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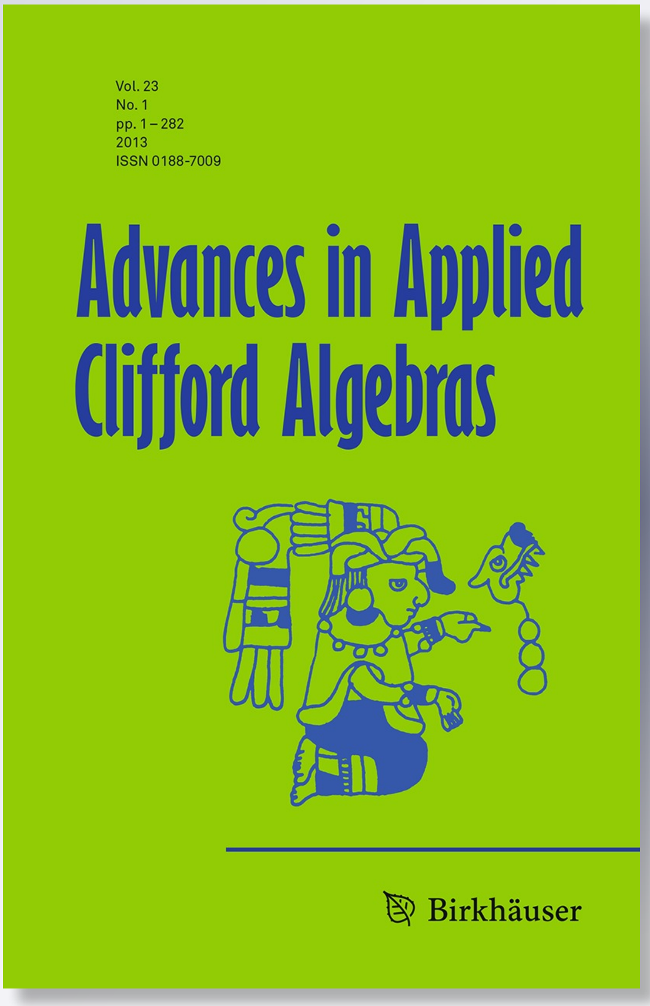
Advances in Applied Clifford Algebras

ISSN 0188-7009
Volume 23
Number 1

Adv. Appl. Clifford Algebras (2013) 23:63-76
DOI 10.1007/s00006-012-0344-6



Adv. Appl. Clifford Algebras 23 (2013), 63–76

0188-7009/010063-14
published online June 29, 2012
DOI 10.1007/s00006-012-0344-6

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Some Identities in Algebras Obtained by the Cayley-Dickson Process

Cristina Flaut[1] and Vitalii Shpakivskyi

Abstract. Polynomial identities in algebras are the central objects of Polynomial Identities Theory. They play an important role in learning of algebras properties. In particular, the Hall identity is fulfilled in the quaternion algebra and does not hold in other non-commutative associative algebras. For this reason, the Hall identity is important for the quaternion algebra. The idea of this work is to generalize the Hall identity to algebras obtained by the Cayley-Dickson process.

Starting from the above remarks, in this paper, we prove that the Hall identity is true in all algebras obtained by the Cayley-Dickson process and, in some conditions, the converse of this statement is also true for split quaternion algebras. From Hall identity, we will find some new properties and identities in algebras obtained by the Cayley-Dickson process.

Keywords. Cayley-Dickson process; Clifford algebras; Hall identity.

0. Introduction

In October 1843, William Rowan Hamilton discovered the quaternions, a 4-dimensional algebra over $\mathbb{R}$ which is associative and noncommutative algebra. In December 1843, John Graves discovered the octonions, an 8-dimensional algebra over $\mathbb{R}$ which is nonassociative and noncommutative algebra. These algebras were rediscovered by Arthur Cayley in 1845 and are also known sometimes as the Cayley numbers. This process, of passing from $\mathbb{R}$ to $\mathbb{C}$, from $\mathbb{C}$ to $\mathbb{H}$ and from$\mathbb{H}$ to $\mathbb{O}$ has been generalized to algebras over fields and over rings. It is called the *Cayley-Dickson doubling process* or the *Cayley–Dickson process.* In 1878, W. K. Clifford discovered Clifford algebras defined to have generators $e_1, e_2, \ldots, e_n$ which anti-commute and satisfy $e_i^2 = a_i \in \mathbb{R}$, for all $i \in \{1, 2, \ldots, n\}$. These algebras generalize the real numbers, complex numbers and quaternions (see [Le; 06]).

[1] To my family: Dan and Ana-Theodora.

Even if are old, quaternions, octonions and Clifford algebras have at present many applications, as for example in physics, coding theory, computer vision, etc. For this reasons these algebras are intense studied. In [Ha; 43], Hall proved that the identity $(xy - yx)^2 z = z(xy - yx)^2$ holds for all elements x, y, z in a quaternion algebra. This identity is called *Hall identity*. Moreover, he also proved the converse: if the Hall identity is true in a skew-field F, then F is a quaternion division algebra. In [Smi; 50], Smiley proved that the Hall identity is true for the octonions and he also proved the converse: if the Hall identity is true in an alternative division algebra A, then A is an octonion division algebra.

In this paper we will prove that the Hall identity is true in all algebras obtained by the Cayley-Dickson process and, in some conditions, the converse is true for split quaternion algebras.

1. Preliminaries

In this paper, we assume that K is a commutative field with $charK \neq 2$ and A is an algebra over the field K. The *center* C of an algebra A is the set of all elements $c \in A$ which commute and associate with all elements $x \in A$. An algebra A is a *simple* algebra if A is not a zero algebra and $\{0\}$ and A are the only ideals of A. The algebra A is called *central simple* if the algebra $A_F = F \otimes_K A$ is simple for every extension F of K. A central simple algebra is a simple algebra. An algebra A is called *alternative* if $x^2 y = x(xy)$ and $xy^2 = (xy)y$, for all $x, y \in A$, *flexible* if $x(yx) = (xy)x = xyx$, for all $x, y \in A$ and *power associative* if the subalgebra $< x >$ of A, generated by any element $x \in A$, is associative. Each alternative algebra is a flexible algebra and a power associative algebra. In each alternative algebra A, the identities

1) $a(x(ay)) = (axa)y$
2) $((xa)y)a = x(aya)$
3) $(ax)(ya) = a(xy)a$

hold, for all $a, x, y \in A$. These identities are called the *Moufang identities.*

A unitary algebra $A \neq K$ such that we have $x^2 + \alpha_x x + \beta_x = 0$, for each $x \in A$, with $\alpha_x, \beta_x \in K$, is called a *quadratic algebra.*

In the following, we briefly present the *Cayley-Dickson process* and the properties of the algebras obtained. For details about the Cayley-Dickson process, the reader is referred to [Sc; 66] and [Sc; 54].

Let A be a finite dimensional unitary algebra over a field K with a *scalar involution*

$$^{-} : A \to A, \quad a \to \overline{a},$$

i.e. a linear map satisfying the following relations:

$$\overline{ab} = \overline{b}\,\overline{a}, \quad \overline{\overline{a}} = a,$$

and

$$a + \overline{a},\ a\overline{a} \in K \cdot 1 \text{ for all } a, b \in A.$$

The element $\overline{a}$ is called the *conjugate* of the element a, the linear form

$$t : A \rightarrow K\,,\ t\,(a) = a + \overline{a}$$

and the quadratic form

$$n : A \rightarrow K,\ n\,(a) = a\overline{a}$$

are called the *trace* and the *norm* of the element a, respectively. Hence, an algebra A with a scalar involution is quadratic.

Let $\gamma \in K$ be a fixed non-zero element. We define the following algebra multiplication on the vector space

$$A \oplus A : (a_1, a_2)\,(b_1, b_2) = \left(a_1 b_1 + \gamma \overline{b_2} a_2, a_2 \overline{b_1} + b_2 a_1\right).$$

We obtain an algebra structure over $A \oplus A$, denoted by (A, γ) and called the *algebra obtained from A by the Cayley-Dickson process.* We have $\dim (A, \gamma) = 2 \dim A$.

Let $x \in (A, \gamma)$, $x = (a_1, a_2)$. The map

$$^{-} : (A, \gamma) \rightarrow (A, \gamma)\ ,\ x \rightarrow \bar{x}\ = (\bar{a}_1,\ \text{-}a_2)\,,$$

is a scalar involution of the algebra (A, γ), extending the involution $^{-}$ of the algebra A. Let

$$t\,(x) = t(a_1)$$

and

$$n\,(x) = n\,(a_1) - \gamma n(a_2)$$

be the *trace* and the *norm* of the element $x \in (A, \gamma)$, respectively.

If we take $A = K$ and apply this process t times, $t \geq 1$, we obtain an algebra over K,

$$A_t = \left(\frac{\gamma_1, \ldots, \gamma_t}{K}\right). \tag{1.1}$$

By induction in this algebra, the set $\{1, e_2, \ldots, e_n\}, n = 2^t$, generates a basis with the properties:

$$e_i^2 = \gamma_i 1,\ _i \in K, \gamma_i \neq 0,\ i = 2, \ldots, n \tag{1.2}$$

and

$$e_i e_j = -e_j e_i = \beta_{ij} e_k,\ \beta_{ij} \in K,\ \beta_{ij} \neq 0, i \neq j, i, j =\ 2, \ldots, n, \tag{1.3}$$

β_{ij} and e_k being uniquely determined by e_i and e_j.

From [Sc; 54], Lemma 4, it results that in any algebra A_t with the basis $\{1, e_2, \ldots, e_n\}$ satisfying relations (1.2) and (1.3) we have:

$$e_i\,(e_i x) = \gamma_i^2 = (x e_i) e_i, \tag{1.4}$$

for all $i \in \{1, 2, \ldots, n\}$ and for every $x \in A$.

It is known that if an algebra A is finite-dimensional, then it is a division algebra if and only if A does not contain zero divisors (see [Sc; 66]).

Algebras A_t of dimension 2^t obtained by the Cayley-Dickson process, described above, are central-simple, flexible and power associative for all $t \geq 1$ and, in general, are not division algebras for all $t \geq 1$. But there are fields on which, if we apply the Cayley-Dickson process, the resulting algebras A_t are division algebras for all $t \geq 1$ (see [Br; 67] and [Fl; 12]). We remark that the field K is the center of the algebra A_t, for $t \geq 2$ (see [Sc; 54]).

Let K be a field containing ω, a primitive n-th root of unity, and A be an associative algebra over K. Let $S = \{e_1, \ldots, e_r\}$ be a set of elements in A such that the following condition are fulfilled: $e_i e_j = \omega e_j e_i$ for all $i < j$ and $e_i^n \in \{1, \omega, \omega^2, \ldots, \omega^{n-1}\}$. A *generalized Clifford algebra* over the field K, denoted by $Cl_r^n(K)$, is defined to be the polynomial algebra $K[e_1, \ldots, e_r]$. We remark that the algebra $Cl_r^n(K)$ is an associative algebra. For details about generalized Clifford algebra, the reader is referred to [Ki, Ou; 99], [Ko; 10] and [Sm; 91].

Example 1.1. 1) For $n = 2$, we obtain $Cl_r^2(K)$ with $\omega = -1$, $e_i e_j = -e_j e_i$ for all $i < j$ and $e_i^2 \in \{-1, 1\}$. If $r = p + q$ and $e_1^2 = \ldots = e_p^2 = 1$, $e_{p+1}^2 = .. = e_q^2 = -1$, then the algebra $Cl_r^2(K)$ will be denoted $Cl_{p,q}(K)$.

2) i) For $p = q = 0$ we have $Cl_{0,0}(K) \simeq K$;

ii) For $p = 0, q = 1$, it results that $Cl_{0,1}(K)$ is a two-dimensional algebra generated by a single vector e_1 such that $e_1^2 = -1$ and therefore $Cl_{0,1}(K) \simeq K(e_1)$. For $K = \mathbb{R}$ it follows that $Cl_{0,1}(\mathbb{R}) \simeq \mathbb{C}$.

iii) For $p = 0, q = 2$, the algebra $Cl_{0,2}(K)$ is a four-dimensional algebra spanned by the set $\{1, e_1, e_2, e_1 e_2\}$. Since $e_1^2 = e_2^2 = (e_1 e_2)^2 = -1$ and $e_1 e_2 = -e_2 e_1$, we obtain that this algebra is isomorphic to the division quaternions algebra $\mathbb{H}$.

iv) For $p = 1, q = 1$ or $p = 2, q = 0$, we obtain the algebra $Cl_{1,1}(K) \simeq Cl_{2,0}(K)$ which is isomorphic with a split quaternion algebra, called *paraquaternion algebra* or *antiquaternion algebra* (see [Iv, Za; 05]).

2. Main Results

Let A be an algebra obtained by the Cayley-Dickson process with the basis $\{e_0 := 1, e_1, \ldots, e_n\}$ such that $e_m e_r = -e_r e_m$, $r \neq m, e_m^2 = \gamma_m \in K, m \in \{1, 2, \ldots, n\}$. For elements $a = \sum\limits_{m=0}^{n} a_m e_m, b = \sum\limits_{m=0}^{n} b_m e_m$ we define an element in K, denoted by $T(a, b)$, $T(a, b) = \sum\limits_{m=0}^{n} e_m^2 a_m b_m$. We denote by $\overrightarrow{A}$ the set of the elements $\{\overrightarrow{a} \mid \overrightarrow{a} = \sum\limits_{m=1}^{n} a_m e_m, a_m \in K\}$. It results that the conjugate of the element a can be written as $\overline{a} = a_0 - \overrightarrow{a}$. Obviously, $\overrightarrow{(\overrightarrow{a})} = \overrightarrow{a}$ and $\overrightarrow{e_m} = e_m$.

Lemma 2.1. *Let A be an algebra obtained by the Cayley-Dickson process. The following equalities are true:*

1)
$$T(a, b) = T(b, a),$$
for all $a, b \in A$.

2)
$$T(\lambda a, b) = \lambda T(a, b),$$
for all $\lambda \in K,\ a, b \in A$.

3)
$$T(a, b + c) = T(a, b) + T(a, c),$$
for all $a, b, c \in A$.

4)
$$T(a, \overline{a}) = a\overline{a} = n(a),$$
for all $a \in A$

5)
$$\overrightarrow{a}\overrightarrow{b} = 2T\left(\overrightarrow{a}, \overrightarrow{b}\right) - \overrightarrow{b}\overrightarrow{a}, \tag{2.1}$$
$$ab = ba - 2\overrightarrow{b}\overrightarrow{a} + 2T\left(\overrightarrow{a}, \overrightarrow{b}\right), \tag{2.2}$$
$$\overrightarrow{\overrightarrow{a}\overrightarrow{b}} = -T\left(\overrightarrow{a}, \overrightarrow{b}\right) + \overrightarrow{a}\overrightarrow{b}. \tag{2.3}$$
$$(\overrightarrow{a})^2 \in K, \tag{2.4}$$
for all $a, b \in A$.

Proof. Relations from 1), 2), 3), 4) are obvious.

5) For $\overrightarrow{a} = \sum_{m=1}^{n} a_m e_m$, $\overrightarrow{b} = \sum_{m=1}^{n} b_m e_m$, we obtain
$$\overrightarrow{a}\overrightarrow{b} = \sum_{m=1}^{n} a_m e_m \cdot \sum_{m=1}^{n} b_m e_m = \sum_{m=1}^{n} e_m^2 a_m b_m + \alpha = T\left(\overrightarrow{a}, \overrightarrow{b}\right) + \alpha, \alpha \in \overrightarrow{A}. \tag{2.5}$$
Computing $\overrightarrow{b}\overrightarrow{a}$, it follows that
$$\overrightarrow{b}\overrightarrow{a} = T\left(\overrightarrow{a}, \overrightarrow{b}\right) - \alpha, \alpha \in \overrightarrow{A}. \tag{2.6}$$
If we add relations (2.5) and (2.6), it results $\overrightarrow{a}\overrightarrow{b} + \overrightarrow{b}\overrightarrow{a} = 2T\left(\overrightarrow{a}, \overrightarrow{b}\right)$, therefore relation (2.1) is obtained.

For $a = a_0 + \overrightarrow{a}$ and $b = b_0 + \overrightarrow{b}$, we compute
$$ab = (a_0 + \overrightarrow{a})\left(b_0 + \overrightarrow{b}\right) = a_0 b_0 + a_0\overrightarrow{b} + b_0\overrightarrow{a} + \overrightarrow{a}\overrightarrow{b}$$
and
$$ba = \left(b_0 + \overrightarrow{b}\right)(a_0 + \overrightarrow{a}) = b_0 a_0 + b_0\overrightarrow{a} + a_0\overrightarrow{b} + \overrightarrow{b}\overrightarrow{a}.$$

Subtracting the last two relations and using relation (2.1), we obtain $ab - ba = \overrightarrow{a}\overrightarrow{b} - \overrightarrow{b}\overrightarrow{a} = 2T\left(\overrightarrow{a}, \overrightarrow{b}\right) - 2\overrightarrow{b}\overrightarrow{a}$, then relation (2.2) is proved.

Relation (2.3) is obvious.

For $\overrightarrow{a} = \sum_{m=1}^{n} a_m e_m$, it results that $(\overrightarrow{a})^2 = \sum_{m=1}^{n} a_m^2 \gamma_m \in K$. $\square$

For quaternion algebras, the above result was proved in [Sz; 09].

Theorem 2.2. *Let A be an algebra obtained by the Cayley-Dickson process such that $e_m^2 = -1$, for all $m \in \{1, 2, \ldots, n\}$. If $n - 1 \in K - \{0\}$, then, for all $x \in A$, we have*

$$\overline{x} = \frac{1}{1-n} \sum_{m=0}^{n} e_m x e_m.$$

Proof. Let $x = \sum\limits_{m=0}^{n} e_m x_m$. From Lemma 2.1 and relation (1.4), we obtain

$$\begin{aligned}
\sum_{m=0}^{n} e_m x e_m &= x + \sum_{m=1}^{n} e_m x e_m \\
&= x + \sum_{m=1}^{n} e_m \left(e_m x - 2e_m \overrightarrow{x} + 2T\left(e_m, \overrightarrow{x}\right)\right) \\
&= x + \sum_{m=1}^{n} e_m^2 x - 2\sum_{m=1}^{n} e_m^2 \overrightarrow{x} + 2\sum_{m=1}^{n} e_m^2 e_m x_m \\
&= x - nx + 2n\overrightarrow{x} - 2\sum_{m=1}^{n} e_m x_m \\
&= (1-n)\, x - 2\,(1-n)\, \overrightarrow{x} = (1-n)\left(x - 2\overrightarrow{x}\right) \\
&= (1-n)\, \overline{x}.
\end{aligned}$$

□

For the real quaternions, the below relation is well known:

$$\overline{x} = -\frac{1}{2}\left(x + ixi + jxj + kxk\right).$$

Theorem 2.3. *Let A be an algebra obtained by the Cayley-Dickson process. Then for all $x, y, z \in A$, it results that*

$$(xy - yx)^2 z = z\,(xy - yx)^2. \tag{2.7}$$

Proof. We will compute both members of the equality $(xy - yx)^2 z = z\,(xy - yx)^2$. Using relation (2.2) from Lemma 1 and since $T\left(\overrightarrow{x}, \overrightarrow{y}\right) \in K$, we obtain

$$\begin{aligned}
&\left(-2\overrightarrow{y}\,\overrightarrow{x} + 2T\left(\overrightarrow{x}, \overrightarrow{y}\right)\right)^2 z = z\left(-2\overrightarrow{y}\,\overrightarrow{x} + 2T\left(\overrightarrow{x}, \overrightarrow{y}\right)\right)^2 \Rightarrow \\
&\Rightarrow \left[4\left(\overrightarrow{y}\,\overrightarrow{x}\right)^2 + 4T^2\left(\overrightarrow{x}, \overrightarrow{y}\right) - 8\left(\overrightarrow{y}\,\overrightarrow{x}\right) T\left(\overrightarrow{x}, \overrightarrow{y}\right)\right] z = \\
&= z\left[4\left(\overrightarrow{y}\,\overrightarrow{x}\right)^2 + 4T^2\left(\overrightarrow{x}, \overrightarrow{y}\right) - 8\left(\overrightarrow{y}\,\overrightarrow{x}\right) T\left(\overrightarrow{x}, \overrightarrow{y}\right)\right] \Rightarrow \\
&\Rightarrow 4\left(\overrightarrow{y}\,\overrightarrow{x}\right)^2 z + 4T^2\left(\overrightarrow{x}, \overrightarrow{y}\right) z - 8T\left(\overrightarrow{x}, \overrightarrow{y}\right)\left(\overrightarrow{y}\,\overrightarrow{x}\right) z = \\
&= 4z\left(\overrightarrow{y}\,\overrightarrow{x}\right)^2 + 4T^2\left(\overrightarrow{x}, \overrightarrow{y}\right) z - 8T\left(\overrightarrow{x}, \overrightarrow{y}\right) z\left(\overrightarrow{y}\,\overrightarrow{x}\right).
\end{aligned}$$

Dividing this last relation by 4 and after reducing the terms, it results that $\left(\overrightarrow{y}\,\overrightarrow{x}\right)^2 z - 2T\left(\overrightarrow{x},\overrightarrow{y}\right)\left(\overrightarrow{y}\,\overrightarrow{x}\right) z = z\left(\overrightarrow{y}\,\overrightarrow{x}\right)^2 - 2T\left(\overrightarrow{x},\overrightarrow{y}\right) z\left(\overrightarrow{y}\,\overrightarrow{x}\right).$

We denote

$$\begin{aligned} E = &\left[\left(\overrightarrow{y}\,\overrightarrow{x}\right)^2 z - z\left(\overrightarrow{y}\,\overrightarrow{x}\right)^2\right] \\ &- \left[2T\left(\overrightarrow{x},\overrightarrow{y}\right)\left(\overrightarrow{y}\,\overrightarrow{x}\right) z - 2T\left(\overrightarrow{x},\overrightarrow{y}\right) z\left(\overrightarrow{y}\,\overrightarrow{x}\right)\right] \end{aligned}$$

and we will prove that $E = 0$.

We set

$$E_1 = \left(\overrightarrow{y}\,\overrightarrow{x}\right)^2 z - 2T\left(\overrightarrow{x},\overrightarrow{y}\right)\left(\overrightarrow{y}\,\overrightarrow{x}\right) z$$

and

$$E_2 = z\left(\overrightarrow{y}\,\overrightarrow{x}\right)^2 - 2T\left(\overrightarrow{x},\overrightarrow{y}\right) z\left(\overrightarrow{y}\,\overrightarrow{x}\right).$$

First, we compute E_1. We obtain

$$E_1 = [\left(\overrightarrow{y}\,\overrightarrow{x}\right)^2 - 2T\left(\overrightarrow{x},\overrightarrow{y}\right)\left(\overrightarrow{y}\,\overrightarrow{x}\right)]z.$$

From Lemma 2.1., relation (2.3), we have $\overrightarrow{y}\,\overrightarrow{x} = T\left(\overrightarrow{y},\overrightarrow{x}\right) + \overrightarrow{\overrightarrow{y}\,\overrightarrow{x}}$. Then $\left(\overrightarrow{y}\,\overrightarrow{x}\right)^2 = T^2\left(\overrightarrow{y},\overrightarrow{x}\right) + \left(\overrightarrow{\overrightarrow{y}\,\overrightarrow{x}}\right)^2 + 2T\left(\overrightarrow{y},\overrightarrow{x}\right)\overrightarrow{\overrightarrow{y}\,\overrightarrow{x}}$. Therefore

$$\begin{aligned} E_1 &= [T^2\left(\overrightarrow{y},\overrightarrow{x}\right) + \left(\overrightarrow{\overrightarrow{y}\,\overrightarrow{x}}\right)^2 \\ &\quad + 2T\left(\overrightarrow{y},\overrightarrow{x}\right)\overrightarrow{\overrightarrow{y}\,\overrightarrow{x}} - 2T\left(\overrightarrow{x},\overrightarrow{y}\right)\left(\overrightarrow{y}\,\overrightarrow{x}\right)]z \\ &= [T^2\left(\overrightarrow{y},\overrightarrow{x}\right) + \left(\overrightarrow{\overrightarrow{y}\,\overrightarrow{x}}\right)^2 \\ &\quad + 2T\left(\overrightarrow{y},\overrightarrow{x}\right)(\overrightarrow{\overrightarrow{y}\,\overrightarrow{x}} - \overrightarrow{y}\,\overrightarrow{x})]z. \end{aligned}$$

Since $\overrightarrow{\overrightarrow{y}\,\overrightarrow{x}} - \overrightarrow{y}\,\overrightarrow{x} = -T\left(\overrightarrow{y},\overrightarrow{x}\right)$, it results that

$$\begin{aligned} &[\left(\overrightarrow{y}\,\overrightarrow{x}\right)^2 - 2T\left(\overrightarrow{x},\overrightarrow{y}\right)\left(\overrightarrow{y}\,\overrightarrow{x}\right)] \\ &= [\left(\overrightarrow{\overrightarrow{y}\,\overrightarrow{x}}\right)^2 - T^2\left(\overrightarrow{y},\overrightarrow{x}\right)] = \alpha \in K, \end{aligned}$$

from Lemma 2.1., relation (2.4). Hence $E_1 = \alpha z$.

Now, we compute E_2. We obtain

$$E_2 = z[\left(\overrightarrow{y}\,\overrightarrow{x}\right)^2 - 2T\left(\overrightarrow{x},\overrightarrow{y}\right)\left(\overrightarrow{y}\,\overrightarrow{x}\right)] = z\alpha = \alpha z \text{ since } \alpha \in K.$$

It follows that $E = E_1 - E_2 = 0$, therefore relation (2.7) is proved. □

Remark 2.4. 1) Identity (2.7) is called the *Hall identity*. From the above theorem, we remark that Hall identity is true for all algebras obtained by the Cayley-Dickson process and in all Clifford algebras $Cl_{p,q}(K)$.

2) Relation (2.7) can be written: $[x,y]^2 z = z[x,y]^2$ or $\left[[x,y]^2, z\right] = 0$, where $[x,y] = xy - yx$ is the commutator of two elements. If $A = \mathbb{H}$, then the identity (2.7) is proved by Hall in [Ha; 43]. If $A = \mathbb{H}$ and, for example, $y =$

$i, z = j$, we have a quadratic quaternionic equation for which any quaternion is a root:

$$xixk + kxix + ixixj - jxixi + x^2j - jx^2 - ix^2k - kx^2i = 0.$$

Proposition 2.5. *Let A be an arbitrary algebra over the field K such that the relation* (2.7) *holds for all $x, y, z \in A$. Then we have*

$$[[x,y]\,[u,y]\,,z] + [[x,y][x,v],z] + [[u,y][x,y],z] + [[x,v][x,y],z] = 0, \qquad (2.8)$$

$$[[x,v][u,y],z] + [[u,y][x,v],z] + [[x,y][u,v],z] + [[u,v][x,y],z] = 0, \qquad (2.9)$$

$$[[u,y][u,v],z] + [[x,v][u,v],z] + [[u,v][u,y],z] + [[u,v][x,v],z] = 0 \qquad (2.10)$$

for all $x, y, z, u, v \in A$.

Proof. We linearize relation (2.7). Let $x, y, z \in A$ be three arbitrary elements such that $(xy - yx)^2\, z = z\,(xy - yx)^2$. For $x + \lambda u, y + \lambda v, z$ we obtain

$$[(x+\lambda u)\,(y+\lambda v) - (y+\lambda v)\,(x+\lambda u)]^2 z$$

$$= z[(x+\lambda u)\,(y+\lambda v) - (y+\lambda v)\,(x+\lambda u)]^2.$$

It results

$$[xy - yx + \lambda(uy + xv - yu - vx) + \lambda^2\,(uv - vu)]^2 z$$

$$= z\left[xy - yx + \lambda(uy + xv - yu - vx) + \lambda^2\,(uv - vu)\right]^2.$$

We obtain

$$\begin{aligned}
&(xy - yx)^2\, z + \lambda^2[(uy - yu) + (xv - vx)]^2 z\\
&+ \lambda^4\,(uv - vu)^2\, z\\
&+ \lambda[(xy - yx)\,((uy - yu) + (xv - vx))]z\\
&+ \lambda[((uy - yu) + (xv - vx))\,(xy - yx)]z\\
&+ \lambda^2[(uv - vu)\,(xy - yx)]z\\
&+ \lambda^2[(xy - yx)\,(uv - vu)]z\\
&+ \lambda^3[[(uy - yu) + (xv - vx)]\,(uv - vu)]z\\
&+ \lambda^3[(uv - vu)\,[(uy - yu) + (xv - vx)]]z\\
&= z\,(xy - yx)^2 + \lambda^2 z[(uy - yu) + (xv - vx)]^2\\
&+ \lambda^4 z\,(uv - vu)^2\\
&+ \lambda z[(xy - yx)\,((uy - yu) + (xv - vx))]\\
&+ \lambda z[((uy - yu) + (xv - vx))\,(xy - yx)]\\
&+ \lambda^2 z[(uv - vu)\,(xy - yx)]\\
&+ \lambda^2 z[(xy - yx)\,(uv - vu)]
\end{aligned}$$

$$+ \lambda^3 z[[(uy - yu) + (xv - vx)]\,(uv - vu)]$$
$$+ \lambda^3 z[(uv - vu)\,[(uy - yu) + (xv - vx)]], \text{ for all } x, y, z, u, v \in A.$$

Since the coefficients of λ are equal in both members of the equality, we obtain:

$$\begin{aligned}&[(xy - yx)\,((uy - yu) + (xv - vx))]z\\&\quad + [((uy - yu) + (xv - vx))\,(xy - yx)]z\\&= z[(xy - yx)\,((uy - yu) + (xv - vx))]\\&\quad + z[((uy - yu) + (xv - vx))\,(xy - yx)].\end{aligned}$$

We can write this last relation under the form:

$$\begin{aligned}&\{[x,y]\,[u,y]\}z + \{[x,y]\,[x,v]\}z + \{[u,y]\,[x,y]\}z + \{[x,v]\,[x,y]\}z\\&= z\{[x,y]\,[u,y]\} + z\{[x,y]\,[x,v]\} + z\{[u,y]\,[x,y]\} + z\{[x,v]\,[x,y]\}.\end{aligned}$$

It results

$$[[x,y]\,[u,y]\,, z] + [[x,y][x,v], z] + [[u,y][x,y], z] + [[x,v][x,y], z] = 0$$

and we obtain relation (2.8).

Since the coefficients of λ^2 are equal in both members of the equality, we obtain:

$$\begin{aligned}&[(uy - yu) + (xv - vx)]^2 z\\&\quad + [(uv - vu)\,(xy - yx)]z\\&\quad + [(xy - yx)\,(uv - vu)]z\\&= z[(uy - yu) + (xv - vx)]^2\\&\quad + z[(uv - vu)\,(xy - yx)]\\&\quad + z[(xy - yx)\,(uv - vu)].\end{aligned}$$

It results that

$$\begin{aligned}&[(uy - yu)\,(xv - vx)]z + [(xv - vx)\,(uy - yu)]z\\&\quad + [(uv - vu)\,(xy - yx)]z + [(xy - yx)\,(uv - vu)]z\\&= z[(uy - yu)\,(xv - vx)] + z[(xv - vx)\,(uy - yu)]\\&\quad + z[(uv - vu)\,(xy - yx)] + z[(xy - yx)\,(uv - vu)].\end{aligned}$$

We can write this last relation under the form:

$$[[x,v]\,[u,y]\,, z] + [[u,y]\,[x,v]\,, z] + [[x,y]\,[u,v]\,, z] + [[u,v]\,[x,y]\,, z] = 0$$

and we obtain relation (2.9).

Since the coefficients of λ^3 are equal in both members of the equality, we obtain:

$$\begin{aligned}&[[(uy-yu)+(xv-vx)]\,(uv-vu)]z\\&\quad+[(uv-vu)\,[(uy-yu)+(xv-vx)]]z\\&=z[[(uy-yu)+(xv-vx)]\,(uv-vu)]\\&\quad+z[(uv-vu)\,[(uy-yu)+(xv-vx)]].\end{aligned}$$

We can write this last relation under the form:

$$[[u,y][u,v],z]+[[x,v][u,v],z]+[[u,v][u,y],z]+[[u,v][x,v],z]=0$$

and we obtain relation (2.10). □

Remark 2.6. 1) In [Ti; 99] and [Fl; 01] some equations over division quaternion algebra and octonion algebra are solved: in [Fl; 01] for general case, when K is a commutative field with $charK\neq 2$ and γ_m are arbitrary and in [Ti; 99] for $K=\mathbb{R}$, $\gamma_m=-1$, with $m\in\{1,2\}$ for quaternions and $m\in\{1,2,3\}$ for octonions. Let A be such an algebra. For example, equation

$$ax=xb,\quad a,b,x\in A, \tag{2.11}$$

for $a\neq\overline{b}$ has general solution under the form $x=\overrightarrow{a}p+p\overrightarrow{b}$, for arbitrary $p\in A$.

2) In [Fl, Şt; 09], authors studied equation $x^2a=bx^2+c, a,b,c\in A$, where A is a generalized quaternion division algebra or a generalized octonion division algebra. If A is an arbitrary algebra obtained by the Cayley-Dickson process and $a,b,c\in A$ with $a=b$ and $c=0$, then, from Theorem 2.3., it results that this equation has infinity of solutions of the form $x=vw-wv$, where $v,w\in A$.

Proposition 2.7. *Let A be a quaternion algebra or an octonion algebra. Then for all $x,y\in A$, there are the elements z,w such that $(xy-yx)^2=\overrightarrow{z}w+w\overrightarrow{z}$.*

Proof. Let z be an arbitrary element in $A-K$. From Theorem 2.3., we have that $(xy-yx)^2z=z(xy-yx)^2$, for all $x,y,z\in A$. Since $z\neq\overline{z}$ and $(xy-yx)^2$ is a solution for the equation (2.11), from Remark 2.6, it results that there is an element $w\in A$ such that $(xy-yx)^2=\overrightarrow{z}w+w\overrightarrow{z}$. □

Proposition 2.8. *Let A be a finite dimensional unitary algebra with a scalar involution*

$$^{-}:A\to A,\quad a\to\overline{a},$$

such that for all $x,y\in A$, the following equality holds:

$$(x\overline{y}+y\overline{x})^2=4\,(x\overline{x})\,(y\overline{y})\,. \tag{2.12}$$

Then the algebra A has dimension 1.

Proof. We remark that $x\overline{y}+y\overline{x}=x\overline{y}+\overline{x\overline{y}}\in K$. First, we prove that $[x\overline{y}+y\overline{x}]^2=4\,(x\overline{x})\,(y\overline{y})$, $\forall x,y\in A$, if and only if $x=ry$, $r\in K$. If $x=ry$, then relation (2.12) is proved. Conversely, assuming that relation (2.12) is true and supposing that there is not an element $r\in K$ such that $x=ry$, then for each two non zero elements $a,b\in K$, we have $ax+by\neq 0$. Indeed, if $ax+by=0$, it results $x=-\frac{b}{a}y$, false. We obtain that

$$(ax+by)\,\overline{(ax+by)}\neq 0. \tag{2.13}$$

Computing relation (2.13) , it follows

$$a^2\,(x\overline{x})+abx\overline{y}+bay\overline{x}+b^2y\overline{y}\neq 0. \tag{2.14}$$

If we put $a=y\overline{y}$ in relation (2.14) and then simplify by a, it results

$$(y\overline{y})\,(x\overline{x})+bx\overline{y}+by\overline{x}+b^2\neq 0. \tag{2.15}$$

Let $b=-\frac{1}{2}\,(x\overline{y}+y\overline{x})\in K, b\neq 0$. If we replace this value in relation (2.15) , we obtain $4\,(x\overline{x})\,(y\overline{y})-(x\overline{y}+y\overline{x})^2\neq 0$, which it is false. Therefore, there is an element $r\in K$ such that $x=ry$.

Assuming that the algebra A has dimension greater or equal with 2, it results that there are two linearly independent vectors, v and w, respectively. Since relation (2.12) is satisfies for v and w, we obtain that there is an element $s\in K$ such that $v=sw$, which it is false. Hence $\dim A=1$. □

Proposition 2.9. *Let A be an alternative division algebra over the field K whose center is K. If $(xy-yx)^2\,z=z\,(xy-yx)^2$ for all $x,y,z\in A$, then A is a quadratic algebra.*

Proof. Let $x,y\in A$ such that $xy\neq yx$. If we denote $z=xy-yx$, it follows that z^2 commutes with all elements from A, then z^2 is in the center of A. We obtain $z^2=\alpha\in K^*$. For $t=x^2y-yx^2$ it results that $t^2=(x^2y-yx^2)^2\in K$ and $t=(xy-yx)\,x+x\,(xy-yx)=zx+xz$. We have $zt=z\,(zx+xz)=z^2x+zxz=\alpha x+zxz$ and $tz=(zx+xz)\,z=zxz+xz^2=\alpha x+zxz$. Therefore $tz=zt$. For $z+t=\left(x^2+x\right)y-y\left(x^2+x\right)$ we have that $(z+t)^2=\beta\in K$, then $z^2+t^2+2tz=\beta$, hence $tz=\gamma\in K$. Since $zx=x(yx)-(yx)\,x$, it follows that $(zx)^2=\delta\in K$. If we multiply the relation $(zx)\,(zx)=\delta$ with z in the left side, we obtain $z\,((zx)\,(zx))=\delta z$. Using alternativity and then flexibility, it results $\left(z^2x\right)(zx)=\delta z$, therefore $\alpha\,(xzx)=\delta z$, hence $xzx=\theta z$, where $\theta=\alpha^{-1}\delta$. It follows that $z\,(xzx)=\theta z^2=\theta\alpha\in K$. Since $z\,(xzx)=(zxz)\,x$, from Moufang identities, we have that $(zxz)\,x=\theta\alpha\in K$. It results that $\gamma x=(tz)\,x=(\alpha x+zxz)\,x=\alpha x^2+(zxz)\,x=\alpha x^2+\theta\alpha$, hence $x^2=ax+b$, where $a=\alpha^{-1}\gamma, b=-\theta$. We obtain that A is a quadratic algebra. □

When A is a division associative algebra, this proposition was proved by Hall in [Ha; 43], Lemma 1.

Theorem 2.10. *Let A be an alternative simple algebra such that the center of A is K and $(xy-yx)^2\,z=z\,(xy-yx)^2$ for all $x,y,z\in A$.*

1) *If A is a division algebra, then $A=K$ or $A=A_t$, $t\in\{1,2,3\}$, where A_t is a division algebra obtained by the Cayley-Dickson process.*

2) *If A is a quadratic but not a division algebra and there are two elements $y, z \in A$ such that $y^2, z^2 \in K - \{0\}$, $yz = -zy \neq 0$, then A is a generalized split quaternion algebra.*

Proof. 1) From Proposition 2.9, it results that A is a quadratic algebra, therefore, from [Al; 49], Theorem 1, we have $\dim A \in \{ 1, 2, 4, 8\}$. If $\dim A = 1$, then $A = K$. If $\dim A = 2$, since the center is K, then we can find an element $x \in A - K$ such that $x^2 \in K$. It results that the set $\{1, x\}$ is a basis in A, therefore $A = K(x)$ is a quadratic field extension of the field K. If $\dim A = 4$, from [Al; 39], p. 145, we have that there are two elements $x, y \in A$, which do not permute, such that $x^2 = x + a$ with $4a + 1 \neq 0$, $xy = y(1 - x), y^2 = b, a, b \in K$. Denoting $z = x - \frac{1}{2}$, we obtain that $z^2 = \left(x - \frac{1}{2}\right)^2 = a - \frac{1}{4} \in K$. Since $zy = \left(x - \frac{1}{2}\right) y = xy - \frac{y}{2} = y - yx - \frac{y}{2} = \frac{y}{2} - yx$ and $yz = y\left(x - \frac{1}{2}\right) = yx - \frac{y}{2}$, we have $yz = -zy$ then $(yz)^2 \in K$. It follows that in the algebra A we can find the elements y, z such that $y^2, z^2, (yz)^2 \in K$ and $yz = -zy$. Therefore, from [Al; 49], Lemma 4, it results that A is a generalized division quaternion algebra.

2) From the above, it results that $A = Q = K + yK + zK + yzK$ is a generalized quaternion algebra, which is split from hypothesis. □

The above Theorem generalizes Theorem 6.2 from [Ha:43].

Corollary 2.11. *Let A be a non-division associative algebra such that the center of A is K. If in algebra A we have $(xy - yx)^2 z = z(xy - yx)^2$ for all $x, y, z \in A$ and there are two elements v, w such that $v^2, w^2 \in K - \{0\}$, $vw = -wv \neq 0$, then A is a generalized split quaternion algebra.* □

Example 2.12. 1) Using notations given in preliminaries, if in Theorem 2.10, we have $t = 1$ and $\alpha_1 = -1$, it results that $A = Cl_{0,1}(K)$ is a quadratic field extension of the field K. If $t = 2$ and $\alpha_1 = \alpha_2 = -1$, we have that $A = Cl_{0,2}(K)$ is a quadratic division quaternion algebra.

2) If we have $v^2, w^2 \in \{-1, 1\}$ in Corollary 2.11, then $A = Cl_{1,1}(K) \simeq Cl_{2,0}(K)$.

Conclusions

In this paper we proved that the Hall identity is true in all algebras obtained by the Cayley-Dickson process and that the converse is also true, in some particular conditions, for split quaternion algebras. Some identities, as above mentioned identity, in algebras obtained by the Cayley-Dickson process can be used to find solutions for some equations in these algebras or to solve them. This is an idea which can constitute the starting point for further research.

Acknowledgments

The authors would like to thank the referee for his/her suggestions, which helped us improve this paper.

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Received: January 14, 2012.
Accepted: May 5, 2012.