(\beta_1, \beta_2)$.

Theorem 3.1. *For the Fibonacci-Narayana quaternion U_n, we have*

$$\text{a)} \sum_{m=0}^{n} U_m = U_{n+3} - U_2,$$

$$\text{b)} \sum_{m=0}^{n} U_{3m} = U_{3n+1} - 1 - e_4.$$

Proof. a)

$$\sum_{m=0}^{n} U_m = \sum_{m=0}^{n} u_m + e_2 \sum_{m=1}^{n+1} u_m + e_3 \sum_{m=2}^{n+2} u_m + e_4 \sum_{m=3}^{n+3} u_m =: (*).$$

Since $u_0 = 0$, we consider that the term $\sum_{m=0}^{n} u_m$ is equal with $\sum_{m=1}^{n} u_m$. We can use property 1) from the introduction and we obtain

$$\begin{aligned}(*) &= u_{n+3} - 1 + e_2(u_{n+4} - 1) + e_3(u_{n+5} - 2) + e_4(u_{n+6} - 3)\\ &= U_{n+3} - (1 + e_2 + 2e_3 + 3e_4) = U_{n+3} - U_2.\end{aligned}$$

b) Since $u_0 = 0$, the term $\sum_{m=0}^{n} u_{3m}$ is equal with $\sum_{m=1}^{n} u_{3m}$, therefore

$$\sum_{m=0}^{n} U_{3m} = \sum_{m=0}^{n} u_{3m} + e_2 \sum_{m=0}^{n} u_{3m+1} + e_3 \sum_{m=0}^{n} u_{3m+2} + e_4 \sum_{m=0}^{n} u_{3m+3} =: (**);$$

using properties 4), 2), 3), and again 4), we have

$$(**) = u_{3n+1} - 1 + u_{3n+2}e_2 + u_{3n+3}e_3 + (u_{3n+4} - 1)e_4 = U_{3n+1} - 1 - e_4. \qquad \square$$

Let $\{u_n\}$ be a Fibonacci-Narayana sequence, and let $U_n = u_n \cdot 1 + u_{n+1}e_2 + u_{n+2}e_3 + u_{n+3}e_4$ be the nth Fibonacci-Narayana quaternion.

The function $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n + \ldots$ is called *the generating function* for the sequence $\{a_0, a_1, a_2, \ldots\}$. In [5], the author found a generating function for Fibonacci quaternions. In the following theorem, we established the generating function for Fibonacci-Narayana quaternions.

Theorem 3.2. *The generating function for the Fibonacci-Narayana quaternion U_n is*

$$G(t) = \frac{U_0 + (U_1 - U_0)t + (U_2 - U_1)t^2}{1 - t - t^3} = \frac{e_1 + e_2 + e_3 + (1 + e_3)t + (e_2 + e_3)t^2}{1 - t - t^3}. \qquad (3.1)$$

Proof. Assuming that the generating function of the quaternion Fibonacci-Narayana sequence $\{U_n\}$ has the form $G(t) = \sum_{n=0}^{\infty} U_n t^n$, we obtain that

$$\begin{aligned}&\sum_{n=0}^{\infty} U_n t^n - t\sum_{n=0}^{\infty} U_n t^n - t^3\sum_{n=0}^{\infty} U_n t^n\\&= U_0 + U_1 t + U_2 t^2 + U_3 t^3 + \ldots - U_0 t - U_1 t^2 - U_2 t^3 - U_3 t^4 - \ldots\\&\quad - U_0 t^3 - U_1 t^4 - U_2 t^5 - U_3 t^6 - \ldots\\&= U_0 + (U_1 - U_0)t + (U_2 - U_1)t^2,\end{aligned}$$

since $U_n = U_{n-1} + U_{n-3}, n \geq 3$ and the coefficients of t^n for $n \geq 3$ are equal with zero.

It results

$$U_0+(U_1 - U_0)t+(U_2 - U_1)t^2=\sum_{n=0}^{\infty} U_n t^n\,(1 - t - t^3),$$

or in equivalent form

$$\frac{U_0+(U_1 - U_0)t+(U_2 - U_1)t^2}{1 - t - t^3}=\sum_{n=0}^{\infty} U_n t^n.$$

The theorem is proved. □

Theorem 3.3 (Binet-Cauchy formula for Fibonacci-Narayana numbers). *Let* $u_n = u_{n-1} + u_{n-3}, n \geq 3$ *be the nth Fibonacci-Narayana number, then*

$$u_n=\frac{1}{(\alpha\text{-}\beta)\,(\beta\text{-}\gamma)\,(\gamma\text{-}\alpha)}\left[\alpha^{n+1}\,(\gamma\text{-}\beta)+\beta^{n+1}\,(\alpha\text{-}\gamma)+\gamma^{n+1}\,(\beta\text{-}\alpha)\right], \tag{3.2}$$

where α, β, γ *are the solutions of the equation* $t^3 - t^2 - 1 = 0$.

Proof. Supposing that $u_n = A\alpha^n + B\beta^n + C\gamma^n, A, B, C \in \mathbb{C}$ and using the recurrence formula for the Fibonacci-Narayana numbers, $u_n = u_{n-1} + u_{n-3}$, it results that α, β, γ are the solutions of the equation $t^3 - t^2 - 1 = 0$. Since $u_0 = 0, u_1 = 1, u_2 = 1$, we obtain the following system:

$$\begin{cases} A + B + C = 0,\\ A\alpha + B\beta + C\gamma = 1,\\ A\alpha^2 + B\beta^2 + C\gamma^2 = 1.\end{cases} \tag{3.3}$$

The determinant of this system is a Vandermonde determinant and can be computed easily. It is $\Delta = (\alpha\text{-}\beta)\,(\beta\text{-}\gamma)\,(\gamma\text{-}\alpha) \neq 0$.

Using Cramer's rule, the solutions of the system (3.3) are

$$A = \frac{\alpha\,(\gamma - \beta)}{(\alpha - \beta)\,(\beta - \gamma)\,(\gamma - \alpha)} = \frac{\alpha}{(\beta - \alpha)\,(\gamma - \alpha)},$$

$$B = \frac{\beta\,(\alpha - \gamma)}{(\alpha - \beta)\,(\beta - \gamma)\,(\gamma - \alpha)} = \frac{\beta}{(\alpha - \beta)\,(\gamma - \beta)},$$

$$C = \frac{\gamma\,(\beta - \alpha)}{(\alpha - \beta)\,(\beta - \gamma)\,(\gamma - \alpha)} = \frac{\gamma}{(\beta - \gamma)\,(\alpha - \gamma)},$$

therefore relation (3.2) is true. □

Theorem 3.4 (Binet-Cauchy formula for the Fibonacci-Narayana quaternions). *Let* $U_n = u_n \cdot 1 + u_{n+1}e_2 + u_{n+2}e_3 + u_{n+3}e_4$ *be the* n*th Fibonacci-Narayana quaternion, then*

$$U_n = D\frac{\alpha^{n+1}}{(\beta\text{-}\alpha)(\gamma\text{-}\alpha)} + E\frac{\beta^{n+1}}{(\alpha\text{-}\beta)(\gamma\text{-}\beta)} + F\frac{\gamma^{n+1}}{(\beta\text{-}\gamma)(\alpha\text{-}\gamma)}, \tag{3.4}$$

where α, β, γ *are the solutions of the equation* $t^3 - t^2 - 1 = 0$ *and*

$$D = 1 + \alpha e_1 + \alpha^2 e_2 + \alpha^3 e_3,$$

$$E = 1 + \beta e_1 + \beta^2 e_2 + \beta^3 e_3,$$

$$F = 1 + \gamma e_1 + \gamma^2 e_2 + \gamma^3 e_3.$$

Proof. Using relation (3.2), we have that

$$\begin{aligned} U_n &= u_n \cdot 1 + u_{n+1}e_2 + u_{n+2}e_3 + u_{n+3}e_4 \\ &= \frac{1}{(\alpha\text{-}\beta)(\beta\text{-}\gamma)(\gamma\text{-}\alpha)}\left[\left(\alpha^{n+1}(\gamma\text{-}\beta) + \beta^{n+1}(\alpha\text{-}\gamma) + \gamma^{n+1}(\beta\text{-}\alpha)\right) \cdot 1\right. \\ &\quad + \left(\alpha^{n+2}(\gamma - \beta) + \beta^{n+2}(\alpha - \gamma) + \gamma^{n+2}(\beta - \alpha)\right) e_1 \\ &\quad + \left(\alpha^{n+3}(\gamma - \beta) + \beta^{n+3}(\alpha - \gamma) + \gamma^{n+3}(\beta - \alpha)\right) e_2 \\ &\quad \left.+ \left(\alpha^{n+4}(\gamma - \beta) + \beta^{n+4}(\alpha - \gamma) + \gamma^{n+4}(\beta - \alpha)\right) e_3\right] \\ &= \frac{1}{(\alpha - \beta)(\beta - \gamma)(\gamma - \alpha)}\left[\alpha^{n+1}(\gamma\text{-}\beta)\left(1+\alpha e_1+\alpha^2 e_2+\alpha^3 e_3\right)\right. \\ &\quad + \beta^{n+1}(\alpha\text{-}\gamma)\left(1 + \beta e_1 + \beta^2 e_2 + \beta^3 e_3\right) \\ &\quad \left.+ \gamma^{n+1}(\beta\text{-}\alpha)\left(1 + \gamma e_1 + \gamma^2 e_2 + \gamma^3 e_3\right)\right]. \end{aligned}$$

□

For negative n, the nth Fibonacci-Narayana number is defined as in the following: $u_n = u_{n+3} - u_{n+2},\ u_0 = 0, u_1 = 1, u_2 = 1$. In the same way is defined the Fibonacci-Narayana quaternion U_n for negative n.

Theorem 3.5. *Let* $U_n = u_n \cdot 1 + u_{n+1}e_2 + u_{n+2}e_3 + u_{n+3}e_4$ *be the* n*th Fibonacci-Narayana quaternion, therefore the following relations are true:*

$$1)\ \sum_{i=0}^{n} C_n^i U_{2n-2i-1} = U_{3n-1}.$$

$$2)\ \sum_{i=0}^{n} C_n^i U_{3n-2i-1} = U_{4n-1}.$$

Proof. 1) Using Newton's formula, it results that

$$\begin{aligned} \left(t^2+1\right)^n &= C_n^0\left(t^2\right)^n + C_n^1\left(t^2\right)^{n-1} + C_n^2\left(t^2\right)^{n-2} + \ldots + C_n^n \\ &= C_n^0 t^{2n} + C_n^1 t^{2n-2} + C_n^2 t^{2n-4} + \ldots + C_n^n. \end{aligned}$$

From here, we have that

$$
\begin{aligned}
&\sum_{i=0}^{n} \mathsf{C}_n^i U_{2n-2i-1} \\
&= \mathsf{C}_n^0 U_{2n-1} + \mathsf{C}_n^1 U_{2n-3} + \mathsf{C}_n^2 U_{2n-5} + \ldots + \mathsf{C}_n^n U_{-1} \\
&= \mathsf{C}_n^0 \left(D\frac{\alpha^{2n}}{(\beta-\alpha)(\gamma-\alpha)} + E\frac{\beta^{2n}}{(\alpha-\beta)(\gamma-\beta)} + F\frac{\gamma^{2n}}{(\beta-\gamma)(\alpha-\gamma)} \right) \\
&\quad + \mathsf{C}_n^1 \left(D\frac{\alpha^{2n-2}}{(\beta-\alpha)(\gamma-\alpha)} + E\frac{\beta^{2n-2}}{(\alpha-\beta)(\gamma-\beta)} + F\frac{\gamma^{2n-2}}{(\beta-\gamma)(\alpha-\gamma)} \right) + \ldots \\
&\quad + \mathsf{C}_n^n \left(D\frac{1}{(\beta-\alpha)(\gamma-\alpha)} + E\frac{1}{(\alpha-\beta)(\gamma-\beta)} + F\frac{1}{(\beta-\gamma)(\alpha-\gamma)} \right) \\
&= D\frac{1}{(\beta-\alpha)(\gamma-\alpha)} (\mathsf{C}_n^0 \alpha^{2n} + \mathsf{C}_n^1 \alpha^{2n-2} + \ldots + \mathsf{C}_n^n 1) \\
&\quad + E\frac{1}{(\alpha-\beta)(\gamma-\beta)} (\mathsf{C}_n^0 \beta^{2n} + \mathsf{C}_n^1 \beta^{2n-2} + \ldots + \mathsf{C}_n^n 1) \\
&\quad + F\frac{1}{(\beta-\gamma)(\alpha-\gamma)} (\mathsf{C}_n^0 \gamma^{2n} + \mathsf{C}_n^1 \gamma^{2n-2} + \ldots \mathsf{C}_n^n 1) \\
&= D\frac{1}{(\beta-\alpha)(\gamma-\alpha)} \left(\alpha^2+1\right)^n + E\frac{1}{(\alpha-\beta)(\gamma-\beta)} \left(\beta^2+1\right)^n \\
&\quad + F\frac{1}{(\beta-\gamma)(\alpha-\gamma)} \left(\gamma^2+1\right)^n \\
&= D\frac{1}{(\beta-\alpha)(\gamma-\alpha)} \alpha^{3n} + E\frac{1}{(\alpha-\beta)(\gamma-\beta)} \beta^{3n} + F\frac{1}{(\beta-\gamma)(\alpha-\gamma)} \gamma^{3n} = U_{3n-1}.
\end{aligned}
$$

We used that $\alpha^3 = \alpha^2 + 1, \beta^3 = \beta^2 + 1, \gamma^3 = \gamma^2 + 1$.

2) Since $t^3 = t^2 + 1$, starting from relation $\left(t^3 + t\right)^n = t^{4n}$, for $t \in \{\alpha, \beta, \gamma\}$, by straightforward calculations as in 2), we obtain the asked relation. □

Conclusions

In this paper we investigated some new properties of generalized Fibonacci quaternions and Fibonacci-Narayana quaternions. Since Fibonacci-Narayana quaternions was not intensive studied until now, we expect to find in the future more and surprising new properties. We studied these elements for the beauty of the relations obtained, but the main reason is that the elements of this type, namely Fibonacci X elements, where $X \in \{$*quaternions, generalized quaternions*$\}$, can provide us many important information in the algebra $\mathbb{H}(\beta_1, \beta_2)$, as for example: sets of invertible elements in algebraic structures without division.

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