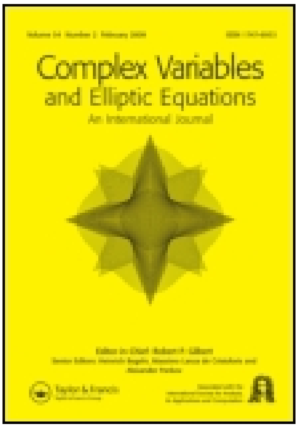

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S.A. Plaksa [a] & V.S. Shpakivskyi [a]

[a] Complex Analysis and Potential Theory , Institute of Mathematics of the National Academy of Sciences of Ukraine , Tereshchenkivska Street 3, Kiev , 01601 , Ukraine

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Cauchy theorem for a surface integral in commutative algebras

S.A. Plaksa* and V.S. Shpakivskyi

Complex Analysis and Potential Theory, Institute of Mathematics of the National Academy of Sciences of Ukraine, Tereshchenkivska Street 3, Kiev 01601, Ukraine

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We prove an analogue of the Cauchy integral theorem for hyperholomorphic functions given in three-dimensional domains with non piece-smooth boundaries and taking values in an arbitrary finite-dimensional commutative associative Banach algebra.

Keywords: Cauchy integral theorem; hyperholomorphic function; commutative associative Banach algebra; Lebesgue area of a surface; quadrable surface; surface integral

AMS Subject Classifications: 30G35; 28B05

1. Introduction

The Cauchy integral theorem is a fundamental result of the classical complex analysis in the complex plane $\mathbb{C}$: if the boundary ∂D of a domain $D \subset \mathbb{C}$ is a closed Jordan rectifiable curve, and a function $F\colon \overline{D} \longrightarrow \mathbb{C}$ is continuous in the closure $\overline{D}$ of D and is holomorphic in D, then $\int_{\partial D} F(z)dz = 0$.

Developing hypercomplex analysis in both commutative and noncommutative algebras needs similar general analogues of the Cauchy integral theorem for several-dimensional spaces.

It is well known that in the case where a simply connected domain has a closed piece-smooth boundary, spatial analogues of the Cauchy integral theorem can be obtained with using the classical Gauss – Ostrogradskii formula, if a given function has specifically continuous partial derivatives of the first-order up to the boundary. In such a way, analogues of the Cauchy integral theorem are proved in the quaternion algebra (see, e.g. [1, p.66]) and in Clifford algebras (see, e.g. [2, p.52]).

Generalizations of the Cauchy integral theorem have relations to weakening requirements to the boundary or the given function. Usually, such generalizations are based on generalized Gauss – Ostrogradskii – Green – Stokes formula (see, e.g. [3,4]) under the condition of continuity of partial derivatives of the given function, but for extended classes of surfaces of integration; see, e.g. [5,6], where rectifiable or regular surfaces are considered. In the papers [7,8], the continuity of partial derivatives is changed by a differentiability of

*Corresponding author. Email: plaksa@imath.kiev.ua

components of the given function taking values in the quaternion algebra. Note that the boundary of domain is remained piece-smooth in [8].

In this paper, we prove an analogue of the Cauchy integral theorem for functions taking values in an arbitrary finite-dimensional commutative associative algebra. Similarly to the paper [8], we weaken requirements to functions given in a domain of three-dimensional space. At the same time, the functions can be given in a domain with non piece-smooth boundary.

2. Quadrable surfaces

A set Σ is called a *surface* in the real space $\mathbb{R}^3$ if Σ is a homeomorphic image of the square $G := [0, 1] \times [0, 1]$ (cf. e.g. [9, pp.24, 131]).

By Σ^ε we denote ε-neighborhood of the surface Σ, i.e. the set $\Sigma^\varepsilon := \{(x, y, z) \in \mathbb{R}^3 : \sqrt{(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2} \le \varepsilon,\ (x_1, y_1, z_1) \in \Sigma\}$.

The *Fréchet distance* $d(\Sigma, \Lambda)$ between the surfaces Σ and Λ is called the infimum of real numbers ε, for which the relations $\Sigma \subset \Lambda^\varepsilon$, $\Lambda \subset \Sigma^\varepsilon$ are fulfilled (see, e.g. [10]). A sequence of polyhedral surfaces Λ_n *converges uniformly* to the surface Σ, if $d(\Lambda_n, \Sigma) \to 0$ as $n \to \infty$ (cf. e.g. [9, p.121]).

The *Lebesgue area* of a surface Σ is

$$\mathfrak{L}(\Sigma) := \inf \liminf_{n\to\infty} \mathfrak{L}(\Lambda_n),$$

where the infimum is taken for all sequences Λ_n convergent uniformly to Σ (see, e.g. [9, p.468]), and $\mathfrak{L}(\Lambda_n)$ is the area of polyhedral surface Λ_n.

Let a surface Σ have the finite Lebesgue area, i.e. $\mathfrak{L}(\Sigma) < \infty$. Then by the L. Cesari theorem [11, p.7], there exists a surface parameterization

$$\Sigma = \big\{ f(u, v) := \big(x(u, v),\ y(u, v),\ z(u, v)\big) : (u, v) \in G \big\}$$

such that the Jacobians

$$A := \frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial y}{\partial v}\frac{\partial z}{\partial u}, \quad B := \frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial z}{\partial v}\frac{\partial x}{\partial u}, \quad C := \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \tag{1}$$

exist a.e. in the square G and

$$\mathfrak{L}(\Sigma) = \int\limits_G \sqrt{A^2 + B^2 + C^2}\, du dv \tag{2}$$

(here and in what follows, all integrals are understood as Lebesgue integrals).

In the case where $\mathfrak{L}(\Sigma) < \infty$ and the equality (2) holds for the given parameterization of Σ, we shall say that a surface Σ is *quadrable*.

Let us formulate certain sufficient conditions for a surface Σ be quadrable.

(1) Let Σ be a *rectifiable surface*, i.e. Σ be a Lipschitz image of the square G. Then it follows from [9, IV.4.28, IV.4.1 (e)] that Σ is quadrable.
(2) Let the components $x(u, v)$, $y(u, v)$, $z(u, v)$ of mapping f be absolutely continuous in the sense of Tonelli (see, e.g. [12, p.169]). Let, furthermore, in Jacobians A, B, C of mapping f in every of the products $\frac{\partial y}{\partial u}\frac{\partial z}{\partial v}$, $\frac{\partial y}{\partial v}\frac{\partial z}{\partial u}$, $\frac{\partial z}{\partial u}\frac{\partial x}{\partial v}$, $\frac{\partial z}{\partial v}\frac{\partial x}{\partial u}$, $\frac{\partial x}{\partial u}\frac{\partial y}{\partial v}$, $\frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$, one partial derivative belong to the class $L_p(G)$ of functions

integrable to the pth power on G and the other partial derivative belong to $L_q(G)$, where $\frac{1}{p}+\frac{1}{q}=1$. Then, Σ is quadrable (see [9, V.2.26]). Note that for a rectifiable surface Σ, components $x(u,v),\ y(u,v),\ z(u,v)$ of mapping f are absolutely continuous in the sense of Tonelli (see, e.g. [12, p.169]).

(3) If two components of the mapping $f(u,v)$ are Lipschitz functions and the third component is absolutely continuous in the sense of Tonelli, then Σ is quadrable (see [9, V.2.28]).

3. Surface integrals

In what follows, we understand the *closed surface* $\Gamma \subset \mathbb{R}^3$ as an image of a sphere under homeomorphic mapping which maps *at least one circle onto a rectifiable curve*. In other words, the closed surface Γ is the union of two surfaces Γ_1, Γ_2 for which $\Gamma_1 \cap \Gamma_2 =: \gamma$ is a closed Jordan rectifiable curve. Let the surfaces Γ_1, Γ_2 be parametrically definable:

$$\Gamma_1 = \left\{ f_1(u,v) := \big(x_1(u,v),\ y_1(u,v),\ z_1(u,v)\big) : (u,v) \in G \right\},$$

$$\Gamma_2 = \left\{ f_2(u,v) := \big(x_2(u,v),\ y_2(u,v),\ z_2(u,v)\big) : (u,v) \in G \right\}.$$

A closed surface Γ is called *quadrable* if the surfaces Γ_1 and Γ_2 are quadrable.

For a closed quadrable surface Γ and a continuous function $F : \Gamma \to \mathbb{R}$, we define integrals on Γ by the equalities

$$\int_\Gamma F(x,y,z)\,dydz := \int_G F\Big(x_1(u,v), y_1(u,v), z_1(u,v)\Big) A_1\,dudv - \int_G F\Big(x_2(u,v), y_2(u,v), z_2(u,v)\Big) A_2\,dudv, \tag{3}$$

$$\int_\Gamma F(x,y,z)\,dzdx := \int_G F\Big(x_1(u,v), y_1(u,v), z_1(u,v)\Big) B_1\,dudv - \int_G F\Big(x_2(u,v), y_2(u,v), z_2(u,v)\Big) B_2\,dudv, \tag{4}$$

$$\int_\Gamma F(x,y,z)\,dxdy := \int_G F\Big(x_1(u,v), y_1(u,v), z_1(u,v)\Big) C_1\,dudv - \int_G F\Big(x_2(u,v), y_2(u,v), z_2(u,v)\Big) C_2\,dudv \tag{5}$$

with the Jacobians A_k, B_k, C_k of mapping f_k of the form (1) for $k = 1, 2$.

It is easy to check up that the definitions (3)–(5) are correct. Indeed, values of integrals on the right-hand sides of equalities (3)–(5) are the same for all parameterizations f_1, f_2 for which the areas $\mathfrak{L}(\Gamma_1), \mathfrak{L}(\Gamma_2)$ are expressed by the equalities of the form (2), and values of integrals on the left-hand sides of equalities (3)–(5) do not depend on a choice of a rectifiable curve γ which divides Γ into two parts.

LEMMA 3.1 *If* Γ *is a closed quadrable surface, then*

$$\int_{\Gamma} dydz = \int_{\Gamma} dzdx = \int_{\Gamma} dxdy = 0. \tag{6}$$

Proof By definition,

$$\int_{\Gamma} dydz = \int_{G} A_1\, dudv - \int_{G} A_2\, dudv. \tag{7}$$

It follows from the Radó results [9, V.2.64 (iii), IV.4.21 (iii_3)] that for the surfaces Γ_1, Γ_2 the following equalities are true:

$$\int_{G} A_k\, dudv = \int_{\partial G} ydz, \qquad k = 1, 2, \tag{8}$$

where the integral on the right-hand side is understood as a Lebesgue – Stieltjes integral and is took along the boundary ∂G of the square G into a positive direction. Now, we obtain from the equalities (7), (8) that the first integral of (6) is equal to zero. The other equalities (6) are proved by analogy. □

4. Hyperholomorphic functions in a commutative Banach algebra

Let $\mathbb{A}$ be a commutative associative Banach algebra over the field of complex numbers $\mathbb{C}$ with the basis $\{e_k\}_{k=1}^{n}$, $3 \le n < \infty$.

Let us single out the linear span $E_3 := \{\zeta = xe_1 + ye_2 + ze_3 : x, y, z \in \mathbb{R}\}$ generated by the vectors e_1, e_2, e_3. Associate with a set $\Omega \subset \mathbb{R}^3$ the set $\Omega_\zeta := \{\zeta = xe_1 + ye_2 + ze_3 : (x, y, z) \in \Omega\}$ in E_3.

Consider a function $\Psi : \Omega_\zeta \to \mathbb{A}$ of the form

$$\Psi(\zeta) = \sum_{k=1}^{n} U_k(x, y, z)e_k + i \sum_{k=1}^{n} V_k(x, y, z)e_k, \tag{9}$$

where $(x, y, z) \in \Omega$ and $U_k : \Omega \to \mathbb{R}$, $V_k : \Omega \to \mathbb{R}$.

We shall say that a function of the form (9) is *hyperholomorphic* in a domain Ω_ζ if its real-valued components U_k, V_k are differentiable in Ω and the following equality is fulfilled in every point of Ω_ζ:

$$\frac{\partial \Psi}{\partial x} e_1 + \frac{\partial \Psi}{\partial y} e_2 + \frac{\partial \Psi}{\partial z} e_3 = 0. \tag{10}$$

In the scientific literature, the different denominations are used for functions satisfying equations of the form (10). For example, in the papers [7,13,14] they are called regular functions, and in the papers [2,15,16] they are called monogenic functions. We use the terminology of the papers [1,8,17].

5. Auxiliary results

Let Ω be a bounded closed set in $\mathbb{R}^3$. For a continuous function $\Psi : \Omega_\zeta \to \mathbb{A}$ of the form (9), we define a volume integral by the equality

$$\int\limits_{\Omega_\zeta} \Psi(\zeta) dx dy dz := \sum_{k=1}^{n} e_k \int\limits_{\Omega} U_k(x, y, z) dx dy dz + i \sum_{k=1}^{n} e_k \int\limits_{\Omega} V_k(x, y, z) dx dy dz.$$

Let Γ be a closed quadrable surface in $\mathbb{R}^3$. For a continuous function $\Psi : \Gamma_\zeta \to \mathbb{A}$ of the form (9), where $(x, y, z) \in \Gamma$ and $U_k : \Gamma \to \mathbb{R}$, $V_k : \Gamma \to \mathbb{R}$, we define a surface integral on Γ_ζ with the differential form $\sigma := dy dz e_1 + dz dx e_2 + dx dy e_3$ by the equality

$$\begin{aligned}\int\limits_{\Gamma_\zeta} \Psi(\zeta)\sigma :=& \sum_{k=1}^{n} e_1 e_k \int\limits_{\Gamma} U_k(x, y, z) dy dz + \sum_{k=1}^{n} e_2 e_k \int\limits_{\Gamma} U_k(x, y, z) dz dx \\ &+ \sum_{k=1}^{n} e_3 e_k \int\limits_{\Gamma} U_k(x, y, z) dx dy + i \sum_{k=1}^{n} e_1 e_k \int\limits_{\Gamma} V_k(x, y, z) dy dz \\ &+ i \sum_{k=1}^{n} e_2 e_k \int\limits_{\Gamma} V_k(x, y, z) dz dx + i \sum_{k=1}^{n} e_3 e_k \int\limits_{\Gamma} V_k(x, y, z) dx dy,\end{aligned}$$

where the integrals on the right-hand side of equality are defined by the equalities (3)–(5).

The next lemma is a result of Lemma 3.1 and the definition of σ.

LEMMA 5.1 *If Γ is a closed quadrable surface, then*

$$\int\limits_{\Gamma_\zeta} \sigma = 0. \tag{11}$$

Let us introduce the Euclidian norm $\|a\| := \left(\sum_{k=1}^{n} |a_k|^2\right)^{1/2}$ in the algebra $\mathbb{A}$, where $a = \sum_{k=1}^{n} a_k e_k$ and $a_k \in \mathbb{C}$ for $k = \overline{1, n}$.

Let Γ be a closed quadrable surface in $\mathbb{R}^3$. For a continuous function $U : \Gamma_\zeta \to \mathbb{R}$, we define a surface integral on Γ_ζ with the differential form $\|\sigma\|$ by the equality

$$\begin{aligned}&\int\limits_{\Gamma_\zeta} U(xe_1 + ye_2 + ze_3)\|\sigma\| \\ &:= \int\limits_{G} U\Big(x_1(u, v)e_1 + y_1(u, v)e_2 + z_1(u, v)e_3\Big)\sqrt{A_1^2 + B_1^2 + C_1^2}\, du dv \\ &\quad - \int\limits_{G} U\Big(x_2(u, v)e_1 + y_2(u, v)e_2 + z_2(u, v)e_3\Big)\sqrt{A_2^2 + B_2^2 + C_2^2}\, du dv.\end{aligned}$$

LEMMA 5.2 *If Γ is a closed quadrable surface and a function $\Psi : \Gamma_\zeta \to \mathbb{A}$ is continuous, then*

$$\left\| \int_{\Gamma_\zeta} \Psi(\zeta)\sigma \right\| \le 3nM \int_{\Gamma_\zeta} \|\Psi(\zeta)\| \|\sigma\|$$

with $M := \max_{1\le m,s\le n} \|e_m e_s\|$.

Proof Using the representation (9), where $(x, y, z) \in \Gamma$, we obtain

$$\begin{aligned}\left\| \int_{\Gamma_\zeta} \Psi(\zeta)\sigma \right\| &\le \sum_{k=1}^{n} \|e_1 e_k\| \int_{\Gamma} \big|U_k(x,y,z) + iV_k(x,y,z)\big|\, dydz \\ &+ \sum_{k=1}^{n} \|e_2 e_k\| \int_{\Gamma} \big|U_k(x,y,z) + iV_k(x,y,z)\big|\, dzdx \\ &+ \sum_{k=1}^{n} \|e_3 e_k\| \int_{\Gamma} \big|U_k(x,y,z) + iV_k(x,y,z)\big|\, dxdy \le 3nM \int_{\Gamma_\zeta} \|\Psi(\zeta)\| \|\sigma\|.\end{aligned}$$

□

If a simply connected domain $\Omega \subset \mathbb{R}^3$ have a closed piece-smooth boundary $\partial\Omega$ and a function $\Psi : \Omega_\zeta \to \mathbb{A}$ is continuous together with partial derivatives of the first-order up to the boundary $\partial\Omega_\zeta$, then the following equality follows from the classical Gauss – Ostrogradskii formula:

$$\int_{\partial\Omega_\zeta} \Psi(\zeta)\sigma = \int_{\Omega_\zeta} \left(\frac{\partial\Psi}{\partial x} e_1 + \frac{\partial\Psi}{\partial y} e_2 + \frac{\partial\Psi}{\partial z} e_3 \right) dxdydz\,. \tag{12}$$

We prove the next theorem similarly to the proof of Theorem 9 [7] and Theorem 1 [8], where functions taking values in the quaternion algebra was considered.

Theorem 5.3 *Let ∂P be the boundary of a closed cube P that is contained in a domain Ω and a function $\Psi : \Omega_\zeta \to \mathbb{A}$ be hyperholomorphic in the domain Ω_ζ. Then, the following equality holds:*

$$\int_{\partial P_\zeta} \Psi(\zeta)\sigma = 0.$$

Proof Suppose that $\left\| \int_{\partial P_\zeta} \Psi(\zeta)\sigma \right\| = K$.

Denote by S the area of surface ∂P. Divide P into eight equal cubes and denote by P^1 such a cube, for which $\left\| \int_{\partial P^1_\zeta} \Psi(\zeta)\sigma \right\| \ge K/8$. Clearly, the surface ∂P^1 have the area $S/4$.

Continuing this process, we obtain a sequence of embedded cubes P^m with the areas $S/4^m$ of the surfaces ∂P^m, that satisfies the inequalities

$$\left\| \int_{\partial P^m_\zeta} \Psi(\zeta)\sigma \right\| \ge K/8^m. \tag{13}$$

By the Cantor principle, there exists the unique point $\zeta_0 := x_0 e_1 + y_0 e_2 + z_0 e_3$ common for all cubes P^m. Inasmuch as the function Ψ is of the form (9) and the real-valued components U_k, V_k are differentiable in Ω, in a neighbourhood of the point ζ_0 we have the expansion

$$\Psi(\zeta) = \Psi(\zeta_0) + \Delta x \frac{\partial \Psi(\zeta_0)}{\partial x} + \Delta y \frac{\partial \Psi(\zeta_0)}{\partial y} + \Delta z \frac{\partial \Psi(\zeta_0)}{\partial z} + \delta(\zeta, \zeta_0)\rho,$$

where $\Delta x := x - x_0$, $\Delta y := y - y_0$, $\Delta z := z - z_0$, and $\delta(\zeta, \zeta_0)$ is an infinitesimal function as $\rho := \|\zeta - \zeta_0\| \to 0$.

Therefore, for all sufficiently small cubes, we have

$$\int\limits_{\partial P^m_\zeta} \Psi(\zeta)\sigma = \Psi(\zeta_0) \int\limits_{\partial P^m_\zeta} \sigma + \frac{\partial \Psi(\zeta_0)}{\partial x} \int\limits_{\partial P^m_\zeta} \Delta x \sigma + \frac{\partial \Psi(\zeta_0)}{\partial y} \int\limits_{P^m_\zeta} \Delta y \sigma$$
$$+ \frac{\partial \Psi(\zeta_0)}{\partial z} \int\limits_{\partial P^m_\zeta} \Delta z \sigma + \int\limits_{\partial P^m_\zeta} \delta(\zeta, \zeta_0)\rho\, \sigma = \sum_{r=1}^{5} I_r .$$

By the formula (12), $I_1 = 0$. Using (12) and taking into account the equality (10), we obtain

$$I_2 + I_3 + I_4 = \frac{\partial \Psi(\zeta_0)}{\partial x} e_1 V_m + \frac{\partial \Psi(\zeta_0)}{\partial y} e_2 V_m + \frac{\partial \Psi(\zeta_0)}{\partial z} e_3 V_m = 0,$$

where by V_m we have denoted the volume of cube P^m.

Note that for an arbitrary $\varepsilon > 0$ there exists the number m_0 such that the inequality $\|\delta(\zeta, \zeta_0)\| < \varepsilon$ is fulfilled for all cubes P^m with $m > m_0$. Note also that ρ is not greater than the diagonal of P^m, i.e. $\rho \leq \frac{\sqrt{S}}{2^m \sqrt{2}}$. Therefore, using Lemma 5.2 and the mentioned inequalities for $\delta(\zeta, \zeta_0)$ and ρ, we obtain

$$\Bigg\| \int\limits_{\partial P^m_\zeta} \Psi(\zeta)\sigma \Bigg\| = \|I_5\| \leq 3nM \int\limits_{\partial P^m_\zeta} \rho \, \|\delta(\zeta, \zeta_0)\| \, \|\sigma\| \leq 3nM \frac{\sqrt{S}}{2^m \sqrt{2}} \frac{S}{4^m} \varepsilon . \qquad (14)$$

It follows from the relations (13) and (14) that $K \leq c\,\varepsilon$, where the constant c does not depend on ε. Passing to the limit in the last inequality as $\varepsilon \to 0$, we obtain the equality $K = 0$, and the theorem is proved. □

6. Main result

Let us establish an analogue of Cauchy integral theorem for the surface integral on the boundary $\partial\Omega_\zeta$ in the case where the function $\Psi : \overline{\Omega_\zeta} \to \mathbb{A}$ is hyperholomorphic in a domain Ω_ζ and continuous in the closure $\overline{\Omega_\zeta}$ of this domain.

For such a function consider the modulus of continuity

$$\omega_{\,\overline{\Omega}_\zeta}(\Psi, \delta) := \sup_{\zeta_1, \zeta_2 \in \overline{\Omega}_\zeta, \|\zeta_1 - \zeta_2\| \leq \delta} \|\Psi(\zeta_1) - \Psi(\zeta_2)\|.$$

The *two-dimensional upper Minkowski content* (see, e.g. [18, p.79]) is

$$\mathcal{M}^*(\partial\Omega) := \limsup_{\varepsilon \to 0} \frac{V(\partial\Omega^\varepsilon)}{2\varepsilon} ,$$

where $V(\partial\Omega^\varepsilon)$ denotes the volume of $\partial\Omega^\varepsilon$.

THEOREM 6.1 *Suppose that the boundary $\partial\Omega$ of a simply connected domain $\Omega \subset \mathbb{R}^3$ is a closed quadrable surface for which $\mathcal{M}^*(\partial\Omega) < \infty$, and Ω has Jordan measurable intersections with planes perpendicular to coordinate axes. Suppose also that a function $\Psi : \overline{\Omega_\zeta} \to \mathbb{A}$ is hyperholomorphic in the domain Ω_ζ and continuous in the closure $\overline{\Omega_\zeta}$ of this domain. Then the following equality holds:*

$$\int_{\partial\Omega_\zeta} \Psi(\zeta)\sigma = 0. \tag{15}$$

Proof Inasmuch as $\mathcal{M}^*(\partial\Omega) < \infty$, there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ the following inequality holds:

$$V(\partial\Omega^\varepsilon) \le c\,\varepsilon, \tag{16}$$

where the constant c does not depend on ε.

Let us take $\varepsilon < \varepsilon_0/\sqrt{3}$. Let us make a partition of the space $\mathbb{R}^3$ onto cubes with an edge of the length ε by planes perpendicular to the coordinate axes. Then, we have the equality

$$\int_{\partial\Omega_\zeta} \Psi(\zeta)\,\sigma = \sum_j \int_{\partial(\Omega_\zeta \cap K^j_\zeta)} \Psi(\zeta)\,\sigma + \sum_k \int_{\partial K^k_\zeta} \Psi(\zeta)\,\sigma, \tag{17}$$

where the first sum is applied to the cubes K^j for which $\overline{K^j} \cap \partial\Omega \ne \varnothing$, and the second sum is applied to the cubes K^k for which $\overline{K^k} \subset \Omega$. By Theorem 5.3, the second sum is equal to zero.

To estimate an integral of the first sum we take a point $\zeta_j \in \Omega_\zeta \cap K^j_\zeta$. Note that the diameter of set $\Omega \cap K^j$ does not exceed $\varepsilon\sqrt{3}$. Inasmuch as Ω has Jordan measurable intersections with planes perpendicular to coordinate axes, the Lebesgue measure of the boundaries of mentioned intersections is equal to 0, and consequently, the set $\partial(\Omega_\zeta \cap K^j_\zeta)$ consists of closed quadrable surfaces. Therefore, taking into account the equality (11) and using Lemma 5.2, we obtain

$$\begin{aligned}\left\| \int_{\partial(\Omega_\zeta \cap K^j_\zeta)} \Psi(\zeta)\sigma \right\| &= \left\| \int_{\partial(\Omega_\zeta \cap K^j_\zeta)} (\Psi(\zeta) - \Psi(\zeta_j))\sigma \right\| \\ &\le 3nM \int_{\partial(\Omega_\zeta \cap K^j_\zeta)} \|\Psi(\zeta) - \Psi(\zeta_j)\|\|\sigma\| \le 3nM\,\omega_{\overline{\Omega_\zeta}}(\Psi, \varepsilon\sqrt{3}) \int_{\partial(\Omega_\zeta \cap K^j_\zeta)} \|\sigma\|.\end{aligned} \tag{18}$$

Thus, the following estimate is a result of the equality (17) and the inequality (18):

$$\Big\| \int_{\partial\Omega_\zeta} \Psi(\zeta)\,\sigma \Big\| \le 3nM\omega_{\overline{\Omega}_\zeta}(\Psi, \varepsilon\sqrt{3}) \sum_j \int_{\partial(\Omega_\zeta \cap K_\zeta^j)} \|\sigma\|$$

$$\le 3nM\omega_{\overline{\Omega}_\zeta}(\Psi, \varepsilon\sqrt{3}) \Big(\int_{\partial\Omega_\zeta} \|\sigma\| + 6 \sum_j \varepsilon^2 \Big). \tag{19}$$

Inasmuch as $\bigcup_j K^j \subset \partial\Omega^{\varepsilon\sqrt{3}}$, taking into account the inequality (16), we obtain the estimation

$$\sum_j \varepsilon^3 \le V\Big(\partial\Omega^{\varepsilon\sqrt{3}}\Big) \le c\varepsilon\sqrt{3}\,,$$

from which it follows that

$$\sum_j \varepsilon^2 \le c\sqrt{3}. \tag{20}$$

Finally, the following inequality is as a result of the estimations (19) and (20):

$$\Big\| \int_{\partial\Omega_\zeta} \Psi(\zeta)\sigma \Big\| \le c_1\, \omega_{\overline{\Omega}_\zeta}(\Psi, \varepsilon\sqrt{3}) \tag{21}$$

where the constant c_1 does not depend on ε.

To complete the proof, note that $\omega_{\overline{\Omega}_\zeta}(\Psi, \varepsilon\sqrt{3}) \to 0$ as $\varepsilon \to 0$ due to the uniform continuity of the function Ψ on $\overline{\Omega}_\zeta$. □

Theorem 6.1 generalizes Theorem 1 [19] that was proved in a three-dimensional commutative algebra for functions which generate solutions of the three-dimensional Laplace equation.

7. Remarks

Note that for a surface Σ in $\mathbb{R}^3$, there exists positive constants c_1 and c_2, such that

$$c_1\varepsilon^3 N_\Sigma(\varepsilon) \le V(\Sigma^\varepsilon) \le c_2\varepsilon^3 N_\Sigma(\varepsilon)\,, \tag{22}$$

where $N_\Sigma(\varepsilon)$ is the least number of ε-balls needed to cover Σ (see [20]).

It is evidently follows from (22) that the inequality (16) is equivalent to the inequality of the form

$$N_\Sigma(\varepsilon)\,\varepsilon^2 \le c, \tag{23}$$

where the constant c does not depend on ε.

Taking into account that a rectifiable surface Σ is a Lipschitz image of the square G and the inequality of the form (23) is fulfilled for G, it is easy to prove the inequality (23) for Σ.

For a surface Σ in $\mathbb{R}^3$ that has a finite two-dimensional Hausdorff measure $\mathcal{H}^2(\Sigma)$, if there exists a positive constant c, such that

$$c\varepsilon^2 \le \mathcal{H}^2\big(\Sigma \cap B(x,\varepsilon)\big) \quad \forall\, x \in \Sigma \;\; \forall\, \varepsilon \in (0; \operatorname{diam} \Sigma]\,, \tag{24}$$

where diam Σ is the diameter of Σ, and $B(x, \varepsilon)$ denotes the open ball with centre x and radius ε, then the inequalities $P_\Sigma(\varepsilon)\varepsilon^2 \le c_1\mathcal{H}^2(\Sigma) < \infty$ is fulfilled, where $P_\Sigma(\varepsilon)$ is the greatest number of disjoint ε-balls with centres in Σ and the constant c_1 does not depend on ε (see [21, p.309]). Taking into account the inequality $N_\Sigma(2\varepsilon) \le P_\Sigma(\varepsilon)$ (see [18, p.78]), we obtain the inequality (23) for a surface Σ satisfying the condition (24).

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